Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 456740, 12 pages
http://dx.doi.org/10.1155/2015/456740

Review Article

Developments of Mindlin-Reissner Plate Elements

Song Cen[1,2,3] and Yan Shang[1,2]

[1] *Department of Engineering Mechanics, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*
[2] *High Performance Computing Center, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*
[3] *Key Laboratory of Applied Mechanics, School of Aerospace Engineering, Tsinghua University, Beijing 100084, China*

Correspondence should be addressed to Song Cen; censong@tsinghua.edu.cn

Received 8 September 2014; Accepted 13 January 2015

Academic Editor: Xin-Lin Gao

Since 1960s, how to develop high-performance plate bending finite elements based on different plate theories has attracted a great deal of attention from finite element researchers, and numerous models have been successfully constructed. Among these elements, the most popular models are usually formulated by two theoretical bases: the Kirchhoff plate theory and the Mindlin-Reissener plate theory. Due to the advantages that only C^0 continuity is required and the effect of transverse shear strain can be included, the latter one seems more rational and has obtained more attention. Through abundant works, different types of Mindlin-Reissener plate models emerged in many literatures and have been applied to solve various engineering problems. However, it also brings FEM users a puzzle of how to choose a "right" one. The main purpose of this paper is to present an overview of the development history of the Mindlin-Reissner plate elements, exhibiting the state-of-art in this research field. At the end of the paper, a promising method for developing "shape-free" plate elements is recommended.

1. Introduction

Plate structure is a kind of important engineering components and has been widely used in many industries, such as aerospace engineering, shipbuilding industry, and civil engineering. To date, the finite element method is usually considered as the most convenient and effective tool in analyses and designs of various plate structures [1–56]. The history of plate bending elements can be traced back to the 1960s [1, 2]. Since then, how to develop high-performance models has attracted a great deal of attentions from finite element researchers [57–87]. Among these elements, the most popular models are usually formulated by two theories: the Kirchhoff plate theory and the Mindlin-Reissner plate theory. Most early researches focused on the models following the classical Kirchhoff thin plate theory, which can be found in related review paper [5]. However, because transverse shear deformation is neglected, the Kirchhoff plate theory is only appropriate for modeling thin structures. Furthermore, since C^1 continuity between adjacent elements is required, the construction procedures of Kirchhoff plate elements are quite difficult and more complicated. Actually, construction of C^1 approximation for arbitrary finite elements is still a very challenging problem nowadays.

On the other hand, in Mindlin-Reissner (thick) plate theory [6, 7], the deflection w and the rotations ψ_x, ψ_y are independently defined, so that the influences of the transverse shear strain can be included, and only C^0 continuity is required when formulating the interpolation functions of kinematics variables. Hence, the construction of thick plate elements is indeed easier than that of the thin plate elements [8], and more and more attention was paid to develop plate bending elements based on the Mindlin-Reissner plate theory [88–136]. However, the low-order standard isoparametric displacement-based plate elements without special treatments only produce poor results in the thin plate case due to the false shear strains which lead to the shear locking phenomenon. This shear locking problem is primarily caused by the inability to model the Kirchhoff-type constraints in thin plate limits. On the discrete level, only if the trial function space is rich enough, the element can possess good properties. Unfortunately, this is not an easy case to standard low-order elements [9]. To overcome this difficulty, different methods have been proposed, such as the reduced integration

[10, 11] and the selected reduced integration [12, 13]. In addition to the shear locking problem, there are two other great numerical challenges in Mindlin-Reissner plate theory, that is, the singularity caused by the plate obtuse corners and the edge effect caused by certain boundary conditions [14, 15]. Although these two do not pose great effects on the entire structure, they do complicate the numerical analysis. But, no enough attention has been paid to them.

Through substantial attempts from scholars, various Mindlin-Reissner plate bending elements [137–188] are available in numerous literatures. Furthermore, by using the higher-order elasticity theories, some models based on the nonclassical Mindlin-Reissner plate theories are proposed [189–191], such as the modified couple stress theory [192, 193]. However, no one has been treated as the so-called “best” one. Thus, how to choose a “right” one is quite puzzling to the FEM users. At the present stage, the major drawback in Mindlin-Reissner plate elements is that the element’s performance is highly sensitive to mesh distortion. How to develop a “shape-free” element is still an open topic. The expression “shape-free” means that the element can always perform well regardless of the mesh quality. In this paper, a review on the development of the Mindlin-Reissner plate elements is firstly proposed. Then, a promising “shape-free” Mindlin-Reissner plate element method is briefly recommended.

2. Developments of Mindlin-Reissner Plate Elements

The first attempt to employ the Mindlin-Reissner plate theory in finite element method was the 8-node isoparametric element derived from the degenerated shell approach [17]. In early time, most Mindlin-Reissner plate elements were the displacement-based models. A common problem encountered was that these elements often gave poor results in thin plate limit if their element stiffness matrices were evaluated using full integration schemes. This phenomenon is the well-known “shear locking” problem. The first explanation on the shear locking was presented by Zienkiewicz and Hinton [18]. They found that, with the use of a penalty function formulation, the product of the shear stiffness matrix and the displacement vector should be zero when the thickness-span ratio was sufficiently small. This discovery led to the development of so-called “locking indicator” [11], which measured the singularity of an element stiffness matrix. Also based on this work [18], Hughes et al. [12, 19] gave an alternative concept, “constraint index,” to predict locking behavior in different finite element meshes. However, this index often deduced an overly pessimistic assessment. Afterwards, Spilker and Munir [20, 21] proposed a modified version, called “rotational constraint index,” which can achieve better correlations. In order to gain more complete understanding and better estimation of the shear locking, Averill and Reddy [23] proposed another analytical technique verifying the roles that shear constraints play in thin plate. These constraints were expressed in terms of the nodal DOFs and then were judged to be the proper Kirchhoff constraints or spurious locking constraints.

In 1971, in order to eliminate the shear locking problem existing in the Mindlin-Reissner plate elements, Zienkiewicz et al. [10] introduced the famous reduced integration scheme which can drastically improve the element performance by reducing the integration order for all stiffness matrix components. This work [10] was based on the element given by [17]. Almost at the same time, an analogous work was proposed by Pawsey and Clough [24]. Although the reduced integration method did not have theoretical explanations, it indeed was a significant improvement for plate bending elements [25]. Another effective approach was the selective reduced integration, whose validity had been subsequently demonstrated by Hughes and other scholars [12, 13, 19, 26]. Its main idea was to split the strain energy into two parts: bending part and shear part. Lower order integration was used for the troublesome shear strain energy, while full integration was employed for the bending energy. The applications of selective reduced integration may need some modifications in code procedures [27], so that the computational cost will be increased. Since then, these two reduced integration schemes have been widely applied, helping eliminate or alleviate the shear locking phenomenon. Prathap and Somashekar [28] proposed a vagarious strategy with (2×1) and (1×2) Gauss integration, respectively, for the terms γ_{xz} and γ_{yz}. The derived element QUAD4-CC was free of locking and had a proper rank in rectangular forms. Unfortunately, this advantage deteriorated when the element was not rectangular.

Although improved results can been obtained by using reduced or selective reduced integration in most cases, these methods cannot always ensure the absence of shear locking. Belytschko et al. [31] proposed a so-called mode decomposition approach combined with one-point quadrature, which can show improved performances compared to a straightforward application of reduced integration. However, it was still found to shear lock with certain mesh patterns. Furthermore, there were even some cases that elements cannot be improved by them at all [29]. Moreover, the most noteworthy and fatal problem was that reduced or selective reduced integration often brought about spurious zero energy modes [32], which will yield unreliable results under certain boundary conditions. An exception that possessed a correct rank and did not lock was the so-called “heterosis” element [26] which utilized an 8-node interpolation for rotations and 9-node interpolation for deflections. In order to eliminate the rank deficiency, the stabilization method was proposed by Belytschko et al. [33–36]. The essential ingredient of this method was to superimpose a “stabilizing” stiffness matrix to the stiffness matrix obtained by reduced integration. The most well-known one was the γ-method initiated by Flanagan and Belytschko [37], in which a vector γ orthogonal to displacement fields was introduced. From the view of computational cost, this procedure seemed attractive and economical for constructing simple elements. However, additional researches were still needed for understanding the physical meanings of the stabilizing parameters more clearly [27].

Another alternative approach to alleviate shear locking, named by “assumed natural strain (ANS)” method, was proposed by Hughes and Tezduyar [38] and MacNeal [39]. In [38], Hughes and Tezduyar were the first to find that shear locking can be avoided without losing important energy modes. The main idea of ANS method was to define the troublesome shear strains independent of the approximation of displacements. Then, these independent assumed strains were connected with the nodal displacements through the strain-displacement relation calculated at certain discrete points. Finally, they were interpolated over the element by using specific shape functions. This method had been proved to be mathematically valid by Brezzi et al. [40], and the key to the success was related to the choices of these preselected points. After then, based on this concept, many models were successfully developed [41–45]. For instance, Tessler and Hughes [42, 43] proposed the quadrilateral and triangular plate elements, named as MIN4 and MIN3, in which "anisoparametric" representations were used together with continuous transverse shear edge constraints. In their work, a concept of "finite-element-appropriate" shear correction factor, derived through multiplying the classical shear factor with a factor relating to the element's geometrical and material properties, was also proposed. Sze et al. [44] developed a triangular element AST6, which had a favorable constraint index of shear locking and optimized strains with respect to a linear pure bending field. Afterwards, they gave an improved model [45] in where the matrix inversion in original formulations can be avoided.

Some variations of the ANS method were then presented, including the well-known mixed interpolated tensorial components (MITC) method proposed by Bathe et al. [40, 46–49]. Their method consisted in interpolating the covariant shear strain components γ_{xz} and γ_{yz} in a particular manner. A series of MITC plate/shell elements were constructed for linear/nonlinear analysis [50–56], among which the 4-node element MITC4 [47] may be the most widely used one in commercial codes [51], although it experienced some slight locking and shear oscillation in distorted meshes. Based on MITC4, Kebari [57] proposed the one point integrated version, in which the Taylor series expansion approach [52] was employed to accommodate the linear variation of stress within the element.

The discrete shear gap (DSG) method [58] was another scheme analogous to the ANS approach. The difference between these two methods was that the lack of collocation points makes the application of the DSG method independent of the element's order [59]. Thus, the DSG method was employed by many elements to eliminate the shear locking [60–63].

Batoz et al. proposed the discrete Kirchhoff technique [29, 64, 65] for constructing thin plate elements. This "discrete" concept was subsequently extended to the thick plate elements, and different types of models were proposed, such as the DRM element [66], the discrete Mindlin triangle (DMT) element [67], the discrete shear triangular (DST) and discrete shear quadrilateral (DSQ) elements [68–72], and the discrete Kirchhoff–Mindlin elements [73, 74]. These "discrete" thick plate elements can model thick plate well, while coinciding with the DKT/DKQ element for the thin plate.

In 1993, Zeinkiewicz et al. proposed the linked interpolation method [75, 76], which has also been applied by other researchers later. The main thought of this concept was to improve the approximated deflection fields by the rotational DOFs. In this method, one order higher polynomials can be used for the deflection than rotations without adding additional element parameters. Thus, it can help meet the requirements in thin plate limit that the rotations were equal to derivatives of the deflections. It is known that the linked interpolation on its own cannot ensure free of shear locking, unless some other techniques were simultaneously employed, such as bubble modes, selective/reduced integration, or others. Even so, many elements were still developed based on the linked interpolation techniques [25, 75–84]. Besides, some researchers started to study the relation between the linked interpolation elements and other models. For example, T3BL was proved to be essentially the same as the MIN3 [87]. Recently, inspired by the higher-order linked interpolation functions developed for the Timoshenko beam, Ribarić and Jelenić [80] found a distinct basis for developing plate elements. Later, they [81] proposed two higher-order 9-node quadrilateral plate finite elements in which the interpolation functions were expended with three bubble parameters for keeping the polynomial completeness. Although the higher-order linked interpolation can make elements perform well in distorted mesh, it indeed led to vast computational cost owing to additional bubble degrees of freedom.

The hybrid/mixed approach was another way which can effectively and thoroughly eliminate the shear locking problem (see [20, 21, 32, 75–77, 88–118] and the references herein). Lee and Pian [88] explained why shear locking appears in the thin limit case and gave some solutions in hybrid/mixed models. To develop hybrid/mixed elements, various variational principles were available, including the principle of minimum complementary energy, the Hellinger-Reissner variational principle, the Hu-Washizu variational principle, and their modified forms. In the hybrid/mixed method, displacements or boundary displacement and stresses/strains within the element were independently assumed. Thus, performances of these hybrid/mixed elements were largely based on the appropriate choice of parameters. However, the lack of specific rules in a priori choosing the "best" stress combination may be a major drawback. A significant work was proposed by Malkus and Hughes [13]. They showed that there was equivalence between mixed models and displacement models with reduced integration. After this work, a series of effective hybrid/mixed elements were proposed, including some models mentioned above. Spilker and Munir [20] developed a high-order hybrid-stress element with use of the 12-node cubic serendipity shape functions and 36 stress parameters. This element had a favorable "rotational constraint index" [21]. Saleeb and Chang [27] proposed a HMPL5 element with three bubble functions represented by a fifth internal node. However, the use of internal node brought inconvenience and seemed unpopular. Simo and Rifai [117] presented a three-field mixed formulation in terms of displacements, stresses, and an enhanced strain field. Since the stress interpolation was assumed L2-orthogonal to the enhanced strain interpolation, the stress field can be eliminated from the final formulations. Thus, it collapsed into a two-field mixed method. This method can be used to develop low-order elements with enhanced accuracy in coarse meshes. Ayad and his cooperators proposed the mixed shear projected (MiSP) elements, including the stand bilinear models [99] and the high-order ones [113, 114]. This approach represented the transversal shear forces and shear strains in a particular manner: the shear forces were derived from the bending moments using the equilibrium equations and the shear strains were defined in terms of the edge tangential strains. Brasile [25] constructed element TIP3, whose stress fields were assumed from a moment-shear uncoupled polynomial approximation and then forced to satisfy some equilibrium conditions so that the minimum number of the stress parameters can be obtained. de Miranda and Ubertini [100] also proposed a self-equilibrium hybrid element. Particularly, its stresses were expressed by uncoupled polynomial expansions in terms of skew coordinates [119]. Pereira and Freitas [115, 116] employed Legendre polynomials to express the resultants within the element domain and the deflections along boundaries. Because of the orthogonality of the Legendre approximation, a highly sparse finite element solving system was derived. By combination of one-point integration and a stabilization matrix, Gruttmann and Wagner [111] proposed a model whose stiffness matrix can be analytically integrated, so that some computational costs can be saved. Even though there were some inherent drawbacks existing in hybrid/mixed method, such as the need of matrices inversion, it still inspired researchers to develop new types of novel hybrid/mixed models.

Recently, some scholars, mostly among mathematicians, began to employ the discontinuous Galerkin (DG) finite element methods [17] to design plate elements [120–126]. This DG method admitted the discontinuities in the element discrete space, leading to new types of conforming or nonconforming elements. Brezzi and Marini [121] used nonconforming piecewise linear functions for both rotations and transversal displacements and added internal degrees of freedom to make the element perform better. Later, Lovadina [123] proposed a simplified version by using piecewise constant shear stress approximation functions. Arnold et al. [120] developed two DG Mindlin-Reissner plate element families: one was the fully discontinuous case and the other was the case of continuous rotations and nonconforming deflections. Hansbo et al. [126, 127] also proposed a quadrilateral element model with continuous displacements and discontinuous rotations. In this model, the rotations had the same order as the parametric derivatives of the displacements in the parametric reference plane, and they obeyed the same transformation law. Boesing et al. [125] developed an interior penalty DG model. In their work, the a priori error analysis of DG methods was also presented. Although the discontinuous Galerkin method seems somewhat unfamiliar to traditional FEM, it was clear that the DG technique can offer a different way to these problems difficult for a classical approach [120].

Unlike that most quadrilateral elements were constructed in Cartesian or isoparametric coordinates, Cen et al. [128] extended the quadrilateral area coordinate method [129, 130] to formulate a new element AC-MQ4 by employing the general conforming theory [148, 149]. A distinct feature of this method was that the transformation relations between the area coordinates and Cartesian coordinates were always linear, so that the order of the displacement fields expressed in form of area coordinates will not deteriorate in distorted meshes. Hence, the resulting element AC-MQ4 was quite insensitive to mesh distortion. When constructing the element, the locking-free Timoshenko's beam formulas were employed to help eliminate the shear-locking problem. Recently, this technique has also been widely used in developing Mindlin-Reissner plate elements [131–136].

The smoothed finite element method (S-FEM), proposed by Liu et al. [137, 138], was also extended to Mindlin-Reissner plate elements. This method utilized the strain smoothing technique of mesh-free methods [139] to improve the accuracy of standard FEM. When developing smoothed Mindlin-Reissner plate models, the locking-free curvature smoothing formulation is mainly used. In S-FEM models, smoothing domains can be located inside the elements or cover parts of adjacent elements. Thus, based on different smoothing procedures, different types of S-FEM elements were derived, including CS-FEM [51, 140], NS-FEM [61], and ES-FEM [62, 141–143]. Nguyen-Thoi et al. [140] applied the cell-based strain smoothing technique to modify the well-known element MIN3 [87] to formulate a new element CS-MIN3. In CS-FEM, the number of supporting nodes was the same as those in elements. Therefore, the bandwidth of stiffness matrix in the CS-FEM was similar to the standard FEM. Nguyen-Xuan et al. [61] proposed a node-based smoothed model (NS-FEM), in which the strain smoothing was associated with the nodes of the elements. This NS-FEM can provide an upper bound solution in energy norm for the elasticity problem [144]. By introducing the curvatures smoothing technique into the well-known MITC4 [47] and incorporating stabilized conforming nodal integration, Nguyen-Xuan et al. [141] proposed the MISC models. These elements can provide more accurate results, especially in distorted meshes. Furthermore, also based on the concept of S-FEM, Liu et al. [60, 145–147] proposed the alpha finite element method (αFEM), in which the element was enhanced by the additional averaged nodal strain terms with an adjustable parameter α, resulting in a properly softer stiffness formulation.

Based on the mechanism of shear locking phenomenon and the potential functional of Mindlin-Reissner plate, Cai et al. [150] extended the generalized mixed variational principle (GMVP) [151], which was a type of parameterized variational principles, to Mindlin-Reissner plate. In their models, splitting factors were introduced to adjust the shear potential and complementary energy components. Through this way, the element stiffness matrix can be artificially changed, which can make the element free of shear locking and possess higher accuracy.

The so-called free formulation method, introduced in 1975 by Bergan and Hanssen [152], was a radically different approach. Instead of various variational principles, this method [152–157] was directly derived from the conditions that the elements satisfy the requirements of the patch test and the rigid body motions. Such requirements can be greatly relaxed by slight modification of the coupling stiffness between fundamental and higher-order displacement modes [152]. Thus, the shear deformations were only associated with the higher modes and can be tackled in a special way [153, 154].

In addition to the methods mentioned above, there are still many other models and methods proposed by numerous literatures [158–184]. But among them only a fraction of typical models can be briefly introduced here. Through an adaptive mesh-refinement method, Holzer et al. [160] proposed a model employing the enough higher-order interpolation for shape functions. Similar to the model given by [151], Arnold and Brezzi [177] also developed a model containing a shear energy splitting parameter. What is different was that this parameter was used to determine how much shear energy will be exactly integrated. Suggestions for appropriately choosing such parameter were also presented by [178]. Similarly, Wang et al. [171, 172] applied the combined hybrid method to design quadrilateral elements, which can make the system's energy optimal by adjusting the combined parameter. Polit et al. [179] proposed an 8-node quadrilateral element, in which each monomial term of the interpolation functions for the normal rotations was matched by the derivatives of its counterpart deflection. Zhou [176] took the linear combination of the mixed and displacement-based formulations to present a new variational formulation. Cheung and Chen extended the refined nonconforming element method [161] to develop Mindlin-Reissner plate elements [162, 163]. Duan and Liang [174] reformulated the classical mixed variational principle by relaxing the continuity of the transverse displacement while strengthening the normal continuity of the shear stresses. Pontaza and Reddy [175] proposed a formulation based on the least-squares variational principle, in which the high-order nodal expansions and the full integration were used. Maunder and de Almeida [170] proposed equilibrium models in which the stress fields were divided into hyperstatic part and spurious kinematic part. Ming and Shi [173] provided a unified variational formulation for three different elements, MIN4 [42], Q9 [183], and FMIN4 [184], and proved their optimal error bounds. Castellazzi and Krysl [164] proposed a new assumed-strain technique with use of the nodal integration, in which the weighted residual method was employed to weakly enforce the equilibrium equation and the kinematic equation. Later, they [167] verified the insensitivity of this method to geometrical distortion in a variety of distorted meshes.

Other than these conventional models mentioned above, there were some transition elements being proposed for the compatible connection of nonmatching meshes [185–188]. Sohn and Im [185] developed a quadrilateral model which can include an arbitrary number of additional nodes on each edge. Choi and Park [186, 187] proposed different types of transition plate bending elements for adaptive h-refinement. Sofuoglu and Gedikli [188] proposed a 5-node transition element by using the refined nonconforming method [161].

Generally, in order to develop a good element, more than one technique mentioned above can be employed in a combination. After decades of development, the Mindlin-Reissner plate element has been in a rather mature status. However, there are still two major deficiencies: (i) the performances of low-order elements will deteriorate as the meshes distort and (ii) the precision of stress is lower than that of displacement. The former is the more important one, especially in the nonlinear analysis. At present, this problem was generally alleviated at the expense of mathematical complexity or high computational cost and no effective and succinct solutions are available. For this problem, a promising solution was proposed recently, that is, the hybrid displacement function (HDF) method [194] for Mindlin-Reissner plate element. The main characteristic of this method is to assume the stress fields within element through the assumption of the displacement function F [195], rather than directly assuming themselves. All variables of Mindlin-Reissner plate theory can be expressed by such displacement function F, and the derived resultants can satisfy all the governing equations. Furthermore, these solutions can degenerate into the ones of thin plate when the thickness is sufficiently small. The HDF elements are free of shear locking and can perform well in the severely distorted mesh, even when the mesh contains degenerated triangle or concave quadrilateral. Here, the outline of this method is introduced.

For a Mindlin-Reissner plate, the deflection w and rotations ψ_x, ψ_y can be expressed by the displacement function F [195] as follows:

$$\begin{aligned} w &= F - \frac{D}{C}\nabla^2 F, \\ \psi_x &= \frac{\partial F}{\partial x}, \\ \psi_y &= \frac{\partial F}{\partial y}, \end{aligned} \tag{1}$$

in which D and C are the bending and shear stiffness and

$$D\nabla^2\nabla^2 F = q. \tag{2}$$

q in (2) denotes the transversely distributed load. Then, by substituting (1) into the related governing equations of Mindlin-Reissner plate, the resultants $\mathbf{R}$ expressed as the functional of displacement function F are derived as follows:

$$\mathbf{R} = \sum_{i=1}^{n} \mathbf{R}_i^0 \beta_i + \mathbf{R}^* = \widetilde{\mathbf{D}}(F), \tag{3}$$

where $\mathbf{R}_i^0$ ($i = 1 \sim n$) are the general solutions of resultants and β_i are the corresponding coefficients; $\mathbf{R}^*$ is the particular part. The fundamental analytical solutions of displacement function F and the corresponding solutions of the resultant forces have been listed in [194]. Then, substitution of these solutions into the complementary energy functional of the Mindlin-Reissner plate element yields

$$\begin{aligned} \Pi_C^e &= \Pi_C^{e*} + V_C^{e*} \\ &= \frac{1}{2}\iint_{A^e} \widetilde{\mathbf{D}}(F)^{\mathrm{T}} \mathbf{C}\widetilde{\mathbf{D}}(F)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{S^e} \left[\mathbf{L}\widetilde{\mathbf{D}}(F)\right]^{\mathrm{T}} \overline{\mathbf{d}}\,\mathrm{d}s, \end{aligned} \tag{4}$$

where Π_C^{e*} is the complementary energy within the element and V_C^{e*} is the complementary energy along the boundary.

Finally, by applying the principle of minimum complementary energy, the element stiffness matrix $\mathbf{K}^e$ and the element nodal equivalent load vector $\mathbf{P}_q^e$ can be obtained as follows:

$$\mathbf{K}^e \mathbf{q}^e = \mathbf{P}_q^e, \tag{5}$$

in which $\mathbf{q}^e$ is the nodal displacement vector.

The elements derived from the HDF method can produce high-precision results for all field variables. An outstanding feature is that they are quite insensitive to mesh distortions, even when a severely distorted mesh containing concave quadrilateral or degenerated triangular elements is employed. Hence, this HDF method can be treated as a kind of "shape-free" finite element method for Mindlin-Reissner plate. In addition, the HDF method can also be applied to construct special elements for dealing with the edge effect problems of the Mindlin-Reissner plate [196], and the displacement function F can also be used for developing displacement-based element models [197].

3. Conclusions

Due to the fact that only C^0 continuity is required and the effect of transverse shear strain can be included, the Mindlin-Reissner plate theory has become dominant in developing plate elements. In early history of Mindlin-Reissner plate elements, the major difficulty is the shear locking problem. To overcome this problem, some effective and novel methods have been presented. Through plenty of attempts from scholars, different kinds of techniques and concepts are proposed, leading to different types of models. However, no one has emerged as the so-called "best" one. Thus, how to choose a right one is quite puzzling to the FEM users. So, the main purpose of this paper is to present a review on the development of the Mindlin-Reissner plate elements, briefly illustrating features of different models.

Furthermore, even though the development of Mindlin-Reissner plate elements has been in a quite good status, there are still some important difficulties that have not been solved yet, such as that the sensitivity problem to mesh distortion, which is more significant in the nonlinear analysis. Thus, how to develop a "shape-free" model ("shape-free" meaning the performance does not depend on the mesh's quality) is still an open topic. Therefore, at the end of this work, a promising "shape-free" Mindlin plate element model is recommended.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is financially supported by the National Natural Science Foundation of China (11272181), the Specialized Research Fund for the Doctoral Program of Higher Education of China (20120002110080), and Tsinghua University Initiative Scientific Research Program (2014z09099).

References

[1] R. J. Melosh, "A stiffness matrix for the analysis of thin plates in bending," *Journal of the Aerospace Sciences*, vol. 28, no. 1, pp. 34–42, 1961.

[2] R. W. Clough and J. L. Tocher, "Finite element stiffness matrices for analysis of plates in bending," in *Proceedings of Conference on Matrix Methods in Structural Analysis*, pp. 515–545, 1965.

[3] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method for Solid and Structural Mechanics*, Elsevier, Oxford, UK, 6th edition, 2005.

[4] A. J. Fricker, "A simple method for including shear deformations in thin plate elements," *International Journal for Numerical Methods in Engineering*, vol. 23, no. 7, pp. 1355–1366, 1986.

[5] M. M. Hrabok and T. M. Hrudey, "A review and catalogue of plate bending finite elements," *Computers & Structures*, vol. 19, no. 3, pp. 479–495, 1984.

[6] R. D. Mindlin, "Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates," *Journal of Applied Mechanics—Transactions of the ASME*, vol. 18, no. 1, pp. 31–38, 1951.

[7] E. Reissner, "The effect of transverse shear deformation on the bending of elastic plates," *Journal of Applied Mechanics-Transactions of the ASME*, vol. 12, no. 2, pp. A69–A77, 1945.

[8] Y.-Q. Long, S. Cen, and Z.-F. Long, *Advanced Finite Element Method in Structural Engineering*, Tsinghua University Press, Beijing, China, Springer, Berlin, Germany, 2009.

[9] D. N. Arnold, "Discretization by finite elements of a model parameter dependent problem," *Numerische Mathematik*, vol. 37, no. 3, pp. 405–421, 1981.

[10] O. C. Zienkiewicz, R. L. Taylor, and J. M. Too, "Reduced integration technique in general analysis of plates and shells," *International Journal for Numerical Methods in Engineering*, vol. 3, no. 2, pp. 275–290, 1971.

[11] E. D. L. Pugh, E. Hinton, and O. C. Zienkiewicz, "A study of quadrilateral plate bending elements with 'reduced' integration," *International Journal for Numerical Methods in Engineering*, vol. 12, no. 7, pp. 1059–1079, 1978.

[12] T. J. R. Hughes, M. Cohen, and M. Haroun, "Reduced and selective integration techniques in the finite element analysis of plates," *Nuclear Engineering and Design*, vol. 46, no. 1, pp. 203–222, 1978.

[13] D. S. Malkus and T. J. R. Hughes, "Mixed finite element methods—reduced and selective integration techniques: a unification of concepts," *Computer Methods in Applied Mechanics and Engineering*, vol. 15, no. 1, pp. 63–81, 1978.

[14] C. A. Xenophontos, "Finite element computations for the Reissner-Mindlin plate model," *Communications in Numerical Methods in Engineering*, vol. 14, no. 12, pp. 1119–1131, 1998.

[15] D. N. Arnold and R. S. Falk, "A uniformly accurate finite element method for the Reissner-Mindlin plate," *SIAM Journal on Numerical Analysis*, vol. 26, no. 6, pp. 1276–1290, 1989.

[16] H. R. H. Kabir, "Shear locking free isoparametric three-node triangular finite element for moderately-thick and thin plates," *International Journal for Numerical Methods in Engineering*, vol. 35, no. 3, pp. 503–519, 1992.

[17] S. Ahmad, B. M. Irons, and O. C. Zienkiewicz, "Analysis of thick and thin shell structures by curved finite elements," *International Journal for Numerical Methods in Engineering*, vol. 2, no. 3, pp. 419–451, 1970.

[18] O. C. Zienkiewicz and E. Hinton, "Reduced integration, function smoothing and non-conformity in finite element analysis (with special reference to thick plates)," *Journal of the Franklin Institute*, vol. 302, no. 5-6, pp. 443–461, 1976.

[19] T. J. R. Hughes, R. L. Taylor, and W. Kanoknukulchai, "Simple and efficient finite element for plate bending," *International Journal for Numerical Methods in Engineering*, vol. 11, no. 10, pp. 1529–1543, 1977.

[20] R. L. Spilker and N. I. Munir, "The hybrid-stress model for thin plates," *International Journal for Numerical Methods in Engineering*, vol. 15, no. 8, pp. 1239–1260, 1980.

[21] R. L. Spilker and N. I. Munir, "A hybrid-stress quadratic serendipity displacement mindlin plate bending element," *Computers and Structures*, vol. 12, no. 1, pp. 11–21, 1980.

[22] G. Prathap and G. R. Bhashyam, "Reduced integration and the shear-flexible beam element," *International Journal for Numerical Methods in Engineering*, vol. 18, no. 2, pp. 195–210, 1982.

[23] R. C. Averill and J. N. Reddy, "Behaviour of plate elements based on the first-order shear deformation theory," *Engineering Computations*, vol. 7, no. 1, pp. 57–74, 1990.

[24] S. F. Pawsey and R. W. Clough, "Improved numerical integration of thick shell finite elements," *International Journal for Numerical Methods in Engineering*, vol. 3, no. 4, pp. 575–586, 1971.

[25] S. Brasile, "An isostatic assumed stress triangular element for the Reissner-Mindlin plate-bending problem," *International Journal for Numerical Methods in Engineering*, vol. 74, no. 6, pp. 971–995, 2008.

[26] T. J. R. Hughes and M. Cohen, "The 'heterosis' finite element for plate bending," *Computers & Structures*, vol. 9, no. 5, pp. 445–450, 1978.

[27] A. F. Saleeb and T. Y. Chang, "An efficient quadrilateral element for plate bending analysis," *International Journal for Numerical Methods in Engineering*, vol. 24, no. 6, pp. 1123–1155, 1987.

[28] G. Prathap and B. R. Somashekar, "Field- and edge-consistency synthesis of a 4-noded quadrilateral plate bending element," *International Journal for Numerical Methods in Engineering*, vol. 26, no. 8, pp. 1693–1708, 1988.

[29] J. L. Batoz, K. J. Bathe, and L. W. Ho, "A study of 3-node triangular plate bending elements," *International Journal for Numerical Methods in Engineering*, vol. 15, no. 12, pp. 1771–1812, 1980.

[30] G. Prathap and S. Viswanath, "An optimally integrated 4-node quadrilateral plate bending element," *International Journal for Numerical Methods in Engineering*, vol. 19, no. 6, pp. 831–840, 1983.

[31] T. Belytschko, H. Stolarski, and N. Carpenter, "A C^0 triangular plate element with one-point quadrature," *International Journal for Numerical Methods in Engineering*, vol. 20, no. 5, pp. 787–802, 1984.

[32] R. L. Spilker, "Invariant 8-node hybrid-stress elements for thin and moderately thick plates," *International Journal for Numerical Methods in Engineering*, vol. 18, no. 8, pp. 1153–1178, 1982.

[33] T. Belytschko, C. S. Tsay, and W. K. Liu, "A stabilization matrix for the bilinear Mindlin plate element," *Computer Methods in Applied Mechanics and Engineering*, vol. 29, no. 3, pp. 313–327, 1981.

[34] T. Belytschko and C. S. Tsay, "A stabilization procedure for the quadrilateral plate element with one point quadrature," *International Journal for Numerical Methods in Engineering*, vol. 19, no. 3, pp. 405–419, 1983.

[35] T. Belytschko, J. S. J. Ong, and W. K. Liu, "A consistent control of spurious singular modes in the 9-node Lagrange element for the laplace and mindlin plate equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 44, no. 3, pp. 269–295, 1984.

[36] W. K. Liu, J. S. Ong, and R. A. Uras, "Finite element stabilization matrices—a unification approach," *Computer Methods in Applied Mechanics and Engineering*, vol. 53, no. 1, pp. 13–46, 1985.

[37] D. P. Flanagan and T. Belytschko, "A uniform strain hexahedron and quadrilateral with orthogonal hourglass control," *International Journal for Numerical Methods in Engineering*, vol. 17, no. 5, pp. 679–706, 1981.

[38] T. J. R. Hughes and T. E. Tezduyar, "Finite elements based upon Mindlin plate theory with particular reference to the four-node bilinear isoparametric element," *Transactions ASME—Journal of Applied Mechanics*, vol. 48, no. 3, pp. 587–596, 1981.

[39] R. H. MacNeal, "Derivation of element stiffness matrices by assumed strain distributions," *Nuclear Engineering and Design*, vol. 70, no. 1, pp. 3–12, 1982.

[40] F. Brezzi, K.-J. Bathe, and M. Fortin, "Mixed-interpolated elements for Reissner-Mindlin plates," *International Journal for Numerical Methods in Engineering*, vol. 28, no. 8, pp. 1787–1801, 1989.

[41] T. J. R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice Hall, Englewood Cliffs, NJ, USA, 1987.

[42] A. Tessler and T. J. R. Hughes, "An improved treatment of transverse shear in the mindlin-type four-node quadrilateral element," *Computer Methods in Applied Mechanics and Engineering*, vol. 39, no. 3, pp. 311–335, 1983.

[43] A. Tessler and T. J. R. Hughes, "A three-node mindlin plate element with improved transverse shear," *Computer Methods in Applied Mechanics and Engineering*, vol. 50, no. 1, pp. 71–101, 1985.

[44] K. Y. Sze, D. Zhu, and D.-P. Chen, "Quadratic triangular C^0 plate bending element," *International Journal for Numerical Methods in Engineering*, vol. 40, no. 5, pp. 937–951, 1997.

[45] K. Y. Sze and D. Zhu, "A quadratic assumed natural strain triangular element for plate bending analysis," *Communications in Numerical Methods in Engineering*, vol. 14, no. 11, pp. 1013–1025, 1998.

[46] K. J. Bathe and F. Brezzi, "On the convergence of a four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation," in *Proceedings of the 5th Conference on Mathematics of Finite Elements and Applications*, New York, NY, USA, 1985.

[47] K.-J. Bathe and E. N. Dvorkin, "A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation," *International Journal for Numerical Methods in Engineering*, vol. 21, no. 2, pp. 367–383, 1985.

[48] E. N. Dvorkin and K.-J. Bathe, "A continuum mechanics based four-node shell element for general non-linear analysis," *Engineering computations*, vol. 1, no. 1, pp. 77–88, 1984.

[49] K.-J. Bathe and E. N. Dvorkin, "A formulation of general shell elements—the use of mixed interpolation of tensorial components," *International Journal for Numerical Methods in Engineering*, vol. 22, no. 3, pp. 697–722, 1986.

[50] F. Brezzi, M. Fortin, and R. Stenberg, "Error analysis of mixed-interpolated elements for Reissner-Mindlin plates," *Mathematical Models & Methods in Applied Sciences*, vol. 1, no. 2, pp. 125–151, 1991.

[51] C. T. Wu and H. P. Wang, "An enhanced cell-based smoothed finite element method for the analysis of Reissner-Mindlin plate bending problems involving distorted mesh," *International Journal for Numerical Methods in Engineering*, vol. 95, no. 4, pp. 288–312, 2013.

[52] W. K. Liu, E. S. Law, D. Lam, and T. Belytschko, "Resultant-stress degenerated-shell element," *Computer Methods in Applied Mechanics and Engineering*, vol. 55, no. 3, pp. 259–300, 1986.

[53] Y. Lee, P.-S. Lee, and K.-J. Bathe, "The MITC3+ shell element and its performance," *Computers and Structures*, vol. 138, pp. 12–23, 2014.

[54] H.-M. Jeon, P.-S. Lee, and K.-J. Bathe, "The MITC3 shell finite element enriched by interpolation covers," *Computers and Structures*, vol. 134, pp. 128–142, 2014.

[55] K. J. Bathe, F. Brezzi, and L. D. Marini, "The MITC9 shell element in plate bending: mathematical analysis of a simplified case," *Computational Mechanics*, vol. 47, no. 6, pp. 617–626, 2011.

[56] P.-S. Lee and K.-J. Bathe, "The quadratic MITC plate and MITC shell elements in plate bending," *Advances in Engineering Software*, vol. 41, no. 5, pp. 712–728, 2010.

[57] H. Kebari, "A one point integrated assumed strain 4-node Mindlin plate element," *Engineering Computations*, vol. 7, no. 4, pp. 284–290, 1990.

[58] K.-U. Bletzinger, M. Bischoff, and E. Ramm, "A unified approach for shear-locking-free triangular and rectangular shell finite elements," *Computers & Structures*, vol. 75, no. 3, pp. 321–334, 2000.

[59] G. Falsone and D. Settineri, "A Kirchhoff-like solution for the Mindlin plate model: a new finite element approach," *Mechanics Research Communications*, vol. 40, pp. 1–10, 2012.

[60] N. Nguyen-Thanh, T. Rabczuk, H. Nguyen-Xuan, and S. Bordas, "An alternative alpha finite element method with discrete shear gap technique for analysis of isotropic Mindlin-Reissner plates," *Finite Elements in Analysis and Design*, vol. 47, no. 5, pp. 519–535, 2011.

[61] H. Nguyen-Xuan, T. Rabczuk, N. Nguyen-Thanh, T. Nguyen-Thoi, and S. Bordas, "A node-based smoothed finite element method with stabilized discrete shear gap technique for analysis of Reissner-Mindlin plates," *Computational Mechanics*, vol. 46, no. 5, pp. 679–701, 2010.

[62] X. Cui, G. R. Liu, G. Y. Li, G. Zhang, and G. Zheng, "Analysis of plates and shells using an edge-based smoothed finite element method," *Computational Mechanics*, vol. 45, no. 2-3, pp. 141–156, 2010.

[63] C. V. Le, "A stabilized discrete shear gap finite element for adaptive limit analysis of Mindlin-Reissner plates," *International Journal for Numerical Methods in Engineering*, vol. 96, no. 4, pp. 231–246, 2013.

[64] J. L. Batoz and M. B. Bentahar, "Evaluation of a new quadrilateral thin plate bending element," *International Journal for Numerical Methods in Engineering*, vol. 18, no. 11, pp. 1655–1677, 1982.

[65] J. L. Batoz, "An explicit formulation for an efficient triangular plate-bending element," *International Journal for Numerical Methods in Engineering*, vol. 18, no. 7, pp. 1077–1089, 1982.

[66] O. C. Zienkiewicz, R. L. Taylor, P. Papadopoulos, and E. Oñate, "Plate bending elements with discrete constraints: new triangular elements," *Computers & Structures*, vol. 35, no. 4, pp. 505–522, 1990.

[67] J. Aalto, "From Kirchhoff to Mindlin plate elements," *Communications in Applied Numerical Methods*, vol. 4, no. 2, pp. 231–241, 1988.

[68] J.-L. Batoz and P. Lardeur, "A discrete shear triangular nine D.O.F. element for the analysis of thick to very thin plates," *International Journal for Numerical Methods in Engineering*, vol. 28, no. 3, pp. 533–560, 1989.

[69] J.-L. Batoz and I. Katili, "On a simple triangular reissner/mindlin plate element based on incompatible modes and discrete constraints," *International Journal for Numerical Methods in Engineering*, vol. 35, no. 8, pp. 1603–1632, 1992.

[70] Y. C. Cai, L. G. Tian, and S. N. Atluri, "A simple locking-free discrete shear triangular plate element," *Computer Modeling in Engineering & Sciences*, vol. 77, no. 3-4, pp. 221–238, 2011.

[71] P. Lardeur, *Développement et Évaluation de Deux Nouveaux Éléments Finis de Plaques et Coques Composites Avec Influence Du Cisaillement Transversal*, UTC, Compiègne, France, 1990.

[72] J. L. Batoz and G. Dhatt, *Modélisation des Structures Par Éléments Finis: Solides Élastiques*, Université Laval, Paris, France, 1990.

[73] I. Katili, "New discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields. Part II. An extended DKQ element for thick-plate bending analysis," *International Journal for Numerical Methods in Engineering*, vol. 36, no. 11, pp. 1885–1908, 1993.

[74] I. Katili, "New discrete Kirchhoff-Mindlin element based on Mindlin-Reissner plate theory and assumed shear strain fields. Part I: an extended DKT element for thick-plate bending analysis," *International Journal for Numerical Methods in Engineering*, vol. 36, no. 11, pp. 1859–1883, 1993.

[75] O. C. Zeinkiewicz, Z. N. Xu, L. F. Zeng, A. Samuelsson, and N.-E. Wiberg, "Linked interpolation for Ressiner-Mindlin plate element: part I—a simple quadrilateral," *International Journal for Numerical Methods in Engineering*, vol. 36, no. 18, pp. 3043–3056, 1993.

[76] R. L. Taylor and F. Auricchio, "Linked interpolation for Reissner-Mindlin plate elements: part II—a simple triangle," *International Journal for Numerical Methods in Engineering*, vol. 36, no. 18, pp. 3057–3066, 1993.

[77] F. Auricchio and R. L. Taylor, "A shear deformable plate element with an exact thin limit," *Computer Methods in Applied Mechanics and Engineering*, vol. 118, no. 3-4, pp. 393–412, 1994.

[78] X. Zhongnian, "A thick-thin triangular plate element," *International Journal for Numerical Methods in Engineering*, vol. 33, no. 5, pp. 963–973, 1992.

[79] R. G. Durán and E. Liberman, "On the convergence of a triangular mixed finite element method for Reissner-Mindlin plates," *Mathematical Models and Methods in Applied Sciences*, vol. 6, no. 3, pp. 339–352, 1996.

[80] D. Ribarić and G. Jelenić, "Higher-order linked interpolation in quadrilateral thick plate finite elements," *Finite Elements in Analysis and Design*, vol. 51, pp. 67–80, 2012.

[81] D. Ribarić and G. Jelenić, "Distortion-immune nine-node displacement-based quadrilateral thick plate finite elements that satisfy constant-bending patch test," *International Journal for Numerical Methods in Engineering*, vol. 98, no. 7, pp. 492–517, 2014.

[82] F. Auricchio and R. L. Taylor, "A triangular thick plate finite element with an exact thin limit," *Finite Elements in Analysis and Design*, vol. 19, no. 1-2, pp. 57–68, 1995.

[83] P. Papadopoulos and R. L. Taylor, "A triangular element based on Reissner-Mindlin plate theory," *International Journal for Numerical Methods in Engineering*, vol. 30, no. 5, pp. 1029–1049, 1990.

[84] Z. Xu, O. C. Zienkiewicz, and L. F. Zeng, "Linked interpolation for Reissner-Mindlin plate elements: Part III. An alternative quadrilateral," *International Journal for Numerical Methods in Engineering*, vol. 37, no. 9, pp. 1437–1443, 1994.

[85] F. Auricchio and C. Lovadina, "Analysis of kinematic linked interpolation methods for Reissner-Mindlin plate problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 190, no. 18-19, pp. 2465–2482, 2001.

[86] C. Lovadina, "Analysis of a mixed finite element method for the Reissner-Mindlin plate problems," *Computer Methods in Applied Mechanics and Engineering*, vol. 163, no. 1–4, pp. 71–85, 1998.

[87] M. Lyly, "On the connection between some linear triangular Reissner-Mindlin plate bending elements," *Numerische Mathematik*, vol. 85, no. 1, pp. 77–107, 2000.

[88] S. W. Lee and T. H. H. Pian, "Improvement of plate and shell finite elements by mixed formulations," *AIAA Journal*, vol. 16, no. 1, pp. 29–34, 1978.

[89] S. W. Lee and J. C. Zhang, "A 6-node finite element for plate bending," *International Journal for Numerical Methods in Engineering*, vol. 21, no. 1, pp. 131–143, 1985.

[90] S. W. Lee and S. C. Wong, "Mixed formulation finite elements for Mindlin theory plate bending," *International Journal for Numerical Methods in Engineering*, vol. 18, no. 9, pp. 1297–1311, 1982.

[91] T. H. H. Pian and K. Sumihara, "Hybrid SemiLoof elements for plates and shells based upon a modified Hu-Washizu principle," *Computers & Structures*, vol. 19, no. 1-2, pp. 165–173, 1984.

[92] S. L. Weissman and R. L. Taylor, "Resultant fields for mixed plate bending elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 79, no. 3, pp. 321–355, 1990.

[93] R. D. Cook, "Two hybrid elements for analysis of thick, thin and sandwich plates," *International Journal for Numerical Methods in Engineering*, vol. 5, no. 2, pp. 277–288, 1972.

[94] J. Robinson and G. W. Haggenmacher, "LORA-an accurate four node stress plate bending element," *International Journal for Numerical Methods in Engineering*, vol. 14, no. 2, pp. 296–306, 1979.

[95] A. F. Saleeb, T. Y. Chang, and S. Yingyeunyong, "A mixed formulation of C^0-linear triangular plate/shell element-the role of edge shear constraints," *International Journal for Numerical Methods in Engineering*, vol. 26, no. 5, pp. 1101–1128, 1988.

[96] M. Gellert, "A new method for derivation of locking-free plate bending finite elements via mixed hybrid formulation," *International Journal for Numerical Methods in Engineering*, vol. 26, no. 5, pp. 1185–1200, 1988.

[97] O. C. Zienkiewicz and D. Lefebvre, "A robust triangular plate bending element of the Reissner–Mindlin type," *International Journal for Numerical Methods in Engineering*, vol. 26, no. 5, pp. 1169–1184, 1988.

[98] P. M. Pinsky and R. V. Jasti, "A mixed finite element formulation for Reissner-Mindlin plates based on the use of bubble functions," *International Journal for Numerical Methods in Engineering*, vol. 28, no. 7, pp. 1677–1702, 1989.

[99] R. Ayad, G. Dhatt, and J. L. Batoz, "A new hybrid-mixed variational approach for Reissner-Mindlin plates. The MISP model," *International Journal for Numerical Methods in Engineering*, vol. 42, no. 7, pp. 1149–1179, 1998.

[100] S. de Miranda and F. Ubertini, "A simple hybrid stress element for shear deformable plates," *International Journal for Numerical Methods in Engineering*, vol. 65, no. 6, pp. 808–833, 2006.

[101] H.-Y. Duan and G.-P. Liang, "Analysis of some stabilized low-order mixed finite element methods for Reissner-Mindlin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 191, no. 3–5, pp. 157–179, 2001.

[102] G. Y. Shi and P. Tong, "Assumed stress C^0 quadrilateral triangular plate elements by interrelated edge displacements," *International Journal for Numerical Methods in Engineering*, vol. 39, no. 6, pp. 1041–1051, 1996.

[103] K. J. Bathe, *Finite Element Procedures*, Prentice Hall, Upper Saddle River, NJ, USA, 1996.

[104] M. A. Aminpour, "Direct formulation of a hybrid 4-node shell element with drilling degrees of freedom," *International Journal for Numerical Methods in Engineering*, vol. 35, no. 5, pp. 997–1013, 1992.

[105] J. Jirousek, A. Wroblewski, and B. Szybinski, "New 12 DOF quadrilateral element for analysis of thick and thin plates," *International Journal for Numerical Methods in Engineering*, vol. 38, no. 15, pp. 2619–2638, 1995.

[106] F. S. Jin and Q. H. Qin, "A variational principle and hybrid Trefftz finite element for the analysis of Reissner plates," *Computers & Structures*, vol. 56, no. 4, pp. 697–701, 1995.

[107] Y. F. Dong and J. A. T. Defreitas, "A quadrilateral hybrid stress element for mindlin plates based on incompatible displacements," *International Journal for Numerical Methods in Engineering*, vol. 37, no. 2, pp. 279–296, 1994.

[108] T. H. H. Pian and D. P. Chen, "Alternative ways for formulation of hybrid stress elements," *International Journal for Numerical Methods in Engineering*, vol. 18, no. 11, pp. 1679–1684, 1982.

[109] G. Shi and G. Z. Voyiadjis, "Efficient and accurate four-node quadrilateral C^0 plate bending element based on assumed strain fields," *International Journal for Numerical Methods in Engineering*, vol. 32, no. 5, pp. 1041–1055, 1991.

[110] G. Shi and G. Z. Voyiadjis, "Simple and efficient shear flexible two-node arch/beam and four-node cylindrical shell/plate finite elements," *International Journal for Numerical Methods in Engineering*, vol. 31, no. 4, pp. 759–776, 1991.

[111] F. Gruttmann and W. Wagner, "A stabilized one-point integrated quadrilateral Reissner-Mindlin plate element," *International Journal for Numerical Methods in Engineering*, vol. 61, no. 13, pp. 2273–2295, 2004.

[112] A. K. Noor and C. M. Andersen, "Mixed models and reduced selective integration displacement models for non-linear shell analysis," *International Journal for Numerical Methods In Engineering*, vol. 18, no. 10, pp. 1429–1454, 1982.

[113] R. Ayad and A. Rigolot, "An improved four-node hybrid-mixed element based upon Mindlin's plate theory," *International Journal for Numerical Methods in Engineering*, vol. 55, no. 6, pp. 705–731, 2002.

[114] R. Ayad, A. Rigolot, and N. Talbi, "An improved three-node hybrid-mixed element for Mindlin/Reissner plates," *International Journal for Numerical Methods in Engineering*, vol. 51, no. 8, pp. 919–942, 2001.

[115] E. M. B. R. Pereira and J. A. T. Freitas, "Numerical implementation of a hybrid-mixed finite element model for Reissner-Mindlin plates," *Computers & Structures*, vol. 74, no. 3, pp. 323–334, 2000.

[116] E. M. B. R. Pereira and J. A. T. Freitas, "A hybrid-mixed finite element model based on Legendre polynomials for Reissner-Mindlin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 136, no. 1-2, pp. 111–126, 1996.

[117] J. C. Simo and M. S. Rifai, "A class of mixed assumed strain methods and the method of incompatible modes," *International Journal for Numerical Methods in Engineering*, vol. 29, no. 8, pp. 1595–1638, 1990.

[118] Y. S. Choo, N. Choi, and B. C. Lee, "A new hybrid-Trefftz triangular and quadrilateral plate elements," *Applied Mathematical Modelling*, vol. 34, no. 1, pp. 14–23, 2010.

[119] K.-Y. Yuan, Y.-S. Huang, and T. H. H. Pian, "New strategy for assumed stresses for 4-node hybrid stress membrane element," *International Journal for Numerical Methods in Engineering*, vol. 36, no. 10, pp. 1747–1763, 1993.

[120] D. N. Arnold, F. Brezzi, and L. D. Marini, "A family of discontinuous Galerkin finite elements for the Reissner-Mindlin plate," *Journal of Scientific Computing*, vol. 22-23, no. 1, pp. 25–45, 2005.

[121] F. Brezzi and L. D. Marini, "A nonconforming element for the Reissner-Mindlin plate," *Computers and Structures*, vol. 81, no. 8-11, pp. 515–522, 2003.

[122] C. Chinosi, C. Lovadina, and L. D. Marini, "Nonconforming locking-free finite elements for Reissner-Mindlin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 195, no. 25–28, pp. 3448–3460, 2006.

[123] C. Lovadina, "A low-order nonconforming finite element for Reissner-Mindlin plates," *SIAM Journal on Numerical Analysis*, vol. 42, no. 6, pp. 2688–2705, 2005.

[124] D. N. Arnold, F. Brezzi, R. S. Falk, and L. D. Marini, "Locking-free Reissner-Mindlin elements without reduced integration," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 37–40, pp. 3660–3671, 2007.

[125] P. R. Boesing, A. L. Madureira, and I. Mozolevski, "A new interior penalty discontinuous Galerkin method for the Reissner-Mindlin model," *Mathematical Models and Methods in Applied Sciences*, vol. 20, no. 8, pp. 1343–1361, 2010.

[126] P. Hansbo and M. G. Larson, "Locking free quadrilateral continuous/discontinuous finite element methods for the Reissner-Mindlin plate," *Computer Methods in Applied Mechanics and Engineering*, vol. 269, pp. 381–393, 2014.

[127] P. Hansbo, D. Heintz, and M. G. Larson, "A finite element method with discontinuous rotations for the Mindlin-Reissner plate model," *Computer Methods in Applied Mechanics and Engineering*, vol. 200, no. 5–8, pp. 638–648, 2011.

[128] S. Cen, Y.-Q. Long, Z.-H. Yao, and S.-P. Chiew, "Application of the quadrilateral area co-ordinate method: a new element for Mindlin-Reissner plate," *International Journal for Numerical Methods in Engineering*, vol. 66, no. 1, pp. 1–45, 2006.

[129] Y. Q. Long, J. X. Li, Z. F. Long, and S. Cen, "Area co-ordinates used in quadrilateral elements," *Communications in Numerical Methods in Engineering*, vol. 15, no. 8, pp. 533–545, 1999.

[130] Z. F. Long, J. X. Li, S. Cen, and Y. Q. Long, "Some basic formulae for area co-ordinates in quadrilateral elements," *Communications in Numerical Methods in Engineering*, vol. 15, no. 12, pp. 841–852, 1999.

[131] A. Ibrahimbegović, "Plate quadrilateral finite element with incompatible modes," *Communications in Applied Numerical Methods*, vol. 8, no. 8, pp. 497–504, 1992.

[132] A. Ibrahimbegović, "Quadrilateral finite elements for analysis of thick and thin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 110, no. 3-4, pp. 195–209, 1993.

[133] A. K. Soh, Z. F. Long, and S. Cen, "A new nine dof triangular element for analysis of thick and thin plates," *Computational Mechanics*, vol. 24, no. 5, pp. 408–417, 1999.

[134] A.-K. Soh, S. Cen, Y.-Q. Long, and Z.-F. Long, "A new twelve DOF quadrilateral element for analysis of thick and thin plates," *European Journal of Mechanics. A. Solids*, vol. 20, no. 2, pp. 299–326, 2001.

[135] C. S. Wang, P. Hu, and Y. Xia, "A 4-node quasi-conforming Reissner-Mindlin shell element by using Timoshenko's beam function," *Finite Elements in Analysis and Design*, vol. 61, pp. 12–22, 2012.

[136] H. X. Zhang and J. S. Kuang, "Eight-node Reissner-Mindlin plate element based on boundary interpolation using Timoshenko beam function," *International Journal for Numerical Methods in Engineering*, vol. 69, no. 7, pp. 1345–1373, 2007.

[137] G. R. Liu, K. Y. Dai, and T. T. Nguyen, "A smoothed finite element method for mechanics problems," *Computational Mechanics*, vol. 39, no. 6, pp. 859–877, 2007.

[138] G. R. Liu, "A generalized gradient smoothing technique and the smoothed bilinear form for Galerkin formulation of a wide class of computational methods," *International Journal of Computational Methods*, vol. 5, no. 2, pp. 199–236, 2008.

[139] J. S. Chen, C. T. Wu, S. Yoon, and Y. You, "A stabilized conforming nodal integration for Galerkin mesh-free methods," *International Journal for Numerical Methods in Engineering*, vol. 50, no. 2, pp. 435–466, 2001.

[140] T. Nguyen-Thoi, P. Phung-Van, H. Luong-Van, H. Nguyen-Van, and H. Nguyen-Xuan, "A cell-based smoothed three-node Mindlin plate element (CS-MIN3) for static and free vibration analyses of plates," *Computational Mechanics*, vol. 51, no. 1, pp. 65–81, 2013.

[141] H. Nguyen-Xuan, T. Rabczuk, S. Bordas, and J. F. Debongnie, "A smoothed finite element method for plate analysis," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 13–16, pp. 1184–1203, 2008.

[142] C. M. Shin and B. C. Lee, "Development of a strain-smoothed three-node triangular flat shell element with drilling degrees of freedom," *Finite Elements in Analysis and Design*, vol. 86, pp. 71–80, 2014.

[143] H. Nguyen-Xuan, G. R. Liu, C. Thai-Hoang, and T. Nguyen-Thoi, "An edge-based smoothed finite element method (ES-FEM) with stabilized discrete shear gap technique for analysis of Reissner-Mindlin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 199, no. 9–12, pp. 471–489, 2010.

[144] G. R. Liu and G. Y. Zhang, "Upper bound solution to elasticity problems: a unique property of the linearly conforming point interpolation method (LC-PIM)," *International Journal for Numerical Methods in Engineering*, vol. 74, no. 7, pp. 1128–1161, 2008.

[145] G. R. Liu, T. Nguyen-Thoi, and K. Y. Lam, "A novel alpha finite element method (αFEM) for exact solution to mechanics problems using triangular and tetrahedral elements," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 45–48, pp. 3883–3897, 2008.

[146] G. R. Liu, H. Nguyen-Xuan, T. Nguyen-Thoi, and X. Xu, "A novel Galerkin-like weakform and a superconvergent alpha finite element method (SαFEM) for mechanics problems using triangular meshes," *Journal of Computational Physics*, vol. 228, no. 11, pp. 4055–4087, 2009.

[147] G. R. Liu, T. Nguyen-Thoi, and K. Y. Lam, "A novel FEM by scaling the gradient of strains with factor α (αFEM)," *Computational Mechanics*, vol. 43, no. 3, pp. 369–391, 2009.

[148] L. Yu-Qiu and X. Ke-Gui, "Generalized conforming element for bending and buckling analysis of plates," *Finite Elements in Analysis and Design*, vol. 5, no. 1, pp. 15–30, 1989.

[149] Y. L. Chen, S. Cen, Z. H. Yao, Y. Q. Long, and Z. F. Long, "Development of triangular flat-shell element using a new thin-thick plate bending element based on semiLoof constrains," *Structural Engineering and Mechanics*, vol. 15, no. 1, pp. 83–114, 2003.

[150] L. Cai, T. Rong, and D. Chen, "Generalized mixed variational methods for Reissner plate and its applications," *Computational Mechanics*, vol. 30, no. 1, pp. 29–37, 2002.

[151] T.-Y. Rong, "Generalized mixed variational principles and new FEM models in solid mechanics," *International Journal of Solids and Structures*, vol. 24, no. 11, pp. 1131–1140, 1988.

[152] P. G. Bergan and L. Hanssen, "A new approach for deriving 'good' finite elements," *The Mathematics of Finite Elements and Applications*, vol. 2, pp. 483–498, 1975.

[153] P. G. Bergan and X. Wang, "Quadrilateral plate bending elements with shear deformations," *Computers and Structures*, vol. 19, no. 1-2, pp. 25–34, 1984.

[154] P. G. Bergan and M. K. Nygard, "Finite elements with increased freedom in choosing shape functions," *International Journal for Numerical Methods in Engineering*, vol. 20, no. 4, pp. 643–663, 1984.

[155] I. W. Liu, T. Kerh, and C. C. Lin, "A conforming quadrilateral plate bending element with shear deformations," *Computers & Structures*, vol. 56, no. 1, pp. 93–100, 1995.

[156] C. A. Felippa and P. G. Bergan, "A triangular bending element based on an energy-orthogonal free formulation," *Computer Methods in Applied Mechanics and Engineering*, vol. 61, no. 2, pp. 129–160, 1987.

[157] P. G. Bergan, "Finite elements based on energy orthogonal functions," *International Journal for Numerical Methods in Engineering*, vol. 15, no. 10, pp. 1541–1555, 1980.

[158] E. Hinton and H. C. Huang, "A family of quadrilateral Mindlin plate elements with substitute shear strain fields," *Computers and Structures*, vol. 23, no. 3, pp. 409–431, 1986.

[159] O. L. Roufaeil, "A new four-node quadrilateral plate bending element," *Computers & Structures*, vol. 54, no. 5, pp. 871–879, 1995.

[160] S. Holzer, E. Rank, and H. Werner, "An implementation of the *hp*-version of the finite element method for Reissner-Mindlin plate problems," *International Journal for Numerical Methods in Engineering*, vol. 30, no. 3, pp. 459–471, 1990.

[161] Y. K. Cheung and W. J. Chen, "Refined nine-parameter triangular thin plate bending element by using refined direct stiffness method," *International Journal for Numerical Methods in Engineering*, vol. 38, no. 2, pp. 283–298, 1995.

[162] W. J. Chen and Y. K. Cheung, "Refined quadrilateral element based on Mindlin/Reissner plate theory," *International Journal for Numerical Methods in Engineering*, vol. 47, no. 1–3, pp. 605–627, 2000.

[163] C. Wanji and Y. K. Cheung, "Refined 9-Dof triangular Mindlin plate elements," *International Journal for Numerical Methods in Engineering*, vol. 51, no. 11, pp. 1259–1282, 2001.

[164] G. Castellazzi and P. Krysl, "Displacement-based finite elements with nodal integration for Reissner-Mindlin plates," *International Journal for Numerical Methods in Engineering*, vol. 80, no. 2, pp. 135–162, 2009.

[165] C.-K. Choi and S.-H. Kim, "Coupled use of reduced integration and non-conforming modes in quadratic Mindlin plate element," *International Journal for Numerical Methods in Engineering*, vol. 28, no. 8, pp. 1909–1928, 1989.

[166] E. Hinton and N. Bićanić, "A comparison of lagrangian and serendipity mindlin plate elements for free vibration analysis," *Computers and Structures*, vol. 10, no. 3, pp. 483–493, 1979.

[167] G. Castellazzi and P. Krysl, "A nine-node displacement-based finite element for Reissner-Mindlin plates based on an improved formulation of the NIPE approach," *Finite Elements in Analysis and Design*, vol. 58, pp. 31–43, 2012.

[168] X. Y. Zhuang, R. Q. Huang, H. H. Zhu, H. Askes, and K. Mathisen, "A new and simple locking-free triangular thick plate element using independent shear degrees of freedom," *Finite Elements in Analysis and Design*, vol. 75, pp. 1–7, 2013.

[169] C. Carstensen, X. P. Xie, G. Z. Yu, and T. X. Zhou, "A priori and a posteriori analysis for a locking-free low order quadrilateral hybrid finite element for Reissner-Mindlin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 200, no. 9–12, pp. 1161–1175, 2011.

[170] E. A. W. Maunder and J. P. M. de Almeida, "A triangular hybrid equilibrium plate element of general degree," *International Journal for Numerical Methods in Engineering*, vol. 63, no. 3, pp. 315–350, 2005.

[171] B. Hu, Z. Wang, and Y. Xu, "Combined hybrid method applied in the Reissner-Mindlin plate model," *Finite Elements in Analysis and Design*, vol. 46, no. 5, pp. 428–437, 2010.

[172] Z. Wang and B. Hu, "Research of combined hybrid method applied in the Reissner-Mindlin plate model," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 49–66, 2006.

[173] P. Ming and Z.-C. Shi, "Analysis of some low order quadrilateral Reissner-Mindlin plate elements," *Mathematics of Computation*, vol. 75, no. 255, pp. 1043–1065, 2006.

[174] H.-Y. Duan and G.-P. Liang, "Mixed and nonconforming finite element approximations of Reissner-Mindlin plates," *Computer Methods in Applied Mechanics and Engineering*, vol. 192, no. 49-50, pp. 5265–5281, 2003.

[175] J. P. Pontaza and J. N. Reddy, "Mixed plate bending elements based on least-squares formulation," *International Journal for Numerical Methods in Engineering*, vol. 60, no. 5, pp. 891–922, 2004.

[176] T. X. Zhou, "The partial projection method in the finite element discretization of the Reissner-Mindlin plate model," *Journal of Computational Mathematics*, vol. 13, no. 2, pp. 172–191, 1995.

[177] D. N. Arnold and F. Brezzi, "Some new elements for the Reissner-Mindlin plate model," in *Boundary Value Problems for Partial Differential Equations and Applications*, pp. 287–292, 1993.

[178] C. Chinosi and C. Lovadina, "Numerical analysis of some mixed finite element methods for Reissner-Mindlin plates," *Computational Mechanics*, vol. 16, no. 1, pp. 36–44, 1995.

[179] O. Polit, M. Touratier, and P. Lory, "A new 8-node quadrilateral shear-bending plate finite element," *International Journal for Numerical Methods in Engineering*, vol. 37, no. 3, pp. 387–411, 1994.

[180] H. R. Dhananjaya, J. Nagabhushanam, P. C. Pandey, and M. Z. Jumaat, "New twelve node serendipity quadrilateral plate bending element based on Mindlin-Reissner theory using integrated force method," *Structural Engineering and Mechanics*, vol. 36, no. 5, pp. 625–642, 2010.

[181] H. R. Dhananjaya, P. C. Pandey, and J. Nagabhushanam, "New eight node serendipity quadrilateral plate bending element for thin and moderately thick plates using integrated force method," *Structural Engineering and Mechanics*, vol. 33, no. 4, pp. 485–502, 2009.

[182] J. Hu and Z.-C. Shi, "Error analysis of quadrilateral Wilson element for Reissner-Mindlin plate," *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 6–8, pp. 464–475, 2008.

[183] M. Lyly, R. Stenberg, and T. Vihinen, "A stable bilinear element for the Reissner-Mindlin plate model," *Computer Methods in Applied Mechanics and Engineering*, vol. 110, no. 3-4, pp. 343–357, 1993.

[184] F. Kikuchi and K. Ishii, "Improved 4-node quadrilateral plate bending element of the Reissner-Mindlin type," *Computational Mechanics*, vol. 23, no. 3, pp. 240–249, 1999.

[185] D. Sohn and S. Im, "Variable-node plate and shell elements with assumed natural strain and smoothed integration methods for nonmatching meshes," *Computational Mechanics*, vol. 51, no. 6, pp. 927–948, 2013.

[186] C. K. Choi and Y. M. Park, "Conforming and nonconforming transition plate bending elements for an adaptive *h*-refinement," *Thin-Walled Structures*, vol. 28, no. 1, pp. 1–20, 1997.

[187] C.-K. Choi and Y.-M. Park, "Transition plate-bending elements for compatible mesh gradation," *Journal of Engineering Mechanics*, vol. 118, no. 3, pp. 462–480, 1992.

[188] H. Sofuoglu and H. Gedikli, "A refined 5-node plate bending element based on Reissner-Mindlin theory," *Communications in Numerical Methods in Engineering*, vol. 23, no. 5, pp. 385–403, 2007.

[189] S. Li, "On the micromechanics theory of Reissner-Mindlin plates," *Acta Mechanica*, vol. 142, no. 1, pp. 47–99, 2000.

[190] A. Eijo, E. Oñate, and S. Oller, "A four-noded quadrilateral element for composite laminated plates/shells using the refined zigzag theory," *International Journal for Numerical Methods in Engineering*, vol. 95, no. 8, pp. 631–660, 2013.

[191] H. M. Ma, X.-L. Gao, and J. N. Reddy, "A non-classical Mindlin plate model based on a modified couple stress theory," *Acta Mechanica*, vol. 220, no. 1–4, pp. 217–235, 2011.

[192] H. M. Ma, X.-L. Gao, and J. N. Reddy, "A microstructure-dependent Timoshenko beam model based on a modified couple stress theory," *Journal of the Mechanics and Physics of Solids*, vol. 56, no. 12, pp. 3379–3391, 2008.

[193] S. K. Park and X.-L. Gao, "Bernoulli-Euler beam model based on a modified couple stress theory," *Journal of Micromechanics and Microengineering*, vol. 16, no. 11, article 2355, 2006.

[194] S. Cen, Y. Shang, C.-F. Li, and H.-G. Li, "Hybrid displacement function element method: a simple hybrid-Trefftz stress element method for analysis of Mindlin-Reissner plate," *International Journal for Numerical Methods in Engineering*, vol. 98, no. 3, pp. 203–234, 2014.

[195] H. C. Hu, *Variational Principles of Theory of Elasticity with Applications*, Science Press, Beijing, China; Gordan and Breach Science, New York, NY, USA, 1984.

[196] Y. Shang, S. Cen, C.-F. Li, and J.-B. Huang, "An effective hybrid displacement function element method for solving the edge effect of Mindlin-Reissner plate," *International Journal for Numerical Methods in Engineering*, 2015.

[197] Y. Shang, S. Cen, C.-F. Li, and X.-R. Fu, "Two generalized conforming quadrilateral Mindlin-Reissner plate elements based on the displacement function," *Finite Elements in Analysis and Design*, vol. 99, pp. 24–28, 2015.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization