Research Article

$H_\infty$ Control for Nonlinear Systems with Time-Varying Delay Using Matrix-Based Quadratic Convex Approach

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$H_\infty$ control problem for nonlinear system with time-varying delay is considered by using a set of improved Lyapunov-Krasovskii functionals including some integral terms, and a matrix-based on quadratic convex, combined with Wirtinger’s inequalities and some useful integral inequality. $H_\infty$ controller is designed via memoryless state feedback control and new sufficient conditions for the existence of the $H_\infty$ state feedback for the system are given in terms of linear matrix inequalities (LMIs). Numerical examples are given to illustrate the effectiveness of the obtained result.

1. Introduction

The phenomena of time delays are often encountered in many practical systems such as process control systems, manufacturing systems, networked control systems, and economic systems. The existence of these delays may be the source of instability and serious deterioration in the performance of the closed-loop systems. In real world systems especially, the delay should be assumed to be time-varying satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$ and $\tau_1$ is not necessarily restricted to be 0, namely, interval time-varying delay. Stability analysis of time-delay system has been investigated extensively in the past decades [1–25].

As of time delays, it is well known that the nonlinear perturbations can also cause instability and poor performance of practical systems. Therefore, the stability problem of time-delay systems with nonlinear perturbations has received increasing attention; see [9, 13, 16] and the references cited therein.

$H_\infty$ control problem has been widely used to minimize the effects of the external disturbances. The purpose of the problem is to design an $H_\infty$ controller to robustly stabilize the systems while guaranteeing a prescribed level of disturbance attenuation $\gamma$ in the $H_\infty$ sense for the systems with external disturbances. A delay-dependent $H_\infty$ controller ensures asymptotic stability and a prescribed $H_\infty$ performance level of the closed-loop systems. The $H_\infty$ performance indexes and the upper bound of the delay are usually two performance indexes to be used to evaluate the conservatism of the derived condition. The conservatism of the delay-dependent $H_\infty$ control is measured by the allowable delay size or $H_\infty$ performance level bound obtained.

Recently, an improved robust stability and $H_\infty$ performance analysis criterion has been reported [3–5, 7, 10–12, 15, 17, 18]. In [17], $H_\infty$ control problem for uncertain linear system with state delay and parameter uncertainties has been studied, but the time-varying delay is only bounded above by a constant. $H_\infty$ control problem for system with interval time-varying delay has been considered in [7], by employing free weighting matrices approach. However, some useful terms in estimating the derivative of Lyapunov-Krasovskii functional are ignored which might lead to some conservatism. In [15], $H_\infty$ control problem for nonlinear systems with interval time-varying delay has been studied by using Jensen’s inequality to estimate some integral terms of Lyapunov-Krasovskii functional and deriving delay-dependent sufficient condition for the existence of $H_\infty$ control by using reciprocally convex combination technique.

In the study of time-delay system, several approaches have been proposed in order to reduce conservatism.
2 Mathematical Model and Preliminaries

The following notations will be used in this paper: \( \mathbb{R}^n \) denotes the set of all nonnegative real numbers; \( \mathbb{R}^n \) denotes the \( n \)-dimensional space with the Euclidean norm \( \| \cdot \| \); \( \mathcal{M}^{n \times r} \) denotes the space of all matrices of \((n \times r)\)-dimensions.

\( A^T \) denotes the transpose of matrix \( A \); \( A \) is symmetric if \( A = A^T \); \( I \) denotes the identity matrix; \( \lambda(A) \) denotes the set of all eigenvalues of \( A \); \( \lambda\max(A) = \max\{\Re\lambda; \lambda \in \lambda(A)\} \).

\( x_t := \{x(t + s) : s \in [-h, t]\}; [x_t]\| = \sup_{s \in [-h, t]}\|x(t + s)\| \) denotes the set of all \( M^{n \times r} \)-valued continuous functions on \([t_0, t] \); \( \mathcal{L}_2([t_0, t], \mathbb{R}^m) \) denotes the set of all the \( M^{n \times r} \)-valued square integrable functions on \([t_0, t] \).

Matrix \( A \) is called semipositive definite (\( A \geq 0 \)) if \( \langle Ax, x \rangle \geq 0 \), for all \( x \in \mathbb{R}^n \); \( A \) is positive definite (\( A > 0 \)) if \( \langle Ax, x \rangle > 0 \), for all \( x \neq 0 \); \( A > B \) means \( A - B > 0 \). The symmetric term in a matrix is denoted by \(*\).

Consider the following system with time-varying delays and control input:

\[
\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + Bu(t) + Cw(t) + f(t, x(t), x(t - \tau(t)), u(t), w(t)),
\]

\[
z(t) = Ex(t) + Gx(t - \tau(t)) + Fu(t) + g(t, x(t), x(t - \tau(t)), u(t)),
\]

\[
x(t_0 + \theta) = \phi(\theta),
\]

\[
\theta \in [-\tau_2, 0], \quad (t_0, \phi) \in \mathcal{L}^2 \times \mathcal{C}([-\tau_2, t_0], \mathbb{R}^n),
\]

where \( x(t) \in \mathbb{R}^n \) is the state; \( u(t) \in \mathbb{R}^m \) is the control input; \( w(t) \in \mathcal{L}_2([0, \infty], \mathbb{R}) \) is a disturbance input; and \( z(t) \in \mathbb{R}^p \) is the observation output. The delay \( \tau(t) \) is time-varying continuous function which satisfies

\[
0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \quad \mu_1 \leq \tau (t) \leq \mu_2.
\]

Let \( x^* = x(t - \tau(t)) \). The nonlinear functions \( f(t, x, x^*, u, w) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( g(t, x, x^*, u) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^p \) satisfy the following growth condition:

\[
\exists a, b, c, d > 0 : \left\| f(t, x, x^*, u, w) \right\|
\leq a \| x \| + b \| x^* \| + c \| u \| + d \| w \|,
\]

\[
\forall (x, x^*, u, w),
\]

\[
\exists a_1, b_1, c_1 > 0 : \left\| g(t, x, x^*, u) \right\|^2
\leq a_1 \| x \|^2 + b_1 \| x^* \|^2 + c_1 \| u \|^2,
\]

\[
\forall (x, x^*, u).
\]

**Definition 1.** Given \( \gamma > 0 \), the \( H_{\infty} \) control problem for system (1) is to seek if there exists a memoryless state feedback controller \( u(t) = Kx(t) \) such that we have the following.

(i) The zero solution of the closed-loop system, where \( w(t) = 0 \),

\[
\dot{x} = (A + BK)x(t) + Dx(t - \tau(t)) + f(t, x(t), x^*, u, 0),
\]

is asymptotically stable.

(ii) The \( H_{\infty} \) performance

\[
\|z(t)\|_2 < \gamma \| w(t)\|_2
\]

of the closed-loop system (5) is guaranteed for all nonzero \( w(t) \in \mathcal{L}_2([0, \infty], \mathbb{R}) \) and a prescribed \( \gamma > 0 \) under the condition \( x(t) = 0, \forall t \in [-\tau_2, t_0] \).

In this case, we say that the feedback control \( u(t) = Kx(t) \) asymptotically stabilizes the system.

We introduce the following technical lemmas, which will be used in the proof of our results.
Lemma 2 (see [14]). For a given matrix $R > 0$, the following inequality holds for any continuously differentiable function $\omega : [a, b] \rightarrow \mathbb{R}^n$:

$$
\int_a^b \omega^T(u) R \dot{\omega}(u) \, du \geq \frac{1}{b-a} \left( \Gamma_1^T R \Gamma_1 + 3 \Gamma_2^T R \Gamma_2 \right),
$$

(7)

where

$$
\Gamma_1 := \omega(b) - \omega(a), \quad \Gamma_2 := \omega(b) + \omega(a) - \frac{2}{b-a} \int_a^b \omega(u) \, du.
$$

(8)

Remark 3. It is clear to see that the inequality in Lemma 2 provides a tighter lower bound for $\int_a^b \omega^T(u) R \dot{\omega}(u) \, du$ than Jensen's inequality since $3 \Gamma_2^T R \Gamma_2 > 0$ for $\Gamma_2 \neq 0$. Thus, inequality (7) is an improvement over Jensen's inequality.

Before we introduce some useful integral inequalities, we denote

$$
\nu_1(t) := \frac{1}{\tau_2 - \tau(t)} \int_{t-\tau(t)}^{t-\tau_2} y(s) \, ds,
$$

$$
\nu_2(t) := \frac{1}{\tau(t) - \tau_1} \int_{t-\tau}^{t-\tau_1} y(s) \, ds,
$$

(9)

$$
\nu_3(t) := \frac{1}{\tau_1} \int_t^{t-\tau_1} y(s) \, ds.
$$

Lemma 4 (see [20]). For a given scalar $\tau_1 \geq 0$ and any $n \times n$ real matrices $Y_1 > 0$ and $Y_2 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \rightarrow \mathbb{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any vector-valued function $\gamma(t) : [0, \infty) \rightarrow \mathbb{R}^n$ and matrices $M_1 \in \mathbb{R}^{k \times n}$ and $N_1 \in \mathbb{R}^{k \times k}$ satisfying $[M_1, N_1^T] \geq 0$,

$$
\varphi_1 := \int_{t-\tau_1}^t (t - \tau + s) \dot{y}^T(s) Y_1 \dot{y}(s) \, ds
\geq -\frac{\tau_1}{2} \tau_1^T(t) M_1 \tau_1(t) - 2 \tau_1 \tau_1^T(t) N_1 \left[ y(t) - \nu_3(t) \right],
$$

(10)

$$
\varphi_2 := \int_{t-\tau_1}^t (t - \tau + s)^2 \dot{y}^T(s) Y_2 \dot{y}(s) \, ds
\geq \tau_1 \left[ y(t) - \nu_3(t) \right]^T Y_2 \left[ y(t) - \nu_3(t) \right],
$$

where $\nu_3(t)$ is defined in (9).

Lemma 5 (see [21]). Let $\tau(t)$ be a continuous function satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$. For any $n \times n$ real matrix $R_2 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \rightarrow \mathbb{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $\phi_1 \in \mathbb{R}^m$ and real matrices $Z_i \in \mathbb{R}^{m \times n}$ satisfying $[Z_i, F_i] \geq 0$, ($i = 1, 2$),

$$
-\int_{t-\tau_2}^{t-\tau_1} (t - \tau + s) \dot{y}^T(s) Z_i \dot{y}(s) \, ds
\leq \frac{1}{2} (\tau_2 - \tau(t))^2 \phi_1^T(t) Z_1 \phi_1(t) + 2 \left( \frac{\tau_2 - \tau_1}{\tau} \right) \phi_1^T(t) F_1 \phi_1(t)
$$

$$
+ \frac{1}{2} \left[ \left( \tau_2 - \tau_1 \right)^2 - \left( \tau_2 - \tau_1 \right)^2 \right] \phi_1^T(t) Z_2 \phi_1(t)
$$

$$
+ 2 \phi_2^T F_2 \left( \left( \tau_2 - \tau(t) \right) \phi_2(t) + \left( \tau(t) - \tau_1 \right) \phi_2(t) \right),
$$

where

$$
\phi_{12} := y(t) - \nu_3(t),
$$

$$
\phi_{22} := y(t) - \nu_3(t) - x(t - \tau(t)),
$$

(12)

$$
\phi_{23} := y(t - \tau_1) - \nu_2(t),
$$

with $\nu_i(t)$ ($i = 1, 2$) being defined in (9).

Lemma 6 (see [20]). Let $\tau(t)$ be a continuous function satisfying $0 \leq \tau_1 \leq \tau(t) \leq \tau_2$. For any $n \times n$ real matrix $R_1 > 0$ and a vector $\dot{y} : [-\tau_2, 0] \rightarrow \mathbb{R}^n$ such that the integration concerned below is well defined, the following inequality holds for any $2n \times 2n$ real matrix $S_1$ satisfying $[\tilde{R}_1, S_1^T] \geq 0$,

$$
-\int_{t-\tau_2}^{t-\tau_1} (t - \tau + s) \dot{y}^T(s) R_1 \dot{y}(s) \, ds
\leq 2 \psi_{11}^T S_1 \psi_{21} - \psi_{11}^T \tilde{R}_1 \psi_{11} - \psi_{21}^T \tilde{R}_1 \psi_{21},
$$

where $\tilde{R}_1 := \text{diag}(R_1, 3 R_1)$ and

$$
\psi_{11} := \left[ \begin{array}{c}
y(t - \tau(t)) - y(t - \tau_2)
y(t - \tau(t)) + y(t - \tau_2) - 2 \nu_1(t)
\end{array} \right],
$$

(14)

$$
\psi_{21} := \left[ \begin{array}{c}
y(t - \tau_1) + y(t - \tau(t)) - 2 \nu_2(t)
y(t - \tau_1) - y(t - \tau(t))
\end{array} \right],
$$

with $\nu_i(t)$ ($i = 1, 2$) being defined in (9).

Remark 7. In Lemma 6, when $\tau_1 = 0$, the inequality reduces to similar one in [14]. It contains slack matrix variable $S_1$ with dimension $2n \times 2n$ comparing to slack matric variable with dimension $2n \times 5n$ introduced in [8].

Lemma 8 (see [20]). Let $\chi_0, \chi_1,$ and $\chi_2$ be $m \times m$ real symmetric matrices and a continuous function $\tau$ satisfy $\tau_1 \leq \tau \leq \tau_2$, where $\tau_1$ and $\tau_2$ are constants satisfying $0 \leq \tau_1 \leq \tau_2$. If $\lambda_0 \geq 0$, then

$$
\tau_1^2 \chi_0 + \tau_2 \chi_1 + \chi_2 \leq 0 \leq \tau_2^2 \chi_0 + \tau_2 \chi_1 + \chi_2 \leq 0 \leq \lambda_0, \quad \forall \tau \in [\tau_1, \tau_2],
$$

(15)

$$
\tau_1^2 \chi_0 + \tau_2 \chi_1 + \chi_2 \geq 0 \geq \tau_2^2 \chi_0 + \tau_2 \chi_1 + \chi_2 \geq 0 \geq \lambda_0, \quad (i = 1, 2),
$$

or

$$
\tau_1^2 \chi_0 + \tau_2 \chi_1 + \chi_2 \geq 0 \geq \tau_2^2 \chi_0 + \tau_2 \chi_1 + \chi_2 \geq 0 \geq \lambda_0, \quad (i = 1, 2).
$$

(16)
3. Main Results

In this section, we give a design of memoryless $H_\infty$ feedback control for system (1). First, we present delay-dependent asymptotical stabilizability analysis conditions for the nonlinear system with time-varying delay (1). Now, we operate the matrix-based quadratic convex approach with the integral inequalities in [20] to formulate a new stability criterion for system (1). For our goal, we choose the following Lyapunov-Krasovskii functional:

$$V(t, x, \dot{x}) = V_1(t) + V_2(t) + V_3(t), \quad (17)$$

where $x_t$ denotes the function $x(t)$ defined on the interval $[t-\tau, t]$. Set $P_1 = P^{-1}$, $y(t) = P_t x(t)$, $\tau_1 := \tau_2 - \tau_1$, and

$$V_1(t) := y^T(t) P y(t) + \int_{t-\tau_1}^{t} y^T(s) Q_0 y(s) \, ds,$$

$$V_2(t) := \int_{t-\tau_1}^{t} \left[ y^T(t) y^T(s) \right] Q_1 \left[ y^T(t) y^T(s) \right]^T \, ds$$

$$+ \int_{t-\tau_1}^{t} \left[ y^T(t) y^T(s) \right] Q_2 \left[ y^T(t) y^T(s) \right]^T \, ds$$

$$+ \int_{t-\tau_1}^{t-\tau_2} \left[ y^T(t) y^T(s) \right] Q_3 \left[ y^T(t) y^T(s) \right]^T \, ds,$$

$$V_3(t) := \int_{t-\tau_1}^{t} \left[ \tau_1(\tau_1-t+s) y^T(s) W_1 y(s) \right. \left. + (\tau_1-t+s) y^T(s) W_2 y(s) \right] \, ds$$

$$+ \int_{t-\tau_1}^{t} \left[ \tau_2(\tau_2-t+s) y^T(s) R_1 y(s) \right. \left. + (\tau_2-t+s) y^T(s) R_2 y(s) \right] \, ds, \quad (18)$$

where $Q_j > 0$, $(j = 0, 1, 2, 3)$, $W_j > 0$, $R_j > 0$, and $P$ are real matrices to be determined. Before introducing the main theorem, for simplicity, we set

$$\epsilon = a + b + c + \frac{4d^2}{y}. \quad (19)$$

Theorem 9. Given $y > 0$, then system (1) is asymptotically stabilizable and satisfies $\|z(t)\|_2 < y\|w(t)\|_2$ for all nonzero $w \in \mathcal{L}_2[0, \infty)$ if there exist positive definite matrices $P, Q_j > 0$,

$$(j = 0, 1, 2, 3), W_1, W_2, R_1, R_2, S_1, Z_1, Z_2, Z_3, N_1, N_2, N_3, and Y such that the following LMIs hold:

$$\begin{bmatrix} \bar{R}_1 & S_1 \\ S_1^T & \bar{R}_1 \end{bmatrix} \geq 0, \quad \begin{bmatrix} Z_i & N_i \\ N_i^T & R_i \end{bmatrix} \geq 0, \quad (i = 1, 2),$$

$$\begin{bmatrix} Z_3 & N_3 \\ N_3^T & W_2 \end{bmatrix} \geq 0, \quad \Xi_2 (r_1, \mu_1) + \Xi_3 (r_1) + \Xi_4 < 0,$$

$$\Xi_2 (r_1, \mu_2) + \Xi_3 (r_2) + \Xi_4 < 0,$$

$$\Xi_2 (r_2, \mu_2) + \Xi_3 (r_2) + \Xi_4 < 0,$$

$$\Xi_2 (r_2, \mu_2) + \Xi_3 (r_2) + \Xi_4 < 0,$$

where $\bar{R}_1 = \text{diag}(R_1, 3R_1)$ and

$$\Xi_2 (\tau(t), \dot{\tau}(t)) := \Xi_{20} + \tau(t) - \Xi_{21} + [\tau_2 - \tau(t)] \Xi_{22},$$

$$\Xi_3 (\tau(t)) := \Xi_{30} + \tau_2 \Xi_{31} + [\tau_2 - \tau(t)] \Xi_{32},$$

$$\Xi_4 := r_1 Z_3 - \Xi_{41} W_1 \Xi_{42} + \Xi_{43} (r_2 W_2) \Xi_{44} + 2r_1 N_3 (e_1 - e_7) + e_9 (AP + PA^T + BY + Y^T B^T) + \frac{4CC^T + \epsilon I}{y} e_1$$

$$+ e_7 (PD) e_2 + e_8 (D^T P) e_1 + e_9 (AP + YB^T) e_9$$

$$+ e_9 (PA^T + Y^T B) e_1 + e_9 (PD^T) e_2$$

$$+ e_9 (Q_0) e_9 + e_9 \left( -2P + \frac{4CC^T + \epsilon I + Q_0}{y} \right) e_9$$

$$+ e_1 (P) e_10 + e_1^T (P) e_1 + e_1^T (P) e_1 + e_1 (Y^T) e_11 + e_1^T (Y) e_1$$

$$+ e_2 (P) e_12 + e_1^T (PE^T) e_12 + e_1^T (EP) e_1$$

$$+ e_2 (P) e_13 + e_1^T (GP) e_2$$

$$+ e_2 (P) e_14 + e_1^T (P) e_2 + e_1^T \left( -\frac{1}{2a + 4d_1} I \right) e_10$$

$$+ e_1^T \left( -\frac{1}{2a + 4d_1} I \right) e_11 + e_1^T \left( -\frac{I}{3} \right) e_{12}$$

$$+ e_1^T \left( -\frac{I}{3} \right) e_{13} + e_1^T \left( -\frac{1}{2b + 4d_2} I \right) e_{14}, \quad (24)$$

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with \( e_i \in \mathbb{R}^{n \times 14n} (i = 1, 2, \ldots, 14) \) denoting the \( i \)th row-block vector of the \( 14n \times 14n \) identity matrix \( W_1 = \text{diag}(W_i, 3W_i) \)
and
\[
\Xi_{20} := [e_i^T \ e_i^T] (Q_2 - Q_1) [e_i^T \ e_i^T]^T
+ r_1 [e_i^T \ 0] Q_1 [e_i^T \ e_i^T]^T + r_1 [e_i^T \ e_i^T] Q_1 [e_i^T \ 0]^T
- (1 - \dot{r}(t)) [e_i^T \ e_i^T] (Q_2 - Q_3) [e_i^T \ e_i^T]^T
- [e_i^T \ e_i^T] Q_3 [e_i^T \ e_i^T]^T + [e_i^T \ e_i^T] Q_1 [e_i^T \ e_i^T]^T,
\]
\[
\Xi_{21} := [e_i^T \ e_i^T] Q_2 [e_i^T \ 0]^T + [e_i^T \ 0] Q_2 [e_i^T \ e_i^T]^T,
\]
\[
\Xi_{22} := [e_i^T \ e_i^T] Q_3 [e_i^T \ 0]^T + [e_i^T \ 0] Q_3 [e_i^T \ e_i^T]^T,
\]
\[
\Xi_{31} := 2N_1 (e_2 - e_3) + 2N_2 (e_3 - e_2)
+ 2 (e_2 - e_3)^T N_2^T + 2 (e_2 - e_3)^T N_1^T,
\]
\[
\Xi_{32} := 2N_1 (e_3 - e_4) + 2 (e_3 - e_4)^T N_1^T,
\]
\[
\bar{q}_1 := \text{col} \left\{ e_4 - e_2, e_2 + e_4 - 2e_1 \right\},
\]
\[
\bar{q}_2 := \text{col} \left\{ e_3 - e_2, e_2 + e_3 - 2e_1 \right\},
\]
\[
\bar{q}_3 := \text{col} \left\{ e_1 - e_3, e_3 + e_1 - 2e_2 \right\}.
\]

(25)

Moreover, the feedback control is given by
\[
u(t) = Y P^{-1} x(t), \quad t \geq 0.
\]

(26)

Proof. Taking the derivative of \( V \) along the solution of system (1), we obtain
\[
V_1 = 2 y^T (t) P \ddot{y} (t) + y^T (t) Q_0 \ddot{y} (t)
- y^T (t - r_1) Q_0 \ddot{y} (t - r_1),
\]
\[
V_2 = \left[ y^T (t) \ y^T (t) \right] Q_1 \left[ y^T (t) \ y^T (t) \right]^T
- \left[ y^T (t) \ y^T (t - r_1) \right] Q_1 \left[ y^T (t) \ y^T (t - r_1) \right]^T
+ \int_{t-r_1}^{t} 2 \left[ y^T (t) \ y^T (s) \right] Q_1 \left[ y^T (t) \ 0 \right]^T ds
+ \left[ y^T (t) \ y^T (t - r_1) \right] Q_2
\cdot \left[ y^T (t) \ y^T (t - r_1) \right]^T
- (1 - \dot{r}(t)) \left[ y^T (t) \ y^T (t - r(t)) \right] Q_2
\cdot \left[ y^T (t) \ y^T (t - r(t)) \right]^T
+ 2 \int_{t-r(t)}^{t} \left[ y^T (t) \ y^T (s) \right] Q_2 \left[ y^T (t) \ 0 \right]^T ds
\nonumber
\nonumber
\nonumber
\nonumber
\nonumber
- \left[ y^T (t) \ y^T (t - r_1) \right] Q_3 \left[ y^T (t) \ y^T (t - r_2) \right]^T
+ (1 - \dot{r}(t)) \left[ y^T (t) \ y^T (t - r(t)) \right] Q_3
\cdot \left[ y^T (t) \ y^T (t - r(t)) \right]^T
+ \int_{t-r(t)}^{t-\tau(t)} 2 \left[ y^T (t) \ y^T (s) \right] Q_2 \left[ y^T (t) \ 0 \right]^T ds.
\]

(27)

\[V_3 = r_1 \dot{y} (t) \tau_1 W_1 \dot{y} (t) + r_2 \dot{y} (t) W_2 \dot{y} (t)
- \int_{t-r_1}^{t} \left[ y^T (s) \right] \tau_1 W_1 \dot{y} (s) ds
- 2 \int_{t-r_1}^{t} \left[ y^T (s) \right] W_2 \dot{y} (s) ds
+ r_2 \dot{y} (t) \tau_1 \dot{y} (t - r_1)
+ r_2 \dot{y} (t) \tau_1 R_2 \dot{y} (t - r_1)
- \int_{t-r_2}^{t-r_1} \left[ y^T (s) \right] R_2 \dot{y} (s) ds
- 2 \int_{t-r_2}^{t-r_1} \left[ y^T (s) \right] R_2 \dot{y} (s) ds.
\]

(28)

From (4) and Cauchy inequality, we get the following inequalities:
\[
2x^T P_1 f (t, x, x_0, u, w)
\leq 2 \| P_1 x \| \| f (t, x, x_0, u, w) \|
\leq 2 \| P_1 x \| (a \| x \| + b \| x_0 \| + c \| u \| + d \| w \|)
\leq a \| P_1 x \|^2 + a \| x \|^2 + b \| P_1 x_0 \|^2 + b \| P_1 x \|^2 + c \| P_1 x \|^2
+ c \| u \|^2 + \frac{4d^2}{y} \| P_1 x \|^2 + \frac{y}{4} \| w \|^2
= a \| x \|^2 + b \| x_0 \|^2 + c \| u \|^2 + \frac{y}{4} \| w \|^2 + e \| P_1 x \|^2.
\]

(29)

Similarly,
\[
2x^T P_1 f (t, x, x_0, u, w)
\leq a \| x \|^2 + b \| x_0 \|^2 + c \| u \|^2 + \frac{y}{4} \| w \|^2 + e \| P_1 x \|^2,
\]
\[
2x^T P_1 C w \leq \frac{y}{4} \| w \|^2 + \frac{4}{y} \| x(t) \|^2 P_1 C C^T P_1 x (t),
\]
\[
2 \dot{x} (t)^T P_1 C w \leq \frac{y}{4} \| w \|^2 + \frac{4}{y} \| \dot{x} (t) \|^2 P_1 C C^T P_1 \dot{x} (t).
\]

(30)
By using the following identity relation:

\[-2\dot{x}^T(t) P_1 [\dot{x}(t) - Ax(t) - Dx(t - \tau(t)) - Bu(t) - Cw(t) - f(\cdot)] = 0,
\]

we obtain the following:

\[0 = -2\dot{x}^T(t) P_1 [\dot{x}(t) - Ax(t) - Dx(t - \tau(t)) - Bu(t) - Cw(t) - f(\cdot)] \leq -2\dot{x}^T(t) P_1 \left[ \dot{x}(t) - Ax(t) - Dx(t - \tau(t)) - BY \dot{x}(t) \right] + 2\dot{x}^T(t) P_1 Cw(t) + 2\dot{x}^T(t) P_1 f(\cdot) \leq -2\dot{V}_1^T(t) P_1 \dot{y}(t) + 2\dot{y}^T(t) A P y(t) + 2\dot{y}^T(t) D P y(t - \tau(t)) + 2\dot{y}^T(t) (BY + Y^T B)\dot{y}(t) + 4\dot{y}^T(t) C C^T \dot{y}(t) + a y^T(t) P^2 y(t) + b y^T(t - \tau(t)) P^2 y(t)
\]

From (27) and (29), (32), \(\dot{V}_1\) is estimated as

\[
\dot{V}_1 \leq \dot{y}^T(t) \left[ AP + PA^T + (BY + Y^T B)\dot{y}(t) + 4\dot{y}^T(t) C C^T \dot{y}(t) + a y^T(t) P^2 y(t) + b y^T(t - \tau(t)) P^2 y(t)
\]

The inequality in (33) is obtained as:

\[
\int_{t-\tau(t)}^{t} 2 \left[ y^T(t) y^T(s) \right] Q_1 \left[ y(t) 0 \right]^T ds \\
\leq 2 \int_{t-\tau(t)}^{t} y^T(t) y^T(s) ds \int_{t-\tau(t)}^{t} y^T(t) y^T(s) ds \int_{t-\tau(t)}^{t} y^T(t) y^T(s) ds \int_{t-\tau(t)}^{t} y^T(t) y^T(s) ds
\]

Therefore, the estimation of \(\dot{V}_2(t)\) is as follows:

\[
\dot{V}_2(t) \leq \Xi_2 + (\tau(t) - \tau(t)) \Xi_2 + (\tau_2(t) - \tau(t)) \Xi_2
\]

where \(\Xi_2\) is defined in (22). Similarly, \(\dot{V}_3(t)\) is estimated as

\[
\dot{V}_3(t) = \xi^T(t) \Xi_3 \xi(t) + \delta_1(t) + \delta_2(t)
\]
where

\[ \Xi_{30} := e_5^T \left( r_1^2 W_1 + r_1^2 W_2 \right) e_5 + e_8^T \left( r_2^2 R_1 + r_2^2 R_2 \right) e_8, \]

\[
\delta_1(t) = - \tau_1 \int_{t-\tau_1}^{t-\tau_1} y^T(s) R_1 y(s) \, ds \\
- 2 \int_{t-\tau_1}^{t-\tau_1} \left( \tau_2 - t + s \right) y^T(s) R_2 y(s) \, ds,
\]

\[ (38) \]

\[
\delta_2(t) = - \tau_1 \int_{t-\tau_1}^{t} y^T(s) W_1 y(s) \, ds \\
- 2 \int_{t-\tau_1}^{t} \left( \tau_1 - t + s \right) y^T(s) W_2 y(s) \, ds.
\]

By Lemmas 5 and 6, we obtain the following:

\[ \Xi_1 := \text{diag}(R_1, 3R_1) \]

\[
\delta_2(t) = - \tau_1 \int_{t-\tau_1}^{t} y^T(s) W_1 y(s) \, ds \\
+ 2 \psi_{11}^T Y_{21} - \psi_{11}^T R_1 \psi_{11} - \psi_{21}^T R_1 \psi_{21},
\]

\[ (39) \]

where \( \psi_{11} := \left[ \begin{array}{c} y(t - \tau(t)) - y(t - \tau_2) \\
 y(t - \tau(t)) + y(t - \tau_2) - 2\nu_1(t) \end{array} \right], \]

\[ \psi_{21} := \left[ \begin{array}{c} y(t - \tau_1) - y(t - \tau(t)) \\
 y(t - \tau_1) + y(t - \tau(t)) - 2\nu_2(t) \end{array} \right]. \]

\[
- \left( \tau_2 - \tau_1 \right) \int_{t-\tau_1}^{t} y^T(s) R_1 y(s) \, ds \\
\leq 2 \psi_{11}^T Y_{21} - \psi_{11}^T R_1 \psi_{11} - \psi_{21}^T R_1 \psi_{21},
\]

\[ (40) \]

Thus,

\[
\delta_1(t) \leq 2 \psi_{11}^T S_{21} - \psi_{11}^T R_1 \psi_{11} - \psi_{21}^T R_1 \psi_{21} \\
- 2 \left[ \frac{1}{2} \left( \tau_2 - \tau(t) \right)^2 \xi(t) Z_2 \xi(t) \\
+ 2 \left( \tau_2 - \tau(t) \right) \xi(t) N_1 \left[ y(t - \tau(t)) - \nu_1 \right] \\
+ \frac{1}{2} \left[ r_{21}^2 \left( \tau_2 - \tau(t) \right)^2 \right] \xi(t) Z_2 \xi(t) + 2 \xi(t) N_2 \\
\cdot \left[ \left( \tau_2 - \tau(t) \right) \left[ y(t - \tau_1) - y(t - \tau(t)) \right] \\
+ \left( \tau(t) - \tau_1 \right) \left[ y(t - \tau_1) - \nu_2 \right] \right] \right].
\]

\[ (41) \]

where \( \Xi_3(\tau(t)) \) is given in (23). From Lemmas 2 and 4, we obtain

\[
- \tau_1 \int_{t-\tau_1}^{t} y^T(s) W_1 y(s) \, ds \\
\leq \left[ y(t) - y(t - \tau(t)) \right]^T W_1 \left[ y(t) - y(t - \tau_1) \right] \\
+ 3\Omega_1^T W_1 \Omega_1,
\]

\[ (42) \]

from which it follows that

\[
\delta_2(t) \leq \left[ y(t) - y(t - \tau(t)) \right]^T W_1 \left[ y(t) - y(t - \tau_1) \right] \\
+ 3\Omega_1^T W_1 \Omega_1 - r_{21}^2 \xi(t) Z_2 \xi(t) \\
- 2 r_1 \xi(t) N_3 \left[ y(t) - \nu_3 \right],
\]

\[ (43) \]

where

\[ \Omega_1 = y(t) + y(t - \tau_1) - 2\nu_3, \]

\[ \Omega_2 = y(t - \tau(t)) + y(t + \tau_2) - \nu_1, \]

\[ \Omega_3 = y(t - \tau(t)) + y(t - \tau(t)) - \nu_2, \]

\[ \Xi_{33} := - \phi_{33}^T \text{diag}[W_1^T, 3W_1] \phi_3 + r_{21}^2 Z_3, \]

\[ (44) \]

\[
\Xi_3 := \Xi_{30} + \Xi_{33}. \]

Hence, from (41) and (43), we obtain

\[
\hat{V}_3 \leq \xi(t) \left[ \Xi_3(\tau(t)) + \Xi_4 \right] \xi(t),
\]

\[ (45) \]

where

\[ \Xi_4 := \Xi_{30} + \Xi_{33}. \]

From (33), (36), and (45), we obtain \( \hat{V}(t, \eta, \dot{\eta}) \) along the solution of system (I) as

\[
\hat{V}(t, \eta, \dot{\eta}) \leq \xi(t) \Delta(\tau(t), \dot{\tau}(t)) \xi(t),
\]

\[ (46) \]

where

\[ \Delta(\tau(t), \dot{\tau}(t)) = \Xi_2(\tau(t), \dot{\tau}(t)) + \Xi_3(\tau(t)) + (\Xi_1 + \Xi_4). \]

\[ (47) \]
Therefore, we have
\[
\dot{V}(t, x_t) \leq \xi^T(t) \Delta \left( \tau(t), \dot{\tau}(t) \right) \xi(t) + \gamma \|w(t)\|^2 \\
+ y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t) \\
- y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t) \\
+ y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)) \\
- y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)) \\
= \xi^T(t) \tilde{\Delta}(\tau(t), \dot{\tau}(t)) \xi(t) + \gamma \|w(t)\|^2 \\
- y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t) \\
- y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)),
\]
(48)

where
\[
\tilde{\Delta}(\tau(t), \dot{\tau}(t)) = \Xi_2(\tau(t), \dot{\tau}(t)) + \Xi_3(\tau(t)) + (\Xi_1 + \Xi_4)
\]
\[
= \Xi_2(\tau(t), \dot{\tau}(t)) + \Xi_3(\tau(t)) + \Xi_4,
\]
\[
\Xi_4 = \Xi_1 + e_1^T \left( (2 + 4c_1) Y^T Y + 3P\dot{E}E + 4a_1 \right) e_1 \\
+ e_2^T \left( 3PG^T G + 4b_1 P^2 \right) e_2,
\]
(49)

and \(\Xi_4\) is defined in (24). Observe that \(\tilde{\Delta}(\tau(t), \dot{\tau}(t))\) may be rewritten as
\[
\tilde{\Delta}(\tau(t), \dot{\tau}(t)) = \tau^2(t) \Delta_0 + \tau(t) \Delta_1 + \Delta_2,
\]
(50)

where \(\Delta_0 = Z_1 - Z_2\) and \(\Delta_1, \Delta_2\) are \(\tau(t)\)-independent real matrices. By Lemma 8, if \(Z_1 - Z_2 \geq 0\) and the inequalities in (21) hold, then \(\tilde{\Delta}(\tau(t), \dot{\tau}(t)) < 0, \forall \tau(t) \in [\tau_1, \tau_2], \forall \dot{\tau}(t) \in [\mu_1, \mu_2]\). Moreover, \(\tilde{\Delta}(\tau(t), \dot{\tau}(t))\) may be rewritten as a convex combination of \(\dot{\tau}(t)\) as follows:
\[
\tilde{\Delta}(\tau(t), \dot{\tau}(t)) = (1 - \dot{\tau}(t)) \Theta_0 + \dot{\tau}(t) \Theta_1 + \Theta_2,
\]
(51)

where \(\Theta_0 = Q_2 - Q_3\) and \(\Theta_1, \Theta_2\) are \(\dot{\tau}(t)\)-independent real matrices. By utilizing the Schur complement lemma, it follows from (21), (50), and (51) that \(\tilde{\Delta}(\tau(t), \dot{\tau}(t)) < 0\) holds, from which it follows from inequality (48) that
\[
V(t, x_t) \leq yw(t) \dot{w}(t) \\
y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t) \\
- y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)).
\]
(52)

Letting \(w(t) = 0\) and from
\[
y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t) \leq 0,
\]
\[
- y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)) \leq 0,
\]
(53)

there exists a scalar \(\varepsilon_3 > 0\) such that
\[
\dot{V}(t, x_t) \leq -\varepsilon_3 \|x(t)\|^2 < 0, \quad \forall t \geq 0.
\]
(54)

Therefore, system (I) with \(w(t) \equiv 0\) is asymptotically stable. To complete the proof of theorem, next we consider the \(H_{\infty}\) performance \(\|z\|_2 < \gamma \|w\|_2\). By assuming that \(x(t) = 0, t \in [-\tau_2, \tau_0]\), it follows from definition of \(z\) that
\[
\|z\|^2 \leq \|E(x)\|^2 + \|Gx - \tau(t)\|^2 + \|u(t)\|^2
\]
\[
+ 2x^T(t) E^T G \dot{x}(t - \tau(t)) + 2x^T(t) E^T g(\cdot)
\]
\[
+ 2x^T(t - \tau(t)) E^T g(\cdot) + 2u^T(t) F^T g(\cdot) + \|g(\cdot)\|^2
\]
\[
\leq 3 \|Ex(t)\|^2 + 3 \|Gx - \tau(t)\|^2
\]
\[
+ 2 \|u(t)\|^2 + 4 \|g(\cdot)\|^2
\]
\[
\leq x^T(t) \left[ 3E^T E + 4a_1 \right] x(t) + x^T(t - \tau(t))
\]
\[
\cdot \left[ 3G^T G + 4b_1 \right] x(t - \tau(t)) + \left[ 3G^T G + 2 + 4c_1 \right] \|u(t)\|^2
\]
\[
= y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t)
\]
\[
y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)).
\]
(55)

From (52), we obtain
\[
\dot{V}(t, x_t) \leq yw(t) \dot{w}(t)
\]
\[
- y^T(t) \left[ 3P\dot{E}E + 4a_1 P^2 + (2 + 4c_1) Y^T Y \right] y(t)
\]
\[
y^T(t - \tau(t)) \left[ 3PG^T G + 4b_1 P^2 \right] y(t - \tau(t)).
\]
(56)

From estimations of \(\dot{V}(t, x_t)\) and \(\|z(t)\|^2\) in (48) and (55), we obtain
\[
\dot{V}(t, x_t) + z^T(t) z(t) - y^2 \dot{w}(t) w(t) < 0.
\]
(57)

Integrating both sides of the above equation from \(t_0\) to \(t\), we get
\[
\int_{t_0}^{t} \dot{V}(t, x_t) + z^T(t) z(t) - y^2 \dot{w}(t) w(t) \, dt < 0.
\]
(58)

It follows that
\[
\int_{t_0}^{t} z^T(t) z(t) - y^2 \dot{w}(t) w(t) \, dt \\
\leq V(t_0, x_{t_0}) - V(t, x_t) \leq 0.
\]
(59)

Therefore, under zero initial condition \(x(t) = 0, t \in [-\tau_2, \tau_0]\), by letting \(t \to +\infty\) in (59), we get
\[
\int_{t_0}^{\infty} z^T(t) z(t) dt < y^2 \int_{t_0}^{\infty} \dot{w}(t) w(t) dt,
\]
which gives \(\|z\|_2 < y \|w\|_2\). This completes the proof. \(\square\)
When \( r_1 = 0, B = 0, \) and \( C = 0, \) the nonlinear function is \( f(t, x, x') \), and (1) reduces to
\[
\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),
\]
where \( 0 \leq \tau(t) \leq \tau_2, \mu_1 \leq \tau(t) \leq \mu_2. \) In the following, we present a stability criterion for the case when \( r_1 = 0. \)

We consider the following Lyapunov-Krasovskii functional candidate:
\[
V(t, x, \dot{x}) = x^T(t) Px(t) + \int_{t-\tau(t)}^{t} \left[ x^T(s) \begin{bmatrix} x^T(t) & x^T(s) \end{bmatrix} + \begin{bmatrix} \tau_2 (x(t) - x(s)) \end{bmatrix} R_1 x(s) \right] ds + \int_{t-\tau_2}^{t} \left[ \tau_2 (\tau_2 - t + s) x^T(s) R_2 x(s) \right] ds.
\]

Corollary 10. For given scalars \( \tau_2, \mu_1, \mu_2, a, \) and \( b, \) (61) is asymptotically stable if there exist symmetric positive definite matrices \( P, R_1, R_2, N_1, N_2, Z_1, Z_2, Q_2, Q_3, \) and \( S_1, \) such that the following LMIs hold:
\[
\begin{aligned}
\tau_2 Y_1 + (1 - \mu_1) Y_3 + Y_4 < 0, \\
\tau_2 Y_1 + (1 - \mu_2) Y_3 + Y_4 < 0, \\
\tau_2 Y_2 + (1 - \mu_1) Y_3 + Y_4 < 0, \\
\tau_2 Y_2 + (1 - \mu_2) Y_3 + Y_4 < 0, \\
\begin{bmatrix} R_1 & S_1 \\ S_1^T & R_1 \end{bmatrix} \succeq 0, \\
\begin{bmatrix} Z_1 & N_1 \\ N_1^T & R_2 \end{bmatrix} \succeq 0, \\
Q_j = \begin{bmatrix} Q_{j1} & Q_{j2} \\ * & Q_{j3} \end{bmatrix} \succeq 0, \quad (i = 1, 2),
\end{aligned}
\]

where \( R_1 = \text{diag}(R_1, 3R_1) \)
\[
Y_1 := \begin{bmatrix} \tilde{e}_1^T & \tilde{e}_2^T \end{bmatrix} Q_3 \begin{bmatrix} \tilde{e}_2^T & 0 \end{bmatrix} + \begin{bmatrix} \tilde{e}_6^T & 0 \end{bmatrix} Q_3 \begin{bmatrix} \tilde{e}_6^T & \tilde{e}_7^T \end{bmatrix}^T \\
+ \tau_2 (Z_1 - Z_2) + 2N_1 [\tilde{e}_2 - \tilde{e}_4] \\
+ 2[\tilde{e}_2 - \tilde{e}_4] N_1^T + 2N_2 [\tilde{e}_1 - \tilde{e}_2] \\
+ 2[\tilde{e}_1 - \tilde{e}_2] N_2^T,
\]
\[
Y_2 := \begin{bmatrix} \tilde{e}_1^T & \tilde{e}_4^T \end{bmatrix} Q_2 \begin{bmatrix} \tilde{e}_6^T & 0 \end{bmatrix} + \begin{bmatrix} \tilde{e}_6^T & 0 \end{bmatrix} Q_2 \begin{bmatrix} \tilde{e}_6^T & \tilde{e}_7^T \end{bmatrix}^T \\
+ 2N_2 [\tilde{e}_1 - \tilde{e}_3] + 2[\tilde{e}_1 - \tilde{e}_3] N_2^T,
\]
\[
\begin{aligned}
Y_3 := [\tilde{e}_1 & \tilde{e}_2] (Q_4 - Q_2) [\tilde{e}_1 & \tilde{e}_2]^T, \\
Y_4 := \begin{bmatrix} \tilde{e}_1^T & \tilde{e}_2^T \end{bmatrix} \begin{bmatrix} A^T P + PA & \tilde{e}_1 + \tilde{e}_2 (a + b) \tilde{e}_1 \\
& \tilde{e}_1 + \tilde{e}_2 (D^T P) \tilde{e}_1 \\
& \tilde{e}_1 + \tilde{e}_2 (PD) \tilde{e}_2 + \tilde{e}_1 (D^T P) \tilde{e}_2 \\
& \tilde{e}_2 (P^T) \tilde{e}_2 \end{bmatrix} \begin{bmatrix} \tilde{e}_1 & \tilde{e}_2 \end{bmatrix}^T,
\end{aligned}
\]
\[
\begin{aligned}
\begin{bmatrix} \tilde{e}_6^T & \tilde{e}_7^T \end{bmatrix} & \begin{bmatrix} \tilde{e}_6^T & \tilde{e}_7^T \end{bmatrix}^T + \tau_2 \tilde{e}_6^2 (R_1 + R_2) \tilde{e}_6 \\
& + \Theta_1^T S_1 \Theta_2 + \Theta_2^T S_2^T \Theta_1 - \Theta_1^T R_1 \Theta_1 - \Theta_2^T R_2 \Theta_2 \\
& + \tau_2 Z_2, \end{aligned}
\]

with \( \tilde{e}_1 = [1 0 0 0 0 0 0 0] .. \tilde{e}_8 = [0 0 0 0 0 0 0 0]. \)

\textbf{Proof.} The proof is the same as in Theorem 9 by using Lyapunov-Krasovskii functional (62). The proof is omitted. \( \square \)

\section{4. Numerical Examples}

In this section, we provide numerical examples to show the effectiveness of theoretical results.

\textbf{Example 1.} Consider the nonlinear system with interval time-varying delays (1) which was considered in [15], where
\[
A = \begin{bmatrix} -1.3 & 0.3 \\
0.5 & 0.1 \end{bmatrix}, \\
D = \begin{bmatrix} -0.01 & 0.02 \\
0.03 & -0.04 \end{bmatrix}, \\
B = \begin{bmatrix} 0.2 & 0 \\
0.3 & 0 \end{bmatrix}, \\
C = \begin{bmatrix} -0.02 & 0.01 \\
0.02 & -0.03 \end{bmatrix}, \\
E = G = \begin{bmatrix} 0.06 & -0.06 \\
-0.08 & 0.08 \end{bmatrix}, \\
F = \begin{bmatrix} 0.8 & 0 \\
0.6 & 0 \end{bmatrix}, \\
f(\cdot) = g(\cdot) = 0.01 \begin{bmatrix} \sqrt{x_1^2(t) + x_2^2(t - \tau(t))} \\
\sqrt{x_1^2(t) + x_2^2(t - \tau(t))} \end{bmatrix},
\]

\[
(65)
\]
Figure 1: The trajectories of $x_1(t)$ and $x_2(t)$ of (1) in Example 1 without feedback control.

Table 1: The minimum allowable value of disturbance attenuation $\gamma$ with $\mu_1 = -0.1$ and $\mu_2 = 0.1$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\min \gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 9</td>
<td>0.1</td>
<td>0.3</td>
<td>0.2377</td>
</tr>
<tr>
<td>Theorem 9</td>
<td>0.1</td>
<td>0.5</td>
<td>0.2474</td>
</tr>
</tbody>
</table>

$a = b = c = d = a_1 = b_1 = c_1 = 0.01$

Table 2: The minimum allowable value of disturbance attenuation $\gamma$ with $\mu_1 = 0.05$ and $\mu_2 = 0.1$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>$\min \gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 9</td>
<td>0.1</td>
<td>0.5</td>
<td>0.2487</td>
</tr>
</tbody>
</table>

$a = b = c = d = a_1 = b_1 = c_1 = 0.05$

Table 3: The maximum allowable upper bound $\tau_2$ in Example 2 with $\tau_1 = 0$.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\max \tau_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>By Corollary 10</td>
<td>$-0.1$</td>
<td>$0.1$</td>
<td>2.1443</td>
</tr>
<tr>
<td>By Corollary 10</td>
<td>$-0.3$</td>
<td>$0.3$</td>
<td>2.1126</td>
</tr>
<tr>
<td>By Corollary 10</td>
<td>$-0.5$</td>
<td>$0.5$</td>
<td>2.0833</td>
</tr>
<tr>
<td>By Corollary 10</td>
<td>$-0.8$</td>
<td>$0.8$</td>
<td>2.0473</td>
</tr>
<tr>
<td>By Corollary 10</td>
<td>$-1$</td>
<td>$1$</td>
<td>2.0332</td>
</tr>
</tbody>
</table>

For simulation, we choose $\tau(t) = 0.4 + 0.1 \cos(t)$, $\phi(t) = [-5 \cos(t), 3 \cos(t)]$, $\forall t \in [0, 10]$, and $\psi(t)$ is the Gaussian noise which is the set of random numbers with fluctuation range between $-1$ and $1$. Figure 1 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of system (I) without feedback control ($u(t) = 0$) and Figure 2 shows the trajectories of solutions $x_1(t)$ and $x_2(t)$ of the system with feedback control $u(t)$. Moreover, in Tables 1 and 2, by using Theorem 9, we give the minimum allowable value $\gamma$ with $\mu_1 = -0.1, \mu_2 = 0.1$, and with $\mu_1 = 0.05, \mu_2 = 0.1$, for some given $\tau_1$ and $\tau_2$, respectively.

Example 2. Consider the following nonlinear system with interval time-varying delays which was considered in [18]:

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),$$  

$$(67)$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$  

$$(68)$$

$a = b = 0.1$, and $0 \leq \tau(t) \leq \tau_2, \mu_1 \leq \hat{\tau}(t) \leq \mu_2$. By using LMI Toolbox in MATLAB, the LMIs in Corollary 10 are feasible. Table 3 shows the maximum allowable upper bound $\tau_2$ with different values of $\mu_1$ and $\mu_2$.

Example 3. Consider the following nonlinear system with interval time-varying delays which was considered in [13]:

$$\dot{x}(t) = Ax(t) + Dx(t - \tau(t)) + f(t, x(t), x(t - \tau(t))),$$  

$$(69)$$
Table 4: Comparison of maximum allowable upper bound $\tau_2$ in Example 3 with $\tau_1 = 0$. 

<table>
<thead>
<tr>
<th>Method</th>
<th>$a = 0.1$</th>
<th>$b = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[25]</td>
<td>1.009</td>
<td>0.714</td>
</tr>
<tr>
<td>[23]</td>
<td>1.284</td>
<td>1.209</td>
</tr>
<tr>
<td>[13]</td>
<td>1.287</td>
<td>1.279</td>
</tr>
<tr>
<td>Corollary 10</td>
<td>1.7706</td>
<td>1.7355</td>
</tr>
</tbody>
</table>

where

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}, \quad (70)$$

$a = b = 0.1$, and $0 \leq \tau(t) \leq \tau_2$, $\mu_1 \leq \dot{\tau}(t) \leq \mu_2$. By using LMI Toolbox in MATLAB, the LMIs in Corollary 10 are feasible. For comparison with other existing results, we now calculate the admissible maximum allowable upper bounds of $\tau_2$ by setting $\mu = \mu_2 = -\mu_1$ for various values of $\mu$. Table 4 shows that the maximum allowable upper bounds of $\tau_2$ with given $\tau_1 = 0$ obtained from Corollary 10 are greater than the upper bounds obtained in [13, 23, 25].

5. Conclusions

In this paper, we have investigated the $H_{\infty}$ control problem for a class of nonlinear systems with interval time-varying delay. A new Lyapunov-Krasovskii functional is constructed to obtain new delay-dependent sufficient condition for the $H_{\infty}$ control and asymptotic stability condition in terms of LMIs. Numerical examples are given to illustrate the effectiveness of the theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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