Research Article

Solutions to BSDEs Driven by Multidimensional Fractional Brownian Motions

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We study more general backward stochastic differential equations driven by multidimensional fractional Brownian motions. Introducing the concept of the multidimensional fractional (or quasi-) conditional expectation, we study some of its properties. Using the quasi-conditional expectation and multidimensional fractional Itô formula, we obtain the existence and uniqueness of the solutions to BSDEs driven by multidimensional fractional Brownian motions, where a fixed point principle is employed. Finally, solutions to linear fractional backward stochastic differential equations are investigated.

1. Introduction

Increasing attention has been paid to fractional Brownian motion recently, since it can more accurately describe many phenomena in mathematical finance, hydrology, queueing theory, and so on. A centered Gaussian process $\{B_t^H, t \geq 0\}$ is called fractional Brownian motion of Hurst parameter $H \in (0, 1)$ if it has the covariance function $\mathbb{E}[B_t B_s] = (1/2)(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$. This process was first introduced by Kolmogorov [1] and studied by Mandelbrot and van Ness [2], where a stochastic integral representation in terms of a standard Brownian motion was established. Unfortunately, $\{B_t^H, t \geq 0\}$ is neither a semimartingale nor a Markov process, so the powerful tools from the classical theories are not applicable when studying $\{B_t^H, t \geq 0\}$. Nevertheless, an efficient stochastic calculus of $\{B_t^H, t \geq 0\}$ has been developed in [3–5]. This calculus uses an Itô type integration with respect to $\{B_t^H, t \geq 0\}$ and white noise theory. The theory was extended to multiparameter fractional Brownian motion fields $B^H(x)$, $x \in \mathbb{R}^d$, in [3] and applied to stochastic differential equations driven by such fractional white noise in [6]. In many applications such as mathematical finance, the stochastic calculus of a fractional Brownian motion is needed (see [7, 8]).

Pardoux and Peng [9] first proved the existence and uniqueness of the solution of general nonlinear BSDEs. Since then, the theory of BSDEs has been extensively studied by many researchers and BSDEs have found applications in many fields, such as finance, risk measure, stochastic control, and so forth. These equations are of the form

$$y_t = \xi + \int_t^T f(s, y_s, z_s) ds + \int_t^T z_s dW_s, \quad (1)$$

where W_s is a standard Brownian motion and the terminal value ξ and generator $f(s, y_s, z_s)$ are given.

In [10] Hu and Peng studied the backward stochastic differential equations driven by a fractional Brownian motion. The equations are as follows:

$$y_t = \xi + \int_t^T f(s, \eta_s, y_s, z_s) ds + \int_t^T z_s dB_s^H, \quad (2)$$

where $\xi = g(\eta_T)$, g is a continuous function, $\eta_s = \eta_0 + b_s + \int_0^s \sigma_r dB_r^H$ with η_0 , b_s , and σ_r being deterministic constants or functions, and B_r^H is a fractional Brownian motion. They introduced the quasi-conditional expectation

and first discussed its properties. They proved the existence and uniqueness of the solutions to above equations (2). When

$$f(s, \eta_s, y_s, z_s) = A(s, \eta_s) y_s + B(s, \eta_s) z_s + C(s, \eta_s), \quad (3)$$

the existence of the solutions was proved in [11]. Zhang [12] studied properties of solutions of linear fractional BSDEs using the quasi-conditional expectation and derived the comparison theorem and comonotonic theorem of the solutions.

Maticiuc and Nie [13] improved the result and omitted the assumption in [10]; they proved an existence and uniqueness of the solution of reflected BSDEs driven by fractional Brownian motion. Jańczak-Borkowska [14] added an additional term integral with respect to an increasing process in [13] and proved that kind of equation has a unique solution.

Fei et al. [15] extended the results in [10] to the BSDEs driven by both standard and fractional Brownian motion:

$$y_t = g(\eta_T) + \int_t^T f(s, \eta_s, y_s, z_{1,s}, z_{2,s}) ds + \int_t^T z_{1,s} dB_s + \int_t^T z_{2,s} dB_s^H, \quad (4)$$

where $\eta_s = \eta_0 + b_s + \int_0^s \sigma_{1,r} dB_r + \int_0^s \sigma_{2,r} dB_r^H$. Using the quasi-conditional expectation, they discussed the existence and uniqueness of solutions to general BSDEs.

In this paper we will deal with the backward stochastic differential equations driven by multidimensional fractional Brownian motions with a more general terminal condition:

$$y_t = g(\eta_T) + \int_t^T f(s, \eta_s, y_s, z_{1,s}, \dots, z_{m,s}) ds + \sum_{j=1}^m \int_t^T z_{j,s} dB_s^{H_j}, \quad (5)$$

where $\eta_s = (\eta_1(s), \dots, \eta_n(s))$ and

$$\eta_i(t) = \eta_i(0) + \int_0^t b_i(s) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dB_s^{H_j}, \quad (6)$$

where $B_s^{H_j}$, $j = 1, \dots, m$, are independent fractional Brownian motions of Hurst parameter $H_1, \dots, H_m$ with $H_j \in (1/2, 1)$. We use a fixed point principle to prove the existence and uniqueness of solutions to BSDEs (5).

The rest of the paper is organized as follows. In Section 2, we recall some basic results of multidimensional fractional Brownian motion. Section 3 is concerned with the multidimensional fractional (or quasi-) conditional expectation. In Section 4, existence and uniqueness of the solutions to the general fractional BSDEs are proved. Solutions of the linear fractional BSDEs are discussed in Section 5. The conclusions are set out in Section 6.

2. Preliminaries

In this section, we first recall some basic results of multidimensional fractional Brownian motions.

Let $\{B_t^{H_i}, t \in [0, T]\}$, $i = 1, \dots, n$, be independent fractional Brownian motions of Hurst parameters $H_1, \dots, H_n$ with $H_i \in (1/2, 1)$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \in [0, T]\}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t, t \in [0, T]\}$ is the σ -algebra generated by $\{B_t^{H_i}, t \in [0, T], i = 1, \dots, n\}$. The process $\{B_t^{H_i}, t \in [0, T], i = 1, \dots, n\}$ satisfies

$$\begin{aligned} \mathbb{E}[B_t^{H_i}] &= 0, \\ \mathbb{E}[B_t^{H_i} B_s^{H_j}] &= \frac{1}{2} (|t|^{2H_i} + |s|^{2H_i} - |t-s|^{2H_i}) \delta_{ij}; \\ 1 \leq i, j \leq n, \quad s, t \in [0, T], \end{aligned} \quad (7)$$

where

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \quad (8)$$

Let f and g be two continuous functions on $[0, T]$. Define

$$\langle f, g \rangle_{i,t} = \int_0^t \int_0^t \phi_i(u, v) f(u) g(v) dudv, \quad (9)$$

where $\phi_i(u, v) = H_i(2H_i - 1)|u - v|^{2H_i-2}$. If $f = g$, then we denote

$$\|f\|_{i,t}^2 = \int_0^t \int_0^t \phi_i(u, v) f(u) f(v) dudv. \quad (10)$$

If $F : \Omega \rightarrow \mathbb{R}$ is a given function, let $\mathbb{D}_t^{H_i} F$ of F at time t be the Malliavin ϕ_i -derivative; then

$$\mathbb{D}_t^{H_i} F = \int_0^t \phi_i(s, t) D_s^{H_i} F ds, \quad (11)$$

where $D_s^{H_i} F$ is Malliavin derivative of F at time s . More detailed study about Malliavin derivative is in [16].

We now give the multidimensional fractional Itô formula.

Lemma 1 (multidimensional fractional Itô formula [3, 17]). *For $i, k = 1, \dots, n$, $j = 1, \dots, m$, let $\sigma_{ij} \in L^2([0, T])$ be deterministic functions and let $\langle \sigma_{ik}, \sigma_{jk} \rangle_{k,t}$ be continuously differentiable as functions of $t \in [0, T]$. Denote $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$:*

$$\xi_i(t) = \xi_i(0) + \int_0^t b_i(s) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dB_s^{H_j}, \quad (12)$$

$$0 \leq t \leq T,$$

where $\xi(0) = (\xi_1(0), \dots, \xi_n(0))$ is a constant vector and $b_i(s)$, $i = 1, \dots, n$, are continuous deterministic functions with $\int_0^T |b_i(s)| ds < \infty$. Let $F(t, x)$ be continuously differentiable

with respect to t and twice continuously differentiable with respect to x . Then

$$\begin{aligned} F(t, \xi(t)) &= F(0, \xi(0)) + \int_0^t \frac{\partial F}{\partial s}(s, \xi(s)) ds \\ &+ \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(s, \xi(s)) d\xi_i(s) \\ &+ \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s, \xi(s)) \sum_{k=1}^m \sigma_{ik}(s) \mathbb{D}_s^{H_k}(\xi_j(s)) ds. \end{aligned} \quad (13)$$

Since $\mathbb{D}_s^{H_k}(\xi_j(s)) = \int_0^s \phi_k(s, v) \sigma_{jk}(v) dv$, we obtain

$$\begin{aligned} F(t, \xi(t)) &= F(0, \xi(0)) + \int_0^t \frac{\partial F}{\partial s}(s, \xi(s)) ds \\ &+ \sum_{i=1}^n \int_0^t \frac{\partial F}{\partial x_i}(s, \xi(s)) b_i(s) ds \\ &+ \sum_{i=1}^n \sum_{j=1}^m \int_0^t \frac{\partial F}{\partial x_i}(s, \xi(s)) \sigma_{ij}(s) dB_s^{H_j} \\ &+ \frac{1}{2} \sum_{i,j=1}^n \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(s, \xi(s)) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] ds, \end{aligned} \quad (14)$$

where

$$\begin{aligned} &\left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] \\ &= 2H_k(2H_k - 1) \sigma_{ik}(s) \int_0^s (s-v)^{2H_k-2} \sigma_{jk}(v) dv. \end{aligned} \quad (15)$$

By the two-dimensional fractional Itô formula, we can obtain the following chain rule.

Lemma 2 (fractional Itô chain rule [3, 17]). *Let $b_1(s)$, $b_2(s)$, $\sigma_{i1}(s)$, $\sigma_{i2}(s)$, $0 \leq s \leq T$, $i = 1, \dots, n$, be real-valued stochastic processes such that*

$$\mathbb{E} \left\{ \int_0^T |b_j(s)|^2 ds + \sum_{i=1}^n \int_0^T |\sigma_{ij}(s)|^2 ds \right\} < \infty, \quad (16)$$

$j = 1, 2.$

Assume also that $\mathbb{D}_t^{H_i} b_j(s)$ and $\mathbb{D}_t^{H_i} \sigma_{ij}(s)$, $i = 1, \dots, n$, $j = 1, 2$, are continuously differentiable with respect to $(s, t) \in [0, T]^2$ for almost all $\omega \in \Omega$. Suppose that $\mathbb{E} \left\{ \int_0^T \int_0^T |\mathbb{D}_t^{H_i} b_j(s)|^2 ds dt \right\} < \infty$ and $\mathbb{E} \left\{ \int_0^T \int_0^T |\mathbb{D}_t^{H_i} \sigma_{ij}(s)|^2 ds dt \right\} < \infty$, $i = 1, \dots, n$, $j = 1, 2$. Denote

$$\begin{aligned} X_j(t) &= X_j(0) + \int_0^t b_j(s) ds + \sum_{i=1}^n \int_0^t \sigma_{ij}(s) dB_s^{H_i}, \\ t &\in [0, T], \quad j = 1, 2. \end{aligned} \quad (17)$$

Then

$$\begin{aligned} X_1(t) X_2(t) &= X_1(0) X_2(0) + \int_0^t X_1(s) dX_2(s) \\ &+ \int_0^t X_2(s) dX_1(s) \\ &+ \sum_{i=1}^n \int_0^t \mathbb{D}_s^{H_i} X_1(s) \sigma_{i2}(s) ds \\ &+ \sum_{i=1}^n \int_0^t \mathbb{D}_s^{H_i} X_2(s) \sigma_{i1}(s) ds, \end{aligned} \quad (18)$$

which can be written as a differential form:

$$\begin{aligned} d(X_1(t) X_2(t)) &= X_1(t) dX_2(t) + X_2(t) dX_1(t) \\ &+ \left[\sum_{i=1}^n \mathbb{D}_t^{H_i} X_1(t) \sigma_{i2}(t) + \sum_{i=1}^n \mathbb{D}_t^{H_i} X_2(t) \sigma_{i1}(t) \right] dt. \end{aligned} \quad (19)$$

We know that fractional Brownian motion and Brownian motion have the following relationship:

$$B_t^{H_i} = \int_0^t K_{H_i}(t, s) dW_s, \quad (20)$$

where $(W_s, s \geq 0)$ is a standard Brownian motion and

$$\begin{aligned} K_{H_i}(t, s) &= C_{H_i} \left[\left(\frac{t}{s} \right)^{H_i-1/2} (t-s)^{H_i-1/2} \right. \\ &\quad \left. - \left(H_i - \frac{1}{2} \right) s^{1/2-H_i} \int_0^t u^{H_i-3/2} (u-s)^{H_i-1/2} du \right], \end{aligned} \quad (21)$$

with C_{H_i} being a constant such that $\int_0^1 K_{H_i}^2(1, s) ds = 1$. With this $K_{H_i}(t, s)$ we associate an operator

$$\mathcal{K}_{H_i}(t) = \int_0^t K_{H_i}(t, s) h(s) ds, \quad 0 \leq t < \infty. \quad (22)$$

This operator has an inverse operator which can be expressed as

$$\mathcal{K}_{H_i}^{-1}(t) = s^{H_i-1/2} D_{0+}^{H_i-1/2} \left(r^{1/2-H_i} h'(r) \right)(t), \quad (23)$$

where (see [6])

$$\begin{aligned} D_{0+}^{H_i-1/2} f(t) &= \frac{1}{\Gamma(3/2-H_i)} \left(\frac{f(t)}{t^{H_i-1/2}} + \left(H_i - \frac{1}{2} \right) \int_0^t \frac{f(t)-f(s)}{(t-s)^{H_i+1/2}} ds \right). \end{aligned} \quad (24)$$

Let us consider the translation

$$\tilde{B}_t^{H_i} = B_t^{H_i} - \int_0^t b_s ds, \quad (25)$$

where b_s is a continuous and adapted process; then

$$\begin{aligned}\bar{B}_t^{H_i} &= \int_0^t K_{H_i}(t, s) dW_s \\ &\quad - \int_0^t K_{H_i}(t, s) \left(K_{H_i}^{-1} \int_0^s b_r dr \right) (s) ds \\ &= \int_0^t K_{H_i}(t, s) d\bar{W}_s,\end{aligned}\quad (26)$$

where

$$\bar{W}_t = W_t - \int_0^t \left(K_{H_i}^{-1} \int_0^s b_r dr \right) (s) ds. \quad (27)$$

By the Novikov condition, if

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \int_0^T \left(K_{H_i}^{-1} \int_0^s b_r dr \right) (s) ds \right\} \right] < \infty, \quad (28)$$

then

$$\begin{aligned}\exp \left\{ \int_0^t \left(K_{H_i}^{-1} \int_0^s b_r dr \right) (s) dW_s \right. \\ \left. - \frac{1}{2} \int_0^t \left(K_{H_i}^{-1} \int_0^s b_r dr \right)^2 (s) ds \right\}, \quad 0 \leq t \leq T,\end{aligned}\quad (29)$$

which is an exponential martingale. Define $\tilde{P}$ by the following:

$$\begin{aligned}\frac{d\tilde{P}}{dP} &= \exp \left\{ \int_0^t \left(K_{H_i}^{-1} \int_0^s b_r dr \right) (s) dW_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \left(K_{H_i}^{-1} \int_0^s b_r dr \right)^2 (s) ds \right\}.\end{aligned}\quad (30)$$

Then, under the probability measure $\tilde{P}$, $\bar{B}_t^{H_i}$ is a new fractional Brownian motion.

3. Quasi-Conditional Expectation

In this section, we introduce multidimensional fractional (or quasi-) conditional expectation.

Let $\|f\|_t = (\iint_0^t \phi_i(u, v) f(u) f(v) du dv)^{1/2}$. $\mathcal{H}_t^i$ denotes the completion space of continuous functions under the Hilbert norm $\|\cdot\|_t$. Let $\mathcal{H}_t^{i \otimes n}$ denote the set of all real symmetric functions f_n of n variables on $[0, t]$ such that

$$\begin{aligned}\sum_{n=0}^{\infty} n! \int_{[0, t]^{2n}} |f_n(s_1, \dots, s_n)| |f_n(t_1, \dots, t_n)| \\ \cdot \prod_{j=1}^n \phi_i(s_j, t_j) ds_1 \cdots ds_n dt_1 \cdots dt_n < \infty.\end{aligned}\quad (31)$$

Let $\hat{L}^2(\Omega, \mathcal{F}, P)$ denote the set of $F \in L^2(\Omega, \mathcal{F}, P)$ such that F has the following chaos expansion:

$$\begin{aligned}F(\omega) &= \sum_{i=1}^m \sum_{n=0}^{\infty} I_n^{H_i}(f_n) \\ &= \sum_{i=1}^m \sum_{n=0}^{\infty} \int_{[0, T]^n} f_n(s_1, \dots, s_n) dB_{s_1}^{H_i} \cdots dB_{s_n}^{H_i} \\ &= \sum_{i=1}^m \sum_{n=0}^{\infty} n! \int_0^T \cdots \int_0^{s_2} f_n(s_1, \dots, s_n) dB_{s_1}^{H_i} \cdots dB_{s_n}^{H_i},\end{aligned}\quad (32)$$

where $f_n \in \mathcal{H}_t^{i \otimes n}$.

Now we introduce the quasi-conditional expectation for multidimensional fractional Brownian motion, which is important in solving backward stochastic differential equations.

Definition 3. The quasi-conditional expectation of some random variable $F \in \hat{L}^2(\Omega, \mathcal{F}, P)$ relative to a fractional Brownian motion with Hurst parameter H_i is defined by

$$\hat{\mathbb{E}}_P^{H_i} [F | \mathcal{F}_t] = \sum_{n=0}^{\infty} I_n^{H_i}(f_n I_{[0, t]}), \quad (33)$$

where $I_{[0, t]}(t_1, \dots, t_n) = I_{[0, t]}(t_1) \cdots I_{[0, t]}(t_n)$.

Let $G \in \hat{L}^2(\Omega, \mathcal{F}, P)$, since $B_t^{H_i}$, $i = 1, \dots, m$, are independent fractional Brownian motions; by Definition 3, we have

$$\hat{\mathbb{E}}_P^H [G | \mathcal{F}_t] = \hat{\mathbb{E}}_P^{H_1} [\cdots \hat{\mathbb{E}}_P^{H_m} [G | \mathcal{F}_t] \cdots | \mathcal{F}_t]. \quad (34)$$

Lemma 4. If f_s is the real stochastic processes such that

$$\begin{aligned}\mathbb{E} \left[\int_t^T \int_t^T \phi_i(u, v) |f(u)| |f(v)| du dv \right. \\ \left. + \int_t^T \int_t^T |\mathbb{D}_u^{H_i} f(v)|^2 du dv \right] < \infty,\end{aligned}\quad (35)$$

and $\int_t^T f(s) dB_s^{H_i} \in \hat{L}^2(\Omega, \mathcal{F}, P)$, then

$$\hat{\mathbb{E}}_P^{H_i} \left[\int_t^T f(s) dB_s^{H_i} | \mathcal{F}_t \right] = 0. \quad (36)$$

We introduce a function

$$P_t f(x) = (2\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-(x-y)^2/2t} f(y) dy. \quad (37)$$

Let W_t be a standard Brownian motion started at 0 and let β be a $n \times n$ -matrix; one defines an $\mathbb{R}^n$ -valued Markov process X by stipulation that if $X_0 = x$ a.s., then $X_t = x + \beta W_t$, where W_t is n -dimensional Brownian motion. If f is continuous, then we have

$$\frac{\partial}{\partial t} P_t f(x) = \frac{1}{2} \sum_{i,j=1}^n \gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} P_t f(x), \quad (38)$$

where $\gamma = \beta \beta^t$, with β^t being the transpose of β (see [18]).

Theorem 5. Let $\eta_i(t) = \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dB_s^{H_j}$, $i = 1, \dots, n$, $F = f(\eta_T)$, and $G = g(\eta_T)$, where $\eta_T = (\eta_1(T), \dots, \eta_n(T))$. For $i, j = 1, \dots, n$, $k = 1, \dots, m$, and $s \in [0, T]$, assume that $\sigma_{ij}(s)$, $i = 1, \dots, n$, $j = 1, \dots, m$, are measurable and $[(d/ds)\langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s}] > 0$. Let $A_t = (A_{ij}(t))$ be $n \times n$ -matrix, where

$$A_{ij}(t) = \sum_{k=1}^m \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,t}. \quad (39)$$

If $F \geq G$ almost surely, then

$$\hat{\mathbb{E}}_P^H[F | \mathcal{F}_t] \geq \hat{\mathbb{E}}_P^H[G | \mathcal{F}_t]. \quad (40)$$

We also have

$$\hat{\mathbb{E}}_P^H[F | \mathcal{F}_t] = P_{A_T-A_t} f(\eta_t). \quad (41)$$

Proof. It suffices to show the case $G = 0$. Applying Lemma 1 to $P_{A_T-A_t} f(\eta_t)$, we have

$$\begin{aligned} f(\eta_T) &= P_{A_T} f(\eta_0) + \int_0^T \frac{\partial}{\partial s} P_{A_T-A_s} f(\eta_s) ds \\ &\quad + \sum_{i=1}^n \int_0^T \frac{\partial}{\partial x_i} P_{A_T-A_s} f(\eta_s) \left[\sum_{j=1}^m \sigma_{ij}(s) dB_s^{H_j} \right] \\ &\quad + \sum_{i,j=n}^m \int_0^T \frac{\partial^2}{\partial x_i \partial x_j} P_{A_T-A_s}(\eta_s) \\ &\quad \cdot \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] ds. \end{aligned} \quad (42)$$

By (38) and the chain rule of composite function, we deduce

$$\begin{aligned} \frac{\partial}{\partial s} P_{A_T-A_s} f(\eta_s) \\ = - \sum_{i,j=n}^m \frac{\partial^2}{\partial x_i \partial x_j} P_{A_T-A_s}(\eta_s) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right]. \end{aligned} \quad (43)$$

By (42) and (43), we obtain

$$\begin{aligned} f(\eta_T) \\ = P_{A_T} f(\eta_0) \\ + \sum_{i=1}^n \int_0^T \frac{\partial}{\partial x_i} P_{A_T-A_s} f(\eta_s) \left[\sum_{j=1}^m \sigma_{ij}(s) dB_s^{H_j} \right]. \end{aligned} \quad (44)$$

Taking the quasi-conditional expectation both sides of (44), we have

$$\begin{aligned} \hat{\mathbb{E}}_P^H[F | \mathcal{F}_t] \\ = P_{A_T} f(\eta_0) \\ + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} P_{A_T-A_s} f(\eta_s) \left[\sum_{j=1}^m \sigma_{ij}(s) dB_s^{H_j} \right]. \end{aligned} \quad (45)$$

But

$$\begin{aligned} f(\eta_t) &= P_{A_t} f(\eta_0) \\ &\quad + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial x_i} P_{A_t-A_s} f(\eta_s) \left[\sum_{j=1}^m \sigma_{ij}(s) dB_s^{H_j} \right], \end{aligned} \quad (46)$$

from the semigroup property of P_{A_t} ; we have

$$P_{A_T-A_s} f(x) = P_{A_T-A_t} P_{A_t-A_s} f(x), \quad (47)$$

and it is easy to check that

$$\frac{\partial}{\partial x} P_{A_T-A_s} f(x) = P_{A_T-A_t} \frac{\partial}{\partial x} P_{A_t-A_s} f(x). \quad (48)$$

Hence

$$\begin{aligned} \hat{\mathbb{E}}_P^H[F | \mathcal{F}_t] \\ = P_{A_T-A_t} P_{A_t} f(\eta_0) \\ + \sum_{i=1}^n P_{A_T-A_t} \int_0^t \frac{\partial}{\partial x_i} P_{A_t-A_s} f(\eta_s) \left[\sum_{j=1}^m \sigma_{ij}(s) dB_s^{H_j} \right] \\ = P_{A_T-A_t} f(\eta_t). \end{aligned} \quad (49)$$

The assertion is completed. $\square$

4. Backward Stochastic Differential Equations Driven by Multidimensional Fractional Brownian Motions

In this section we will consider a backward stochastic differential equation driven by multidimensional fractional Brownian motions.

Denote $x_t = (x_1(t), x_2(t), \dots, x_n(t))$, where

$$\begin{aligned} x_i(t) &= x_i(0) + \int_0^t b_i(s) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dB_s^{H_j}, \\ i &= 1, \dots, n, \end{aligned} \quad (50)$$

and $B_s^{H_1}, \dots, B_s^{H_m}$ are independent fractional Brownian motions of Hurst parameters $H_1, \dots, H_m$.

Assume that

(H₁) $x_i(0)$, $i = 1, \dots, n$, are given constants.

(H₂) For $i = 1, \dots, n$, $b_i : [0, T] \rightarrow \mathbb{R}$ are continuous deterministic functions.

(H₃) For $i, j = 1, \dots, n$, $k = 1, \dots, m$, $\sigma_{ik} : [0, T] \rightarrow \mathbb{R}$ are continuous deterministic functions, and $[(d/dt)\langle \sigma_{ik}, \sigma_{jk} \rangle_{k,t}] > 0$, where $[(d/dt)\langle \sigma_{ik}, \sigma_{jk} \rangle_{k,t}] = 2H_k(2H_k - 1)\sigma_{ik}(t) \int_0^t (t-s)^{(2H_k-2)} \sigma_{jk}(s) ds$.

(H₄) For $i = 1, \dots, n$, $j = 1, \dots, m$,

$$\inf_{0 \leq t \leq T} \frac{\hat{\sigma}_{ij}(t)}{\sigma_{ij}(t)} \geq c_0 > 0, \quad (51)$$

where $\hat{\sigma}_{ij}(t) = \int_0^t \phi_j(t, r) \sigma_{ij}(r) dr$.

Now we consider the following backward stochastic differential equation:

$$\begin{aligned} dy_t &= -f(t, x_t, y_t, z_t) dt - \sum_{j=1}^m z_{j,t} dB_t^{H_j}, \\ y_T &= g(x_T). \end{aligned} \quad (52)$$

That is,

$$\begin{aligned} y_t &= g(x_T) + \int_t^T f(s, x_s, y_s, z_s) ds + \sum_{j=1}^m \int_t^T z_{j,s} dB_s^{H_j}, \\ 0 \leq t \leq T, \end{aligned} \quad (53)$$

where $z_s = (z_{1,s}, \dots, z_{m,s})$.

We further assume that

(H₅) $g(x)$ is a continuously differentiable function with respect to x and satisfies polynomial growth.

(H₆) $f(t, x, y, z)$ is a continuous function with respect to t and twice continuously differentiable with respect to x, y , and z , so there exists a constant $L > 0$, such that, for all $t \in [0, T]$, $x \in \mathbb{R}$, $y, y' \in \mathbb{R}$, and $z, z' \in \mathbb{R}^m$, we have

$$\begin{aligned} |f(t, x, y, z) - f(t, x, y', z')| \\ \leq L(|y - y'| + |z - z'|). \end{aligned} \quad (54)$$

The partial differential equation associated with above equation (52) is

$$\begin{aligned} u_t &= -\frac{1}{2} \sum_{i,j=1}^n \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] u_{x_i x_j} - \sum_{i=1}^n b_i(t) u_{x_i} \\ &\quad - f\left(t, x, u, -\sum_{i=1}^n \sigma_{i1}(t) u_{x_i}, \dots, -\sum_{i=1}^n \sigma_{im}(t) u_{x_i}\right), \end{aligned} \quad (55)$$

$$u(T, x) = g(x).$$

Theorem 6. If (55) has a solution $u(t, x)$ which is continuously differentiable with respect to t and twice continuously differentiable with respect to x , then

$$\begin{aligned} (y_t, z_{1,t}, \dots, z_{m,t}) \\ = \left(u(t, x_t), -\sum_{i=1}^n \sigma_{i1}(t) u_{x_i}(t, x_t), \dots, -\sum_{i=1}^n \sigma_{im}(t) u_{x_i}(t, x_t) \right). \end{aligned} \quad (56)$$

Proof. From Itô formula (14), we get

$$\begin{aligned} u(t, x_t) &= u(0, x_0) + \int_0^t \frac{\partial u}{\partial s}(s, x_s) ds \\ &\quad + \sum_{i=1}^n \int_0^t \frac{\partial u}{\partial x_i}(s, x_s) b_i(s) ds \\ &\quad + \sum_{i=1}^n \sum_{j=1}^m \int_0^t \frac{\partial u}{\partial x_i}(s, x_s) \sigma_{ij}(s) dB_s^{H_j} \\ &\quad + \int_0^t \sum_{i,j=1}^n \frac{1}{2} \frac{\partial^2 u}{\partial x_i \partial x_j}(s, x_s) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] ds \\ &= u(0, x_0) \\ &\quad + \int_0^t \left\{ \frac{\partial u}{\partial s}(s, x_s) + \sum_{i=1}^n \frac{\partial u}{\partial x_i}(s, x_s) b_i(s) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(s, x_s) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] \right\} ds \\ &\quad + \sum_{j=1}^m \sum_{i=1}^n \int_0^t \frac{\partial u}{\partial x_i}(s, x_s) \sigma_{ij}(s) dB_s^{H_j}. \end{aligned} \quad (57)$$

Since $u(t, x)$ satisfies (55), we deduce

$$\begin{aligned} u(t, x_t) &= g(x_T) \\ &\quad + \int_t^T f\left(s, x, u, -\sum_{i=1}^n \sigma_{i1}(s) u_{x_i}, \dots, -\sum_{i=1}^n \sigma_{im}(s) u_{x_i}\right) ds \\ &\quad - \sum_{j=1}^m \sum_{i=1}^n \int_t^T \sigma_{ij}(s) u_{x_i} dB_s^{H_j}. \end{aligned} \quad (58)$$

From (58), we get

$$\begin{aligned} (y_t, z_{1,t}, \dots, z_{m,t}) &= \left(u(t, x_t), -\sum_{i=1}^n \sigma_{i1}(t) u_{x_i}(t, x_t), \dots, \right. \\ &\quad \left. -\sum_{i=1}^n \sigma_{im}(t) u_{x_i}(t, x_t) \right), \end{aligned} \quad (59)$$

which satisfies BSDE (52), and the proof is complete. $\square$

Lemma 7. Let $a(s, x), \theta_j(s, x)$, $j = 1, \dots, m$, be continuous with respect to t and continuously differentiable with respect to x and let them be of polynomial growth. Assume that $(H_2) - (H_3)$, if

$$\int_0^t a(s, x_s) ds + \sum_{j=1}^m \int_0^t \theta_j(s, x_s) dB_s^{H_j} = 0, \quad (60)$$

$$\forall t \in [0, T],$$

then

$$a(s, x) = \theta_1(s, x) = \dots = \theta_m(s, x) = 0. \quad (61)$$

Proof. Similar to the proof of Lemma 3.2 in [19], we can obtain Lemma 7. $\square$

Theorem 8. Assume that (H_3) and let (52) have a solution of the form $(y_t = u(t, x_t), z_{1,t} = v_1(s, x_s), \dots, z_{m,t} = v_m(s, x_s))$. Then

$$\begin{aligned} v_1(s, x_s) &= -\sum_{i=1}^n \sigma_{i1}(t) u_{x_i}(t, x_t) \\ &\vdots \\ v_m(s, x_s) &= -\sum_{i=1}^n \sigma_{im}(t) u_{x_i}(t, x_t). \end{aligned} \quad (62)$$

Proof. From the proof of Theorem 6, we know

$$\begin{aligned} &u(t, x_t) \\ &= g(x_T) \\ &\quad - \int_t^T \left\{ \frac{\partial u}{\partial s}(s, x_s) + \sum_{i=1}^n \frac{\partial u}{\partial x_i}(s, x_s) b_i(s) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(s, x_s) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] \right\} ds \\ &\quad - \sum_{j=1}^m \sum_{i=1}^n \int_t^T \frac{\partial u}{\partial x_i}(s, x_s) \sigma_{ij}(s) dB_s^{H_j}. \end{aligned} \quad (63)$$

Furthermore, we deduce

$$\begin{aligned} & - \int_t^T \left\{ \frac{\partial u}{\partial s}(s, x_s) + \sum_{i=1}^n \frac{\partial u}{\partial x_i}(s, x_s) b_i(s) \right. \\ & \quad \left. + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(s, x_s) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] \right\} ds \\ & - \sum_{j=1}^m \sum_{i=1}^n \int_t^T \frac{\partial u}{\partial x_i}(s, x_s) \sigma_{ij}(s) dB_s^{H_j} \\ & = \int_t^T f(s, x, u(s, x_s), v_1(s, x_s), \dots, v_m(s, x_s)) ds \\ & \quad + \sum_{j=1}^m \int_t^T v_j(s, x_s) dB_s^{H_j}, \end{aligned} \quad (64)$$

which is also true for $t = 0$. Therefore, we have

$$\begin{aligned} & - \int_0^t \left\{ \frac{\partial u}{\partial s}(s, x_s) + \sum_{i=1}^n \frac{\partial u}{\partial x_i}(s, x_s) b_i(s) \right. \\ & \quad \left. + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j}(s, x_s) \sum_{k=1}^m \left[\frac{d}{ds} \langle \sigma_{ik}, \sigma_{jk} \rangle_{k,s} \right] \right\} ds \\ & - \sum_{j=1}^m \sum_{i=1}^n \int_0^t \frac{\partial u}{\partial x_i}(s, x_s) \sigma_{ij}(s) dB_s^{H_j} \\ & = \int_0^t f(s, x, u(s, x_s), v_1(s, x_s), \dots, v_m(s, x_s)) ds \\ & \quad + \sum_{j=1}^m \int_0^t v_j(s, x_s) dB_s^{H_j}. \end{aligned} \quad (65)$$

From Lemma 7, we get

$$\begin{aligned} v_1(s, x_s) &= -\sum_{i=1}^n \sigma_{i1}(t) u_{x_i}(t, x_t) \\ &\vdots \\ v_m(s, x_s) &= -\sum_{i=1}^n \sigma_{im}(t) u_{x_i}(t, x_t), \end{aligned} \quad (66)$$

which completes the proof. $\square$

In order to show the existence and uniqueness of solution to (52), we introduce the following set.

$\mathcal{C} = \{\varphi(t, x); \varphi(t, x) \text{ is continuously differentiable with respect to } t \text{ and twice continuously differentiable with respect to } x\}.$

And let $\mathcal{V}_T$ denote the set of the process Y_t of the form

$$\mathcal{V}_T = \{Y = \varphi(\cdot, x); \varphi(t, x_t) \in \mathcal{C}, \forall t \in [0, T]\}. \quad (67)$$

Let $\widetilde{\mathcal{V}}_T$ be the completion of $\mathcal{V}_T$ under the following β -norm:

$$\|Y\|_\beta^2 = \int_0^T e^{\beta t} \mathbb{E} |Y_t|^2 dt = \int_0^T e^{\beta t} \mathbb{E} |\varphi(t, x_t)|^2 dt, \quad (68)$$

where β is a positive arbitrary real-valued parameter.

We want to show that the solution to (52) in $\widetilde{\mathcal{V}}_T$ exists uniquely. Let $y_t = \varphi(t, x_t)$, $z_t = (z_{1,t}, \dots, z_{m,t}) = (\psi_1(t, x_t), \dots, \psi_m(t, x_t))$ be given. Consider the following backward stochastic differential equation:

$$dY_t = -f(t, x_t, y_t, z_t) dt - \sum_{j=1}^m Z_{j,t} dB_t^{H_j}, \quad (69)$$

$$Y_T = g(x_T).$$

Theorem 9. Assume that (H_5) – (H_6) and let $y_t = \varphi(t, x_t)$ and $z_{1,t} = \psi_1(t, x_t), \dots, z_{m,t} = \psi_m(t, x_t) \in \mathcal{C}$; then the solution $(Y_t, Z_{1,t}, \dots, Z_{m,t})$ to (69) satisfies $Y, Z_{1,t}, \dots, Z_{m,t} \in \mathcal{V}_T$.

Proof. By (69), we know

$$\begin{aligned} Y_t &= g(x_T) \\ &+ \int_t^T f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) ds \\ &+ \sum_{j=1}^m \int_t^T Z_{j,s} dB_s^{H_j}. \end{aligned} \quad (70)$$

Taking the quasi-conditional expectation, we have

$$\begin{aligned} Y_t &= \widehat{\mathbb{E}}_P^H \left[g(x_T) \right. \\ &\quad \left. + \int_t^T f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \right. \\ &\quad \left. \psi_m(s, x_s)) ds \mid \mathcal{F}_t \right] \\ &= P_{A_T-A_t} g(x_t) + \int_t^T P_{A_T-A_t} \tilde{f}(s, x_s) ds, \end{aligned} \quad (71)$$

where $\tilde{f}(s, x_s) = f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s))$. Obviously, we have Y_t being in $\mathcal{C}$.

By (70), let $t = 0$; we get

$$\begin{aligned} Y_0 &= g(x_T) \\ &+ \int_0^T f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) ds \\ &+ \sum_{j=1}^m \int_0^T Z_{j,s} dB_s^{H_j}. \end{aligned} \quad (72)$$

Taking the quasi-conditional expectation, we deduce

$$\begin{aligned} Y_0 &= \widehat{\mathbb{E}}_P^H \left[\left(g(x_T) \right. \right. \\ &\quad \left. \left. + \int_0^T f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \right. \right. \\ &\quad \left. \left. \psi_m(s, x_s)) ds \right) \mid \mathcal{F}_t \right] \end{aligned} \quad (73)$$

$$+ \sum_{j=1}^m \int_0^t Z_{j,s} dB_s^{H_j}.$$

By the fractional Clark formula [20], we have

$$\begin{aligned} Z_{j,t} &= -\widehat{\mathbb{E}}_P^H \\ &\cdot \left[\left(D_t^{H_j} g(x_T) + D_t^{H_j} \right. \right. \\ &\quad \cdot \int_t^T f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \\ &\quad \dots, \psi_m(s, x_s)) ds \Big) \mid \mathcal{F}_t \Big] \\ &= -\widehat{\mathbb{E}}_P^H \left\{ \sum_{i=1}^n \sigma_{ij}(t) \frac{\partial}{\partial x_i} g(x_T) \right. \\ &\quad \left. + \int_t^T \left[\sum_{i=1}^n \sigma_{ij}(t) \frac{\partial}{\partial x_i} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \right. \right. \\ &\quad \left. \left. \dots, \psi_m(s, x_s)) \right. \right. \\ &\quad \left. \left. + \frac{\partial}{\partial y} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \right. \right. \\ &\quad \left. \left. \dots, \psi_m(s, x_s)) \right. \right. \\ &\quad \left. \cdot \sum_{i=1}^n \sigma_{ij}(t) \varphi_{x_i}(s, x_s) \right. \\ &\quad \left. + \sum_{j=1}^m \frac{\partial}{\partial z_j} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \right. \\ &\quad \left. \dots, \psi_m(s, x_s)) \right. \\ &\quad \left. \cdot \sum_{i=1}^n \sigma_{ij}(t) \psi_{j x_i}(s, x_s) \right] ds \mid \mathcal{F}_t \Big\} \end{aligned}$$

$$\begin{aligned}
&= -\sum_{i=1}^n \sigma_{ij}(t) \widehat{\mathbb{E}}_P^H \\
&\quad \cdot \left\{ \frac{\partial}{\partial x_i} g(x_T) \right. \\
&\quad + \int_t^T \left[\frac{\partial}{\partial x_i} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) \right. \\
&\quad + \frac{\partial}{\partial y} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) \\
&\quad \cdot \varphi_{x_i}(s, x_s) \\
&\quad + \sum_{j=1}^m \frac{\partial}{\partial z_j} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) \\
&\quad \cdot \psi_{jx_i}(s, x_s) \left. \right] ds \mid \mathcal{F}_t \Bigg\} \\
&= -\sum_{i=1}^n \sigma_{ij}(t) \widehat{\mathbb{E}}_P^H \left[\frac{\partial}{\partial x_i} g(x_T) \mid \mathcal{F}_t \right] \\
&\quad - \sum_{i=1}^n \sigma_{ij}(t) \int_t^T \widehat{\mathbb{E}}_P^H [\tilde{f}_i(s, x_s) \mid \mathcal{F}_t] ds,
\end{aligned} \tag{74}$$

where

$$\begin{aligned}
&\tilde{f}_i(s, x_s) \\
&= \frac{\partial}{\partial x_i} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) \\
&\quad + \frac{\partial}{\partial y} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) \\
&\quad \cdot \varphi_{x_i}(s, x_s) \\
&\quad + \sum_{j=1}^m \frac{\partial}{\partial z_j} f(s, x_s, \varphi(s, x_s), \psi_1(s, x_s), \dots, \psi_m(s, x_s)) \\
&\quad \cdot \psi_{jx_i}(s, x_s) ds.
\end{aligned} \tag{75}$$

From the expressions of $Z_{j,t}$, $j = 1, \dots, m$, we easily infer that $Z_{j,\cdot} \in \mathcal{V}_T$, $j = 1, \dots, m$. We complete the proof. $\square$

Now we give the theorem of existence and uniqueness of the solution to BSDE (52).

Theorem 10. Assume that $(H_1)-(H_6)$; then BSDE (52) has a unique solution in $\mathcal{V}_T$.

Proof. Let $y_t^i, z_{j,t}^i \in \mathcal{C}$, $j = 1, \dots, m$, $i = 1, 2$. From Theorem 8, we know that $Y_t^i = \varphi^i(t, x_t)$, $Z_{j,t} = \psi_j^i(t, x_t) \in \mathcal{C}$, $j = 1, \dots, m$, $i = 1, 2$, satisfy (70) with possibly different terminal conditions:

$$\begin{aligned}
Y_t^i &= g^i(x_T) \\
&\quad + \int_t^T f(s, x_s, \varphi^i(s, x_s), \psi_1^i(s, x_s), \dots, \psi_m^i(s, x_s)) ds \\
&\quad + \sum_{j=1}^m \int_t^T Z_{j,s}^i dB_s^{H_j}.
\end{aligned} \tag{76}$$

Denote

$$\begin{aligned}
\widehat{Y}_t &= Y_t^1 - Y_t^2, \\
\widehat{g}(x_T) &= g^1(x_T) - g^2(x_T), \\
\widehat{Z}_{j,t} &= Z_{j,t}^1 - Z_{j,t}^2, \quad j = 1, \dots, m, \\
\widehat{f}_t &= f(s, x_s, \varphi^1(s, x_s), \psi_1^1(s, x_s), \dots, \psi_m^1(s, x_s)) \\
&\quad - f(s, x_s, \varphi^2(s, x_s), \psi_1^2(s, x_s), \dots, \psi_m^2(s, x_s)).
\end{aligned} \tag{77}$$

Then we get

$$d\widehat{Y}_t = -\widehat{f}_t dt - \sum_{j=1}^m \widehat{Z}_{j,t} dB_t^{H_j}, \tag{78}$$

and we deduce

$$d\widehat{Y}_t^2 = -2\widehat{Y}_t \widehat{f}_t dt - 2\widehat{Y}_t \sum_{j=1}^m \widehat{Z}_{j,t} dB_t^{H_j} - 2\sum_{j=1}^m \widehat{Z}_{j,t} \mathbb{D}_t^{H_j} \widehat{Y}_t dt. \tag{79}$$

However, if $Y_t^1 = \bar{\varphi}(t, x_t)$, by Theorem 8, we have

$$Z_{j,t}^1 = -\sum_{i=1}^n \sigma_{ij}(t) \bar{\varphi}_{x_i}(t, x_t), \quad j = 1, \dots, m. \tag{80}$$

By the property of Malliavin derivative relative to the fractional Brownian motion, we obtain

$$\begin{aligned}
\mathbb{D}_t^{H_j} Y_t^1 &= \sum_{i=1}^n \bar{\varphi}_{x_i}(t, x_t) \int_0^t \phi_j(t, s) D_s^{H_j} x_i(t) ds \\
&= \sum_{i=1}^n \bar{\varphi}_{x_i}(t, x_t) \int_0^t \phi_j(t, s) \sigma_{ij}(s) ds \\
&= -\sum_{i=1}^n \frac{\bar{\sigma}_{ij}(t)}{\sigma_{ij}(t)} Z_{j,t}^1.
\end{aligned} \tag{81}$$

Moreover, by (79) and (81), we get

$$\begin{aligned} d\widehat{Y}_t^2 &= -2\widehat{Y}_t\widehat{f}_t dt - 2\widehat{Y}_t \sum_{j=1}^m \widehat{Z}_{j,t} dB_t^{H_j} \\ &\quad + 2 \sum_{j=1}^m \sum_{i=1}^n \frac{\widehat{\sigma}_{ij}(t)}{\sigma_{ij}(t)} \widehat{Z}_{j,t}^2 dt. \end{aligned} \quad (82)$$

By (19), we have

$$\begin{aligned} d(e^{\beta t} \widehat{Y}_t^2) &= \beta e^{\beta t} \widehat{Y}_t^2 dt - 2e^{\beta t} \widehat{Y}_t \widehat{f}_t - 2e^{\beta t} \widehat{Y}_t \sum_{j=1}^m \widehat{Z}_{j,t} dB_t^{H_j} \\ &\quad + 2e^{\beta t} \sum_{j=1}^m \sum_{i=1}^n \frac{\widehat{\sigma}_{ij}(t)}{\sigma_{ij}(t)} \widehat{Z}_{j,t}^2 dt. \end{aligned} \quad (83)$$

Taking the expectations and integrating on both sides of (83), we obtain

$$\begin{aligned} e^{\beta t} \mathbb{E} \widehat{Y}_t^2 + 2 \int_t^T e^{\beta s} \sum_{j=1}^m \mathbb{E} \left(\sum_{i=1}^n \frac{\widehat{\sigma}_{ij}(s)}{\sigma_{ij}(s)} \widehat{Z}_{j,s}^2 \right) ds \\ = e^{\beta T} \mathbb{E} \widehat{Y}_T^2 - \beta \int_t^T e^{\beta s} \mathbb{E} \widehat{Y}_s^2 ds + 2 \int_t^T e^{\beta t} \mathbb{E} (\widehat{Y}_s \widehat{f}_s) ds. \end{aligned} \quad (84)$$

From (51) and (84), we have

$$\begin{aligned} e^{\beta t} \mathbb{E} \widehat{Y}_t^2 + 2nc_0 \sum_{j=1}^m \int_t^T e^{\beta s} \mathbb{E} \widehat{Z}_{j,s}^2 ds \\ \leq e^{\beta T} \mathbb{E} \widehat{Y}_T^2 - \beta \int_t^T e^{\beta s} \mathbb{E} \widehat{Y}_s^2 ds + 2 \int_t^T e^{\beta t} \mathbb{E} (\widehat{Y}_s \widehat{f}_s) ds \\ \leq e^{\beta T} \mathbb{E} \widehat{Y}_T^2 - \beta \int_t^T e^{\beta s} \mathbb{E} \left[\widehat{Y}_s - \frac{\widehat{f}_s}{\beta} \right]^2 ds \\ + \frac{1}{\beta} \int_t^T e^{\beta t} \mathbb{E} \widehat{f}_s^2 ds \leq e^{\beta T} \mathbb{E} \widehat{Y}_T^2 + \frac{1}{\beta} \int_t^T e^{\beta t} \mathbb{E} \widehat{f}_s^2 ds. \end{aligned} \quad (85)$$

By (85), we get

$$e^{\beta t} \mathbb{E} \widehat{Y}_t^2 \leq e^{\beta T} \mathbb{E} \widehat{Y}_T^2 + \frac{1}{\beta} \int_0^T e^{\beta t} \mathbb{E} \widehat{f}_s^2 ds, \quad (86)$$

$$\begin{aligned} \sum_{j=1}^m \int_0^T e^{\beta s} \mathbb{E} \widehat{Z}_{j,s}^2 ds &\leq \frac{1}{2nc_0} e^{\beta T} \mathbb{E} \widehat{Y}_T^2 \\ &\quad + \frac{1}{2nc_0 \beta} \int_0^T e^{\beta t} \mathbb{E} \widehat{f}_s^2 ds. \end{aligned} \quad (87)$$

Integrating from 0 to T on both sides of (86)

$$\int_0^T e^{\beta s} \mathbb{E} \widehat{Y}_s^2 ds \leq T e^{\beta T} \mathbb{E} \widehat{Y}_T^2 + \frac{T}{\beta} \int_0^T e^{\beta t} \mathbb{E} \widehat{f}_s^2 ds. \quad (88)$$

By (87) and (88), we derive

$$\begin{aligned} \|\widehat{Y}\|_\beta^2 + \sum_{j=1}^m \|\widehat{Z}_j\|_\beta^2 &\leq \left(\frac{e^{\beta T}}{2nc_0} + T e^{\beta T} \right) \mathbb{E} |\widehat{Y}_T|^2 \\ &\quad + \frac{1}{\beta} \left(T + \frac{1}{2nc_0} \right) \|\widehat{f}\|_\beta^2. \end{aligned} \quad (89)$$

By the hypotheses of Theorem 10, we know that $f(t, x, y, z)$ is uniformly globally Lipschitzian with respect to y, z , and $\widehat{Y}_T = 0$, so we have

$$\begin{aligned} \|\widehat{Y}\|_\beta^2 + \sum_{j=1}^m \|\widehat{Z}_j\|_\beta^2 \\ \leq \frac{L}{\beta} \left(T + \frac{1}{2nc_0} \right) \left[\|\widehat{y}\|_\beta^2 + \sum_{j=1}^m \|\widehat{z}_j\|_\beta^2 \right], \end{aligned} \quad (90)$$

where L is Lipschitzian constant of the function f .

Choosing β sufficiently large, we see that the mapping $(y, z_1, \dots, z_m) \rightarrow (Y, Z_1, \dots, Z_m)$ determined by (70) is a contraction in $\widetilde{\mathcal{V}}_T$. Using the fixed point principle, we deduce that the solution to (52) exists uniquely, and the proof is completed. $\square$

5. Linear Backward Stochastic Differential Equations

In this section we discuss the following linear backward stochastic differential equation:

$$dy_t = - \left[\alpha_t + \beta_t y_t + \sum_{i=1}^m \gamma_{i,t} z_{i,t} \right] dt - \sum_{i=1}^m z_{i,t} dB_t^{H_i}, \quad (91)$$

$$y_T = \xi,$$

where α_t, β_t , and $\gamma_{i,t}$, $i = 1, \dots, m$, are given, continuous, and adapted processes and $\xi \in \mathcal{F}_T$; in particular we allow the terminal condition ξ to be general form. When the coefficients are deterministic and $i = 1$, the equation is studied in [10–12]. When the coefficients are deterministic, $i = 2$ and $H_1 = 1/2$, the equation is studied in [15]. This paper further studies more general linear backward stochastic differential equation. Let

$$\rho_t = \exp \left\{ \int_0^T \beta_s ds \right\}; \quad (92)$$

that is,

$$d\rho_t = \beta_t \rho_t dt. \quad (93)$$

From fractional Itô chain rule (18), we have

$$\begin{aligned}
 d(\rho_t y_t) &= \rho_t dy_t + y_t d\rho_t - \sum_{i=1}^m z_{i,t} \mathbb{D}_t^{H_i} \rho_t dt \\
 &\quad - \left[\alpha_t \rho_t + \sum_{i=1}^m \gamma_{i,t} \rho_t z_{i,t} + \sum_{i=1}^m z_{i,t} \mathbb{D}_t^{H_i} \rho_t \right] dt \\
 &\quad - \sum_{i=1}^m z_{i,t} \rho_t dB_t^{H_i} \\
 &= -\alpha_t \rho_t \\
 &\quad - \sum_{i=1}^m \rho_t z_{i,t} \left[\left(\gamma_{i,t} + \frac{\mathbb{D}_t^{H_i} \rho_t}{\rho_t} \right) dt + dB_t^{H_i} \right] \\
 &= -\alpha_t \rho_t - \sum_{i=1}^m \rho_t z_{i,t} d\widehat{B}_t^{H_i},
 \end{aligned} \tag{94}$$

where

$$\begin{aligned}
 d\widehat{B}_t^{H_i} &= \left(\gamma_{i,t} + \frac{\mathbb{D}_t^{H_i} \rho_t}{\rho_t} \right) dt + dB_t^{H_i} \\
 &= dB_t^{H_i} + \left(\gamma_{i,t} + \int_0^t \mathbb{D}_s^{H_i} \beta_s ds \right) dt,
 \end{aligned} \tag{95}$$

which is a fractional Brownian motion under the new probability measure Q defined by

$$\begin{aligned}
 \frac{dQ}{dP} &= \exp \left\{ \sum_{i=1}^m \left(\int_0^T \left(\mathcal{K}_{H_i}^{-1} \int_0^\cdot b_{i,r} dr \right) (s) dW_s \right. \right. \\
 &\quad \left. \left. - \frac{1}{2} \int_0^T \left(\mathcal{K}_{H_i}^{-1} \int_0^\cdot b_{i,r} dr \right)^2 (s) ds \right) \right\},
 \end{aligned} \tag{96}$$

where $b_{i,r} = \gamma_{i,r} + \int_0^r \mathbb{D}_s^{H_i} \beta_s ds$ and $\mathcal{K}_{H_i}^{-1}(t)$ is defined in (23). From the definitions of the probability measures Q in (96), according to the Girsanov theorem, we know that $\widehat{B}_t^{H_i}$, $i = 1, \dots, m$, is a fractional Brownian motion relative to probability Q . Thus we have

$$\rho_t y_t = \rho_T \xi + \int_0^T \alpha_s \rho_s ds + \sum_{i=1}^m \int_0^T \rho_t z_{i,s} d\widehat{B}_s^{H_i}. \tag{97}$$

Therefore

$$\rho_t y_t = \widehat{\mathbb{E}}_Q^H \left[\rho_T \xi + \int_t^T \alpha_s \rho_s ds \mid \mathcal{F}_t \right]; \tag{98}$$

that is,

$$y_t = \rho_t^{-1} \widehat{\mathbb{E}}_Q^H \left[\rho_T \xi + \int_t^T \alpha_s \rho_s ds \mid \mathcal{F}_t \right]. \tag{99}$$

From above deduction, we have the following theorem.

Theorem 11. Let α_t , β_t , and $\gamma_{i,t}$, $i = 1, \dots, m$, be given, continuous, and adapted processes and $\xi \in \mathcal{F}_T$. Suppose

$$\mathbb{E} \left[\exp \left\{ \frac{1}{2} \sum_{i=1}^m \int_0^T \left(\mathcal{K}_{H_i}^{-1} \int_0^\cdot b_{i,r} dr \right)^2 (s) ds \right\} \right] < \infty, \tag{100}$$

where $b_{i,r} = \gamma_{i,r} + \int_0^r \mathbb{D}_s^{H_i} \beta_s ds$. Then the solution to (91) exists uniquely. Moreover, we have

$$y_t = \rho_t^{-1} \widehat{\mathbb{E}}_Q^H \left[\rho_T \xi + \int_t^T \alpha_s \rho_s ds \mid \mathcal{F}_t \right]. \tag{101}$$

Proof. The existence and uniqueness for y_t can be obtained from the above argument. For the existence and uniqueness of $z_{i,t}$, $i = 1, \dots, m$, we apply Theorem 12.1 of [21]. We complete the proof of the theorem. $\square$

6. Conclusion

We study the solutions to the backward stochastic differential equations driven by multidimensional fractional Brownian motions, with a more general terminal condition $y_T = g(x_1(T), \dots, x_n(T))$, where $x_i(t) = x_i(0) + \int_0^t b_i(s) ds + \sum_{j=1}^m \int_0^t \sigma_{ij}(s) dB_s^{H_j}$, $i = 1, \dots, n$. Using the quasi-conditional expectation and multidimensional fractional Itô formula, we obtain the existence and uniqueness of the solutions to BSDEs, where a fixed point principle is employed. Finally, solutions to linear backward stochastic differential equations driven by multidimensional fractional Brownian motions are investigated. Thus we extend some results of [10–12, 15]. Nowadays the theory and applications of fractional stochastic differential equations play a crucial role in characterizing many social, physical, biological, and engineering problems. In the future, we may extend our work to integrodifferential inclusions [22] and fractional integrodifferential inclusions [23].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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