Research Article

Tree-Antimagicness of Disconnected Graphs

Martin Bača,¹ Zuzana Kimáková,¹ Andrea Semaničová-Feňovčíková,¹
and Muhammad Awais Umar²

¹Department of Applied Mathematics and Informatics, Technical University, 042 00 Košice, Slovakia

²Abdus Salam International Center of Mathematics, Information Technology University, Lahore, Pakistan

Correspondence should be addressed to Martin Bača; martin.baca@tuke.sk

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A simple graph G admits an H -covering if every edge in $E(G)$ belongs to a subgraph of G isomorphic to H . The graph G is said to be (a, d) - H -antimagic if there exists a bijection from the vertex set $V(G)$ and the edge set $E(G)$ onto the set of integers $\{1, 2, \dots, |V(G)| + |E(G)|\}$ such that, for all subgraphs H' of G isomorphic to H , the sum of labels of all vertices and edges belonging to H' constitute the arithmetic progression with the initial term a and the common difference d . G is said to be a super (a, d) - H -antimagic if the smallest possible labels appear on the vertices. In this paper, we study super tree-antimagic total labelings of disjoint union of graphs.

1. Introduction

We consider finite undirected graphs without loops and multiple edges. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. An *edge-covering* of G is a family of subgraphs $H_1, H_2, \dots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i, i = 1, 2, \dots, t$. In this case we say that G admits an $(H_1, H_2, \dots, H_t)$ -(*edge*) *covering*. If every subgraph H_i is isomorphic to given graph H , then the graph G admits an H -covering. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$ which we call a *total labeling* and the associated H -weight of subgraph H is $wt_f(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$. An (a, d) - H -antimagic total labeling of graph G admitting an H -covering is a total labeling with the property that, for all subgraphs H' isomorphic to H , the H' -weights form an arithmetic progression $a, a + d, a + 2d, \dots, a + (t - 1)d$, where $a > 0$ and $d \geq 0$ are two integers and t is the number of all subgraphs of G isomorphic to H . Such a labeling is called a *super* if the smallest possible labels appear on the vertices. A graph that admits a (super) (a, d) - H -antimagic total labeling is called a (super) (a, d) - H -antimagic. For $d = 0$, it is called H -magic and H -supermagic, respectively.

The H -(super)magic labelings were first studied by Gutiérrez and Lladó [1]. These labelings are the generalization of the edge-magic and super edge-magic labelings that were

introduced by Kotzig and Rosa [2] and Enomoto et al. [3], respectively. For further information about (super) edge-magic labelings, one can see [4–8]. Gutiérrez and Lladó [1] proved that certain classes of connected graphs are H -(super)magic, such as the star $K_{1,n}$ and the complete bipartite graphs $K_{n,m}$ are $K_{1,h}$ -supermagic for some h . They also proved that the cycle C_n is P_h -supermagic for any $2 \leq h \leq n - 1$ such that $\gcd(n, h(h - 1)) = 1$. Lladó and Moragas [9] studied the cycle-(super)magic behavior of several classes of connected graphs. They proved that wheels, windmills, books, and prisms are C_h -magic for some h . Maryati et al. [10] and also Salman et al. [11] proved that certain families of trees are path-supermagic. Ngurah et al. [12] proved that chains, wheels, triangles, ladders, and grids are cycle-supermagic.

The (a, d) - H -antimagic total labeling is introduced by Inayah et al. [13]. In [14], the super (a, d) - H -antimagic labelings are investigated for some shackles of connected graph H .

When H is isomorphic to K_2 , a super (a, d) - K_2 -antimagic total labeling is also called super (a, d) -edge-antimagic total. The notion of (a, d) -edge-antimagic total labeling was introduced by Simanjuntak et al. in [15] as a natural extension of the *edge-magic labeling* defined by Kotzig and Rosa in [2] as *magic valuation*.

The (super) (a, d) - H -antimagic total labeling is related to a super d -antimagic labeling of type $(1, 1, 0)$ of a plane graph.

A labeling of type $(1, 1, 0)$, that is, a total labeling, of a plane graph is said to be d -antimagic if for every positive integer s the set of weights of all s -sided faces is $W_s = \{a_s, a_s + d, a_s + 2d, \dots, a_s + (f_s - 1)d\}$ for some integers $a_s > 0$ and $d \geq 0$, where f_s is the number of the s -sided faces. Note that there are allowed different sets W_s for different s . The *weight* of a face under a labeling of type $(1, 1, 0)$ is the sum of labels of all the edges and vertices surrounding that face. If $d = 0$, then Lih [16] calls such labeling *magic* and describes magic (0-antimagic) labelings of type $(1, 1, 0)$ for wheels, friendship graphs, and prisms. In [17], Ahmad et al. investigate the existence of super d -antimagic labelings of type $(1, 1, 0)$ for disjoint union of plane graphs for several values of difference d .

In this paper we mainly investigate the existence of super $(a, 1)$ -tree-antimagic total labelings for disconnected graphs. We concentrate on the following problem: if graph G admits a (super) $(a, 1)$ -tree-antimagic total labeling, does the disjoint union of m copies of the graph G , denoted by mG , admit a (super) $(b, 1)$ -tree-antimagic total labeling as well?

2. Super $(a, 1)$ -Tree-Antimagic Total Labeling

In this section, we will study the super $(a, 1)$ -tree-antimagicness for disconnected graphs. The main result is the following.

Theorem 1. *Let $m, t \geq 1$ be positive integers. Let G_i with a $(T_i^1, T_i^2, \dots, T_i^t)$ -covering be a super $(a, 1)$ - T -antimagic graph of order p and size q , $i = 1, 2, \dots, m$, where T is a tree and every tree T_i^j , $j = 1, 2, \dots, t$, is isomorphic to T . Then the disjoint union $\bigcup_{i=1}^m G_i$ is also a super $(b, 1)$ - T -antimagic graph.*

Proof. Let $m \geq 1$ and $t \geq 1$ be positive integers and let T be a tree of order k . Let G_i , $i = 1, 2, \dots, m$, be a graph with p vertices and q edges that admits a $(T_i^1, T_i^2, \dots, T_i^t)$ -covering. Note that G_i is not necessarily isomorphic to G_j for $i \neq j$. Assume that every G_i , $i = 1, 2, \dots, m$, has a super $(a, 1)$ - T -antimagic total labeling $f_i : V(G_i) \cup E(G_i) \rightarrow \{1, 2, \dots, p\} \cup \{p+1, p+2, \dots, p+q\}$; thus

$$\{wt_{f_i}(T_i^j) : j = 1, 2, \dots, t\} = \{a, a+1, \dots, a+t-1\} \quad (1)$$

is the set of the corresponding T_i^j -weights, where every tree T_i^j , $j = 1, 2, \dots, t$, is isomorphic to T .

Define the labeling f for the vertices and edges of $\bigcup_{i=1}^m G_i$ in the following way:

$$f(x) = \begin{cases} m(f_i(x) - 1) + i, & \text{if } x \in V(G_i), \\ mf_i(x) + 1 - i, & \text{if } x \in E(G_i). \end{cases} \quad (2)$$

First we will show that the vertices of $\bigcup_{i=1}^m G_i$ under the labeling f , use integers from 1 up to pm ; that is, if $i = 1$, then the vertices of G_1 successively attain values $[1, m+1, 2m+1, \dots, (p-1)m+1]$, if $i = 2$, then the vertices of G_2 successively assume values $[2, m+2, 2m+2, \dots, (p-1)m+2]$, ..., the values of vertices of G_i are equal successively to $[i, m+i, 2m+i, \dots, (p-1)m+i]$, ..., and if $i = m$, then the vertices of G_m successively assume values $[m, 2m, 3m, \dots, pm]$.

Second we can see that the edges of $\bigcup_{i=1}^m G_i$ under the labeling f use integers from $pm+1$ up to $(p+q)m$. It means that if $i = 1$, then the edges of G_1 successively assume values $[(p+1)m, (p+2)m, (p+3)m, \dots, (p+q)m]$, if $i = 2$, then the edges of G_2 successively attain values $[(p+1)m-1, (p+2)m-1, (p+3)m-1, \dots, (p+q)m-1]$, ..., the values of edges of G_i are equal successively to $[(p+1)m+1-i, (p+2)m+1-i, (p+3)m+1-i, \dots, (p+q)m+1-i]$, ..., and if $i = m$, then the edges of G_m successively attain values $[pm+1, (p+1)m+1, (p+2)m+1, \dots, (p+q-1)m+1]$.

It is not difficult to see that the labeling f is a bijective function which assigns the integer $\{1, 2, \dots, (p+q)m\}$ to the vertices and edges of $\bigcup_{i=1}^m G_i$; thus f is a total labeling.

For the weight of every subgraph T_i^j isomorphic to the tree T under the labeling f , we have

$$\begin{aligned} wt_f(T_i^j) &= \sum_{v \in V(T_i^j)} f(v) + \sum_{e \in E(T_i^j)} f(e) \\ &= \sum_{v \in V(T_i^j)} (m(f_i(v) - 1) + i) \\ &\quad + \sum_{e \in E(T_i^j)} (mf_i(e) + 1 - i) \\ &= m \sum_{v \in V(T_i^j)} f_i(v) - m|V(T_i^j)| + i|V(T_i^j)| \\ &\quad + m \sum_{e \in E(T_i^j)} f_i(e) + |E(T_i^j)| - i|E(T_i^j)| \\ &= m \left(\sum_{v \in V(T_i^j)} f_i(v) + \sum_{e \in E(T_i^j)} f_i(e) \right) \\ &\quad - m|V(T_i^j)| + |E(T_i^j)| + i|V(T_i^j)| - i|E(T_i^j)| \\ &= mwt_{f_i}(T_i^j) - m|V(T_i^j)| + |E(T_i^j)| \\ &\quad + i|V(T_i^j)| - i|E(T_i^j)|. \end{aligned} \quad (3)$$

As every T_i^j , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, t$, is isomorphic to the tree T , it holds

$$|V(T_i^j)| = |V(T)| = k, \quad (4)$$

$$|E(T_i^j)| = |E(T)| = |V(T)| - 1 = k - 1.$$

Thus for the T -weights we get

$$wt_f(T_i^j) = mwt_{f_i}(T_i^j) + k(1 - m) - 1 + i. \quad (5)$$

According to (1), we get that the T -weights in components are the following:

if $i = 1$, then the T_1^j -weights in G_1 are $m(a - k) + k$, $m(a - k + 1) + k, \dots$, and $m(a - k + t - 1) + k$;

if $i = 2$, then the T_2^j -weights in G_2 are $m(a-k)+k+1$, $m(a-k+1)+k+1, \dots$, and $m(a-k+t-1)+k+1$;

if $i = 3$, then the T_3^j -weights in G_3 are $m(a-k)+k+2$, $m(a-k+1)+k+2, \dots$, and $m(a-k+t-1)+k+2$;

$\vdots$

if $i = m-1$, then the T_{m-1}^j -weights in G_{m-1} successively attain values $m(a-k)+k+m-2$, $m(a-k+1)+k+m-2, \dots$, and $m(a-k+t-1)+k+m-2$;

if $i = m$, then the T_m^j -weights in G_m successively assume values $m(a-k)+k+m-1$, $m(a-k+1)+k+m-1, \dots$, and $m(a-k+t-1)+k+m-1$.

It is easy to see that the set of all T -weights in $\bigcup_{i=1}^m G_i$ consists of distinct and consecutive integers:

$$\begin{aligned} & \{wt_f(T_i^j) : i = 1, 2, \dots, m; j = 1, 2, \dots, t\} \\ &= \{m(a-k)+k, m(a-k) \\ & \quad + k+1, \dots, m(a-k+t)+k-1\}. \end{aligned} \quad (6)$$

Thus the graph $\bigcup_{i=1}^m G_i$ is a super $(m(a-k)+k, 1)$ - T -antimagic. $\square$

Immediately from the previous theorem we get that arbitrary number of copies of a super $(a, 1)$ - T -antimagic graph is a super $(b, 1)$ - T -antimagic.

Corollary 2. Let G be a super $(a, 1)$ - T -antimagic graph, where T is a tree. Then the disjoint union of arbitrary number of copies of G , that is, mG , $m \geq 1$, also admits a super $(b, 1)$ - T -antimagic total labeling.

Moreover, for m copies of a graph G which is $(a, 1)$ - T -antimagic but is not super, we can derive the following result.

Theorem 3. Let G be an $(a, 1)$ - T -antimagic graph of order p and size q , where T is a tree. Then mG , $m \geq 1$, is also a $(b, 1)$ - T -antimagic graph.

Proof. Let T be a tree of order k and let G be an $(a, 1)$ - T -antimagic graph of order p and size q with corresponding labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ and the corresponding T -weights $a, a+1, a+2, \dots, a+t-1$.

For every vertex v in G , we denote by symbol v_i the corresponding vertex in the i th copy of G in mG . Analogously, let $u_i v_i$ be the edge corresponding to the edge uv in the i th copy of G in mG .

For $m \geq 1$, we define labeling g of mG as follows:

$$\begin{aligned} g(v_i) &= m(f(v)-1) + i \quad \text{for } v \in V(G), i = 1, 2, \dots, m, \\ g(u_i v_i) &= mf(uv) + 1 - i \quad \text{for } uv \in E(G), i = 1, 2, \dots, m. \end{aligned} \quad (7)$$

Let $r \in \{1, 2, \dots, p+q\}$. We consider two cases.

Case 1. If the number r is assigned by the labeling f to a vertex of G , then the corresponding vertices v_i , $i = 1, 2, \dots, m$, in the copies G_i in mG will receive labels

$$\begin{aligned} & \{g(v_i) : g(v_i) = m(r-1) + i, i = 1, 2, \dots, m\} \\ &= \{m(r-1) + 1, m(r-1) + 2, \dots, mr\}. \end{aligned} \quad (8)$$

Case 2. If the number r is assigned by the labeling f to an edge of G , then the corresponding edges $u_i v_i$, $i = 1, 2, \dots, m$, in the copies G_i in mG will have labels

$$\begin{aligned} & \{g(u_i v_i) : g(u_i v_i) = mr + 1 - i, i = 1, 2, \dots, m\} \\ &= \{mr, mr-1, \dots, m(r-1) + 1\}. \end{aligned} \quad (9)$$

We can see that the vertex labels and edge labels in mG are not overlapping, and the maximum used label is $m(p+q)$. Thus g is a total labeling.

Analogously as in the proof of Theorem 1 we get that the weight of every subgraph T_i^j , $i = 1, 2, \dots, m$, $j = 1, 2, \dots, t$, under the labeling g attains the value

$$\begin{aligned} wt_g(T_i^j) &= \sum_{v \in V(T_i^j)} g(v) + \sum_{uv \in E(T_i^j)} g(uv) \\ &= \sum_{v \in V(T_i^j)} (m(f(v)-1) + i) \\ & \quad + \sum_{uv \in E(T_i^j)} (mf(uv) + 1 - i) \\ &= mwt_f(T_i^j) + k(1-m) - 1 + i \end{aligned} \quad (10)$$

and the set of all T -weights in mG successively attain consecutive values $m(a-k)+k, m(a-k)+k+1, \dots, m(a-k+t)+k-1$. Thus the resulting labeling g is a $(b, 1)$ - T -antimagic total labeling. $\square$

3. Disjoint Union of Certain Families of Graphs

In [18], the following results are proved on path-antimagicness of cycles and paths.

Proposition 4 (see [18]). Let k and $n \geq 3$ be positive integers. The cycle C_n is a super $(a, 1)$ - P_k -antimagic for every $k = 2, 3, \dots, n$.

Proposition 5 (see [18]). Let k and $n \geq 3$ be positive integers. The path P_n is a super $(a, 1)$ - P_k -antimagic for every $k = 2, 3, \dots, n$.

In light of Propositions 4 and 5 and Corollary 2 we immediately obtain the following.

Corollary 6. Let $k, m \geq 1$, and $n \geq 3$ be positive integers. Then m copy of a cycle C_n , that is, the graph mC_n , is a super $(a, 1)$ - P_k -antimagic for every $k = 2, 3, \dots, n$.

Corollary 7. Let $k, m \geq 1$, and $n \geq 3$ be positive integers. Then m copy of a path P_n , that is, the graph mP_n , is a super $(a, 1)$ - P_k -antimagic for every $k = 2, 3, \dots, n$.

4. Conclusion

In this paper we have shown that the disjoint union of multiple copies of a (super) $(a, 1)$ -tree-antimagic graph is also a (super) $(b, 1)$ -tree-antimagic. It is a natural question whether the similar result holds also for other differences and other H -antimagic graphs. For further investigation we propose the following open problems.

Open Problem 1. Let G be a (super) (a, d) - T -antimagic graph, where T is a tree different from K_2 . For the graph mG , determine if there is a (super) (a, d) - T -antimagic total labeling, for $d \neq 1$ and all $m \geq 1$.

Open Problem 2. Let G be a (super) (a, d) - H -antimagic graph. For the graph mG , determine if there is a (super) (a, d) - H -antimagic total labeling, for certain values of d and all $m \geq 1$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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