A Posteriori Error Estimate for the Finite Volume Element Method of the Second-Order Hyperbolic Equations

Chuanjun Chen,1 Xin Zhao,2 and Yuanyuan Zhang1

1School of Mathematics and Information Sciences, Yantai University, Yantai 264005, China
2School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China

Correspondence should be addressed to Chuanjun Chen; cjchen@ytu.edu.cn

Received 6 September 2015; Accepted 18 November 2015

Abstract
We establish a posteriori error estimate for the finite volume element method of the second-order hyperbolic equation. Residual-type a posteriori error estimator is derived. The computable upper and lower bounds on the error in the $H^1$-norm are established.

1. Introduction

The finite volume element method is a class of important numerical tools for solving partial differential equations. Due to the local conservation property and some other attractive properties, it is wildly used in many engineering fields, such as heat and mass transfer, fluid mechanics, and petroleum engineering, especially for those arising from conservation laws including mass, momentum, and energy. For the second-order hyperbolic equations, Li et al. [1] have proved the optimal order of convergence in $H^1$-norm. In [2], Kumar et al. have proved optimal order of convergence in $L^2$ and $H^1$-norm for the semidiscrete scheme and quasi-optimal order of convergence in maximum norm.

Since the pioneering work of Babuška and Rheinboldt [3], the adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. Adaptive algorithm is among the most important means to boost accuracy and efficiency of the finite element discretization. The main idea of adaptive algorithm is to use the error indicator as a guide which shows whether further refinement of meshes is necessary. A computable a posteriori error estimator plays a crucial role in an adaptive procedure. A posteriori error analysis for the finite volume element method has been studied in [4–6] for the second-order elliptic problem, in [7–9] for the convection-diffusion equations, in [10] for the parabolic problems, in [11] for a model distributed optimal problem governed by linear parabolic equations, in [12] for the Stokes problem in two dimensions, and in [13] for the second-order hyperbolic equations.

However, to the best of our knowledge, there are few works related to the a posteriori error estimates of the finite volume element method for the second-order hyperbolic problems. The aim of this paper is to establish residual-type a posteriori error estimator of the finite volume element method for the second-order hyperbolic equation. We first construct a computable a posteriori error estimator of the finite volume element method. Then we analyze the residual-type a posteriori error estimates and obtain the computable upper and lower bounds on the error in the $H^1$-norm.

The organization of this paper is stated as follows. In Section 2, we present the framework of the finite volume element method for the second-order hyperbolic equation. In Section 3, we establish the residual-type a posteriori error estimator of the finite volume element method and derive the upper and lower bounds on the error in the $H^1$-norm. We provide some numerical experiments to illustrate the performance of the error estimator in Section 4.

2. Finite Volume Element Formulation

We use standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with the norm $\|u\|_{s,p,\Omega}$ [14]. In order to simplify the notation, we
denote $W^{2,2}(\Omega)$ by $H^2(\Omega)$ and omit the index $p = 2$ and $\Omega$ whenever possible.

In this paper, we consider the following second-order hyperbolic problem:

$$u_{tt} - \nabla \cdot (a(x) \nabla u) = f(x,t), \quad \text{in } \Omega \times (0,T],$$

$$u(x,t) = 0, \quad \text{on } \partial \Omega \times (0,T],$$

$$u(x,0) = u_0(x),$$

$$u_t(x,0) = v_0(x),$$

in $\Omega$,

where $\Omega \subset \mathbb{R}^2$ is a polygonal bounded cross section, possessed with a Lipschitz boundary $\partial \Omega$. For simplicity, the right-hand side $f$ is assumed to be measurable and square-integrable on $\Omega \times (0,T]$ and to be continuous with respect to time. The initial datum $u_0$ and $v_0$ are assumed to be measurable and square-integrable on $\Omega$. $a(x,t) = (a_{ij}(x,t))_{i,j=1}^2$ is a real-valued smooth matrix function, uniformly symmetric, and positive definite in $\Omega$.

The corresponding variational problem is to find $u \in H^1_0(\Omega)$, for $t > 0$, satisfying

$$(u_t, v) + a(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega),$$

where the bilinear form $a(\cdot, \cdot)$ is defined by

$$a(u, v) = \int_{\Omega} a(x) \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1_0(\Omega).$$

Denote by $T_h$ the primal quasi-uniform triangulation of $\Omega$ with $h = \max h_K$, where $h_K$ is the diameter of the triangle $K \in T_h$. Let $\mathcal{U}_h$ be the standard conforming finite element space of piecewise linear functions, defined on the triangulation $T_h$:

$$\mathcal{U}_h = \left\{ u \in C\left(\overline{\Omega}\right) : u|_K \text{ is linear and } u|_{\partial \Omega} = 0, \forall K \in T_h \right\}.$$  

Denote by $T_h^*$ the dual partition which is constructed in the same way as in [1, 15]. Let $z_K$ be the barycenter of $K$. We connect $z_K$ with the midpoints of the edges of $K$ by a straight line, thus partitioning $K$ into three quadrilaterals $K_z$, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of $K$. Then with each vertex $z \in Z_h = \cup_{K \in T_h} Z_h(K)$, we associate a control volume $V_z$, which consists of the union of the subregions $K_z$ sharing the vertex $z$ (see Figure 1).

The partition $T_h^*$ is regular or quasi-uniform, if there exists a positive constant $C > 0$ such that

$$C^{-1} h^2 \leq \text{meas}(V_z) \leq C h^2, \quad \forall V_z \in T_h^*.$$ (5)

The dual partition $T_h^*$ will also be quasi-uniform [5] if the finite element triangulation $T_h$ is quasi-uniform. The test function space $\mathcal{V}_h$ is defined by

$$\mathcal{V}_h = \left\{ v \in L^2(\Omega) : v|_{V_z} \text{ is constant and } v|_{\partial \Omega} = 0 \forall V_z \in T_h^* \right\}.$$ (6)

For any $u_h \in \mathcal{U}_h$, we define an interpolation operator $\Pi_h : \mathcal{U}_h \to \mathcal{V}_h$, such that

$$\Pi_h u_h = \sum_{z \in Z_h^*} u_h(z) \psi_z,$$ (7)

where $\psi_z$ is the characteristic function of the control volume $V_z$.

According to [16], for each $u_h \in \mathcal{U}_h$, there exists a positive constant $C$ independent of $h$, such that $\Pi_h$ satisfies the following inequality:

$$\|u_h - \Pi_h u_h\|_{0,K} \leq C h_K \|u_h\|_{1,K}, \quad \forall K \in T_h.$$ (8)

Introduce the following adjoint elliptic problem:

$$-\nabla \cdot (a(x) \nabla u) = f \quad \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \partial \Omega.$$ (9)

Denote by $\mathcal{T} : L^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)$ the solution operator of this problem, so that

$$a(\mathcal{T} f, \varphi) = (f, \varphi), \quad \forall \varphi \in H^1_0(\Omega).$$ (10)
Define negative norms by
\[
\|v\|_{-s} = \sup \left\{ \frac{(v, \varphi)}{\|\varphi\|_s} ; \varphi \in H^s(\Omega) \right\},
\]
for \( s \geq 0 \) integer.

In fact, by Cauchy-Schwarz inequality, we obtain
\[
\frac{(v, \varphi)}{\|\varphi\|_s} \leq \|v\| \leq \|v\|_s, \text{ for } s \geq 0.
\]

For our error analysis in the next section, it will be convenient to introduce such a norm defined by
\[
\|v\|_{-s} = (\mathcal{F}^s v, v)^{1/2}, \text{ for } s \geq 0 \text{ integer.}
\]

According to Thomée [17], we have the following lemma.

**Lemma 1.** The norm \(\|v\|_{-s}\) is equivalent to \(\|v\|_s\) and \(\|\mathcal{F} f, g\|\) = \((f, \mathcal{F} g)\), where \(s\) is a nonnegative integer. Particularly, \(\|\mathcal{F} f\|_1\) is equivalent to \(\|v\|_{-1}\) when \(s = 1\).

In order to get the fully discrete finite volume element method of (1), we give a partition of the time interval \([0, T]\) as \(0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\). Let \(\tau_n = t_n - t_{n-1}\), \(\tau = \max_{1 \leq n \leq N} \tau_n\), \(U_h^n = U_h(t_n)\), and \(U_h^{n+1/2} = (U_h^n + U_h^{n-1})/2\).

With the help of \(P_h\), we obtain the fully discrete finite volume element method of (1): to find \(U_h^n \in \mathcal{V}_h\), for \(1 \leq n \leq N\), such that
\[
\left( \partial_t U_h^n, \Pi_h \chi \right) + a(U_h^{n+1/2}, \Pi_h \chi) = (f^n, \Pi_h \chi),
\]
\[
\forall \chi \in \mathcal{V}_h,
\]
\[
U_h^0 = u_0,
\]
\[
\partial_t U_h^1 = v_0,
\]
where
\[
\frac{\partial_t U_h^n - \partial_t U_h^{n-1}}{\tau_n} = \left( \frac{U_h^{n+1} - U_h^n}{\tau_n} \right) / \tau_n.
\]

By setting \(v = \partial u/\partial t = u_t\) and \(\mathcal{F} = (u_v)\), the notation \(\nabla \cdot (a(x) \nabla \varphi) = \nabla \cdot (a(x) \nabla \varphi)\), (1) can equivalently be written as
\[
\mathcal{F}_t - \left( \nabla \cdot (a(x) \nabla) \right) 1 = F,
\]
where \(F = (0, 1)\).

Let \(V_h^n = \mathcal{F} U_h^n\); we define
\[
U_t = \frac{t - t_{n-1}}{\tau_n} U_h^n + (1 - \frac{t - t_{n-1}}{\tau_n}) U_h^{n-1}, \quad 1 \leq n \leq N,
\]
\[
V_t = \frac{t - t_{n-1}}{\tau_n} V_h^n + (1 - \frac{t - t_{n-1}}{\tau_n}) V_h^{n-1}, \quad 1 \leq n \leq N.
\]

The residual system, with \(Y_t = (U_t, V_t)\), is defined as follows:
\[
(Y - Y_t) - \left( \nabla \cdot (a(x) \nabla) \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = P_n(y - Y_t) \quad \text{in } \Omega \times [0, T],
\]
\[
u - U_t = 0 \quad \text{on } \partial \Omega \times [0, T],
\]
\[
(Y - Y_t) (\cdot, 0) = 0 \quad \text{in } \Omega,
\]
where the quantities \(P_n\) in \(L^1(0, T; L^2(\Omega))\) and \(P_n\) in \(L^1(0, T; H^{-1}(\Omega))\) are affine functions on each interval \([t_{n-1}, t_n]\), \(1 \leq n \leq N\), that
\[
P_n(\cdot, t) = \left\{ \begin{array}{ll}
V_t - V_t^{n-1}, & 2 \leq n \leq N,
0, & n = 1.
\end{array} \right.
\]

And the quantities \(P_n\) are defined as follows.

From the fully discrete algorithm (14), for any \(\varphi \in H^1_0(\Omega), \psi \in \mathcal{V}_h\), we have
\[
\left( \partial_t U_h^n, \varphi \right) + a(U_h^{n+1/2}, \Pi_h \varphi)
\]
\[
= - \left( f^n - \mathcal{F} U_h^n, \varphi - \Pi_h \varphi \right) + (f^n, \varphi)
\]
\[
+ a(U_h^{n+1/2}, \varphi) - a(U_h^{n+1/2}, \Pi_h \varphi).
\]

Since \((V_t), \partial_t U_h^n\), by (2) and (20), for \(t \in (t^{n-1}, t^n]\), we get
\[
\left( (v - V_t), \varphi \right) + a(u - U_h^{n+1/2}, \varphi)
\]
\[
= \left( f^n - \mathcal{F} U_h^n, \varphi - \Pi_h \varphi \right) + (f - f^n, \varphi)
\]
\[
- a(U_h^{n+1/2}, \varphi) + a(U_h^{n+1/2}, \Pi_h \varphi).
\]

Adding the term \(a(U_h^{n+1/2} - U_t, \varphi)\) into the two hand sides of (21), we get
\[
\left( (v - V_t), \varphi \right) + a(u - U_t, \varphi)
\]
\[
= \left( f^n - \mathcal{F} U_h^n, \varphi - \Pi_h \varphi \right) + (f - f^n, \varphi)
\]
\[
- a(U_h^{n+1/2} - U_t, \varphi).
\]

So on each interval \([t_{n-1}, t_n]\) \((2 \leq n \leq N)\), we have
\[
(P_n \varphi) = \left( f^n - \mathcal{F} U_h^n, \varphi - \Pi_h \varphi \right)
\]
\[
- a(U_h^{n+1/2} - U_t, \varphi)
\]
\[
+ a(U_h^{n+1/2} - U_t, \varphi) + (f - f^n, \varphi),
\]
\[
\text{for } \varphi \in H^1_0(\Omega), \psi \in \mathcal{V}_h.
\]
We define
\[
(L^n, \phi) = \left( f^n - \partial_t U^n_h, \phi - \Pi_h \phi \right)
- \left[ a(U^n_h, \phi) - a(U^n_h, \Pi_h \phi) \right].
\] (24)

Then the term \( P_v \) on the interval \([t_{n-1}, t_n] (2 \leq n \leq N)\) can be written as
\[
(P_v, \phi) = (L^n, \phi) + a(U^n_{1/2} - U_t, \phi) + (f - f^n, \phi),
\] \( \forall \phi \in H^1_0(\Omega), \ \nu \in \mathcal{H}_h. \)

When \( t \in [0, t_1] \),
\[
P_v(\cdot, t) = f(\cdot, t) + \nabla \cdot (a(x) \nabla (u_0 + t v_0)).
\] (26)

3. Residual-Type A Posteriori Error Estimates

In this section, we will construct the residual-type a posteriori error estimates of the finite volume element method for (1). We introduce the jump of a vector-valued function across the edge \( E \in \mathcal{E}_h \) which will be used in the residual-type a posteriori error estimates. Let \( E \) be an interior edge shared by elements \( K \) and \( K_\gamma \). Define the unit normal vectors \( \mathbf{n}_{K_\gamma} \) and \( \mathbf{n}_K \) on \( E \) pointing exterior to \( K_\gamma \) and \( K, \) respectively. Let \( \nu \) be a vector-valued function that is smooth inside each of the elements \( K_\gamma \) and \( K, \) \( \nu^+ \) and \( \nu^- \) denote the traces of \( \nu \) on \( E \) taken from within the interior of \( K_\gamma \) and \( K, \) respectively. Then the jump of \( \nu \) on the edge \( E \) is defined by \( [\nu]_E = \nu^+ - \nu^- \). We denote space refinement indicator by \( \eta^n_\nu \) defined by
\[
\mathcal{R}^n_{E} = f^n - \partial_t U^n_h + \nabla \cdot (a(x) \nabla U^n_{1/2}^h),
\]
\[
\mathcal{R}^n_{K} = - \left[ a(x) \nabla U^n_{1/2}^h \right]_E,
\]
\[
\eta^n_{\nu} = \left( \sum_{K \in \mathcal{E}_h} h^2_K \mathcal{R}^2_{K,0,E} + \sum_{E \in \mathcal{E}_h} h^2_E \mathcal{R}^2_{E,0,E} \right)^{1/2}.
\] (27)

We define time refinement indicator \( \eta^n_{t} \) as
\[
\eta^n_{t} = \tau_n \left\| U^n_h - U^{n-1}_{h} \right\|_1 + \tau_n \left\| V^n_h - V^{n-1}_{h} \right\|_1.
\] (28)

3.1. Upper Bound. The Scott-Zhang interpolation function \( \mathcal{I}_h : H^1_0(\Omega) \rightarrow \mathcal{V}_h \) is introduced in the following lemma [18].

Lemma 2. For each \( \phi \in H^1_0(\Omega) \), a positive constant \( C \) is independent of \( h_K \) and \( h_E \) such that, for any \( K \in \mathcal{T}_h, E \in \mathcal{E}_h \),
\[
\left\| \mathcal{I}_h \phi \right\|_{1,\Omega} \leq C \left\| \phi \right\|_{1,\Omega},
\]
\[
\left\| \phi - \mathcal{I}_h \phi \right\|_{0,\Omega} \leq C h_K \left\| \phi \right\|_{1,\omega_K},
\]
\[
\left\| \phi - \mathcal{I}_h \phi \right\|_{0,\Omega} \leq C h^2_K \left\| \phi \right\|_{1,\omega_K},
\]
where \( \omega_K = \bigcup_{K \cap E \neq \emptyset} K' \) and \( \omega_E = \bigcup_{K \cap E \neq \emptyset} K. \)

We also introduce the trace theorem [14].

Lemma 3 (trace theorem). There exists a positive constant \( C \) independent of \( h_E \) such that
\[
\left\| \phi \right\|_{0,E}^2 \leq C \left[ (h_E^{-1} \left\| \phi \right\|_{0,K} + h_E \left\| \nabla \phi \right\|_{0,K}) \right],
\] (30)

\[ \forall \phi \in H^1(K), \ E \in \partial K, \ \forall K \in \mathcal{T}_h. \]

Then we can get the following theorem for the upper bound of the error.

Theorem 4. The following a posteriori error estimate holds between the solution \( u \) of (1) and the solution \( \hat{U}_h^{n} \) of (14), for \( 2 \leq m \leq N \):
\[
\left\| u - \hat{U}_h^{n} \right\| + \sum_{k=1}^{m} \left\| \int_{t_{k-1}}^{t_k} (U_{\tau} - u) \right\| dt\]
\[ \leq C \sum_{n=2}^{m} (\tau_n (\eta^n_{\nu} + \eta^n_{t})) + C \sum_{n=2}^{m} \int_{t_{n-1}}^{t_n} \left\| f(\cdot, t) - f^n \right\| dt \]
\[ + C \int_{0}^{t_1} \left\| f(\cdot, t) + \nabla \cdot (a(x) \nabla (u_0 + t v_0)) \right\| dt. \] (31)

Proof. Taking the inner product of (18) with \( \left( u^{t_{k-1}}, \mathcal{I}(\nu - \nu^+) \right) \) and setting
\[
Z(t) = \left( \left\| u - U_{\tau} \right\|^2 + \left\| \nu - V_{\tau} \right\|_{1,1}^2 \right)^{1/2},
\] (32)
we obtain, for \( t \in [t_{n-1}, t_n] \),
\[
\frac{1}{2} \frac{dZ^2}{dt} = (P_u, u - U_\tau) + (P_v, \mathcal{I}(\nu - \nu^+)) \leq \left\| P_u \right\| \left\| u - U_\tau \right\| + \left\| \nabla \cdot (a(x) \nabla (U_{h_{n-1}^{1/2} - U_{\tau})) \right\|_1
\]
\[ \cdot \left( \tau_n (\eta^n_{\nu} + \eta^n_{t}) + C \sum_{n=2}^{m} \int_{t_{n-1}}^{t_n} \left\| f(\cdot, t) - f^n \right\| dt \right)
\]
\[ - f^n \left\| \mathcal{I}(\nu - \nu^+) \right\| \leq \left\| P_u \right\| \left\| u - U_\tau \right\| + C \left\| \nabla \cdot (a(x) \nabla (U_{h_{n-1}^{1/2} - U_{\tau}) \right\|_1
\]
\[ \cdot \left\| \nu - V_{\tau} \right\|_{1,1} + C \left\| f(\cdot, t) - f^n \right\| \left\| \nu - V_{\tau} \right\|_{1,1}
\]
\[ \leq C \left( \left\| P_u \right\|^2 + \left\| L^n \right\|_1^2 + \left\| f(\cdot, t) - f^n \right\|^2 \right.
\]
\[ + \left. \left\| \nabla \cdot (a(x) \nabla (U_{h_{n-1}^{1/2} - U_{\tau}) \right\|_{1,1}^2 \right)^{1/2} Z, \right\}
\] (33)

hence,
\[
\frac{dZ^2}{dt} \leq C \left( \left\| P_u \right\|^2 + \left\| L^n \right\|_1^2 + \left\| f(\cdot, t) - f^n \right\|^2 \right.
\]
\[ + \left. \left\| \nabla \cdot (a(x) \nabla (U_{h_{n-1}^{1/2} - U_{\tau}) \right\|_{1,1}^2 \right)^{1/2} \leq C \left( \left\| P_u \right\|^2 + \left\| L^n \right\|_1 + \left\| f(\cdot, t) - f^n \right\| \right. \right. \]
\[ + \left. \left. \left\| \nabla \cdot (a(x) \nabla (U_{h_{n-1}^{1/2} - U_{\tau}) \right\|_{1,1} \right). \right) \] (34)
Integrating the inequality from $t_{n-1}$ to $t_n$ ($2 \leq n \leq N$), we have

$$Z(t_n) - Z(t_{n-1}) \leq C \int_{t_{n-1}}^{t_n} \left[ \left\| P_n \right\| + \left\| L^n \right\|_1 + \left\| f(\cdot, t) - f^n \right\| + \left\| \nabla \cdot (a(x) \nabla (t^{n,1/2} - U_T)) \right\|_1 \right] dt.$$

Using Lemma 1, we obtain

$$\int_{t_{n-1}}^{t_n} \left\| \nabla \cdot (a(x) \nabla (U^{n,1/2}_h - U_T)) \right\|_1 dt \leq C \int_{t_{n-1}}^{t_n} \left[ 1 - \frac{t - t_{n-1}}{\tau_n} \right] dt$$

$$+ C \frac{\tau_n}{2} \left\| t^{n-1}_h - U_h^n \right\|_1 + C \frac{\tau_n}{2} \left\| U_h^n - U_h^{n-1} \right\|_1$$

$$\leq C \tau_n \left\| U_h^{n-1} - U_h^n \right\|_1 + C \tau_n \left\| U_h^n - U_{h-1} \right\|_1,$$

$$\int_{t_{n-1}}^{t_n} \left\| P_n(\cdot, t) \right\| dt = \left\| V_n - V_{h-1} \right\|_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\tau_n} dt$$

$$= \frac{\tau_n}{2} \left\| V_n - V_{h-1} \right\|_1.$$ 

By the definition of $\eta^n$, we get

$$Z(t_n) - Z(t_{n-1})$$

$$\leq C \left[ \tau_n \eta^n + \tau_n \left\| L^n \right\|_1 + \int_{t_{n-1}}^{t_n} \left\| f(\cdot, t) - f^n \right\| dt \right].$$

In order to estimate $\left\| L^n \right\|_1$, we choose $v = \mathcal{F}_h \varphi$ in (24); then

$$(L^n, \varphi) = (f^n - \partial_t \bar{\mathcal{F}}_h U^n_h, \varphi - \Pi_n \varphi)$$

$$- \left[ a \left( U^{n,1/2}_h, \varphi \right) - a \left( U_h^{n,1/2}, \Pi_n \varphi \right) \right]$$

$$= \left( f^n - \partial_t \bar{\mathcal{F}}_h U^n_h, \varphi - \nu \right)$$

$$+ \left( f^n - \partial_t \bar{\mathcal{F}}_h U^n_h, \varphi - \Pi_n \varphi \right)$$

$$+ \left( f^n - \partial_t \bar{\mathcal{F}}_h U^n_h, - \Pi_n \varphi \right)$$

$$- \left[ a \left( U^{n,1/2}_h, \varphi \right) - a \left( U_h^{n,1/2}, \nu \right) \right]$$

$$- \left[ a \left( U^{n,1/2}_h, \varphi \right) - a \left( U_h^{n,1/2}, \varphi \right) \right]$$

$$= \mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4.$$

Using Greens formula, we have

$$\mathcal{F}_3 = - \left( a(x) \nabla U^{n,1/2}_h, \nabla (\varphi - \nu) \right)$$

$$= - \sum_{K \in T_h} \left( a(x) \nabla U_h^{n,1/2}, \varphi - \nu \right)_{0,K}$$

$$= - \sum_{E \in \partial E_h} \left( a(x) \nabla U_h^{n,1/2} \right)_E, \varphi - \nu \right)_{0,E}.$$

By the definition of $\mathcal{K}_h^0$, $\mathcal{R}_E$, we get

$$\mathcal{F}_1 + \mathcal{F}_3$$

$$= \sum_{K \in T_h} \left( f^n - \partial_t \bar{\mathcal{F}}_h U^n_h + \nabla \cdot (a(x) \nabla U_h^{n,1/2} \cdot \varphi - \nu) \right)_{0,K}$$

$$- \sum_{E \in \partial E_h} \left( \left[ a(x) \nabla U_h^{n,1/2} \right]_E, \varphi - \nu \right)_{0,E}$$

From Cauchy-Schwarz inequality and Lemma 2, we can get

$$\left| \mathcal{F}_1 + \mathcal{F}_3 \right| \leq C \sum_{K \in T_h} \left\{ h_K \left\| \mathcal{K}_h^0 \right\|_{0,K} \left\| \varphi \right\|_{1,E_h} \right\}$$

$$+ C \sum_{E \in \partial E_h} \left\{ h_E^{1/2} \left\| \mathcal{R}_E \right\|_{0,E} \left\| \varphi \right\|_{1,E_h} \right\}.$$

For $\mathcal{F}_4$, since $\Pi_h \nu$ is a constant in $K \cap K^*_z$, $z \in Z_h(K)$, $K^*_z \in T_h$, we have

$$\int_K a(x) \nabla U_h^{n,1/2} \cdot \nabla \nu d x$$

$$= \sum_{z \in Z_h(K)} \int_{K \cap K^*_z} a(x) \nabla U_h^{n,1/2} \cdot \nabla (\nu - \Pi_h \nu) d x$$

$$= - \sum_{z \in Z_h(K)} \int_{K \cap K^*_z} \nabla \cdot (a(x) \nabla U_h^{n,1/2}) \cdot (\nu - \Pi_h \nu) d x$$

$$+ \sum_{z \in Z_h(K)} \int_{\partial (K \cap K^*_z)} a(x) \nabla U_h^{n,1/2} \cdot \nu (\nu - \Pi_h \nu) d s.$$

$$= - \int_K \nabla \cdot (a(x) \nabla U_h^{n,1/2}) \cdot (\nu - \Pi_h \nu) d x$$

$$+ \int_{\partial K} a(x) \nabla U_h^{n,1/2} \cdot \nu (\nu - \Pi_h \nu) d s$$

$$+ \sum_{z \in Z_h(K)} \int_{\partial (K \cap K^*_z)} a(x) \nabla U_h^{n,1/2} \cdot \nu (\nu - \Pi_h \nu) d s.$$
Since $a(x)v U^n_h$ and $v$ are continuous inside each element $K \in T_h$, we have
\[
\sum_{z \in Z_0(K)} \int_{K \setminus \partial K} a(x) \nabla U^n_h \cdot \nu ds = 0,
\]
\[
\sum_{z \in Z_0(K)} \int_{K \setminus \partial K} a(x) \nabla U^{1/2}_h \cdot \nu ds = 0.
\]

Thus,
\[
\mathcal{J}_4 = \sum_{K \in T_h} \left( \nabla \cdot (a(x) \nabla U^{1/2}_h), v - \Pi_h v \right)_{0,K}
- \sum_{E \in E_h} \left( [a(x) \nabla U^{1/2}_h]_E, v - \Pi_h v \right)_{0,E}.
\]

Then we get
\[
\mathcal{J}_3 + \mathcal{J}_4 = \sum_{K \in T_h} \left( f^n - \partial_t \tilde{u}^n_h + \nabla \cdot (a(x) \nabla U^{1/2}_h), v - \Pi_h v \right)_{0,K}
- \sum_{E \in E_h} \left( [a(x) \nabla U^{1/2}_h]_E, v - \Pi_h v \right)_{0,E}.
\]

By (8) and Cauchy-Schwarz inequality, we obtain
\[
\left| \sum_{K \in T_h} \left( \mathcal{R}_K, v - \Pi_h v \right)_{0,K} \right|
\leq C \sum_{K \in T_h} \{ h_K \| \mathcal{R}_K \|_{0,K} \| v \|_{1,K} \}
\leq C \left( \sum_{K \in T_h} h_k^2 \| \mathcal{R}_K \|_{0,K}^2 \right)^{1/2} \left( \sum_{K \in T_h} \| v \|_{1,K}^2 \right)^{1/2}
= C \left( \sum_{K \in T_h} h_k^2 \| \mathcal{R}_K \|_{0,K}^2 \right)^{1/2} \| \mathcal{J}_4 \|_{1, \Omega}
\leq C \left( \sum_{K \in T_h} h_k^2 \| \mathcal{R}_K \|_{0,K}^2 \right)^{1/2} \| \mathcal{J}_4 \|_{1, \Omega}.
\]

Since $\Pi_h v$ is a piecewise constant function, by Lemma 3 and (8), we get
\[
\sum_{E \in E_h} |h_E^{-1/2} v - \Pi_h v|_{0,E}^2
\leq C \sum_{E \in E_h} \left( h_E^{-1/2} \| v - \Pi_h v \|_{0,K}^2 + \| v^2 \|_{1,K}^2 \right) \leq C \| v \|_1^2
\]
\[
\leq C \| \mathcal{J}_4 \|_{1}^2.
\]

Substituting the estimate of $\mathcal{J}_1$ into (38) and by the definition of $\eta^n_h$, we have
\[
(L^n, \varphi) \leq C \eta^n_h \| \varphi \|_1 ;
\]

hence
\[
\| L^n \|_{-1} \leq C \eta^n_h.
\]

Substituting the estimation of $\| L^n \|_{-1}$ into (37), we get
\[
Z(t_m) - Z(t_{n-1}) \leq C \left( \sum_{n=2}^{m} \left( \eta^n_h \right) \right)
+ C \sum_{n=2}^{m} \left( t^n \right) \| f (\cdot, t) - f^m \| dt.
\]

For $n = 1$, we have
\[
Z(t_1) - Z(t_0) \leq C \int_0^{t_1} \| f (\cdot, t) + \nabla \cdot (a(x) \nabla (u_0 + v_0)) \| dt.
\]

Noting that $Z(t_0) = Z(0) = 0$, then
\[
Z(t_m) \leq C \sum_{n=2}^{m} \left( \eta^n_h \right) + C \sum_{n=2}^{m} \left( t^n \right) \| f (\cdot, t) - f^m \| dt
+ C \int_0^{t_1} \| f (\cdot, t) + \nabla \cdot (a(x) \nabla (u_0 + v_0)) \| dt.
\]

By the fact that $(1/ \sqrt{2})(a+b) \leq \sqrt{a^2 + b^2} \leq a + b (a, b > 0)$, we have
\[
|u^m - \tilde{u}^n_h| + |v^m - \tilde{v}^n_h| \leq C \sum_{n=2}^{m} \left( \eta^n_h \right) + C \sum_{n=2}^{m} \left( t^n \right) \| f (\cdot, t) - f^m \| dt
+ C \int_0^{t_1} \| f (\cdot, t) + \nabla \cdot (a(x) \nabla (u_0 + v_0)) \| dt.
\]
In view of the definition of the operator $\mathcal{T}$, we have

$$
\mathcal{T} \frac{\partial V}{\partial t} + u = \mathcal{T} f(\cdot, t), \quad (55)
$$

Subtracting (56) from (55), we get

$$
\mathcal{T} \frac{\partial (V - V_e)}{\partial t} + \left( u - U_h^{m+1/2} \right) = \mathcal{T} \left( f(\cdot, t) - f(\cdot, t^m) \right),
$$

Integrating (58) from $k=1$ to $k=m$, we obtain

$$
\mathcal{T} \left( V^m - V_h^m \right) - \mathcal{T} \left( V^{m-1} - V_h^{m-1} \right) = \int_{t_{m-1}}^{t_m} \mathcal{T} \left( f(\cdot, t^m) - f(\cdot, t) \right) dt + \int_{t_{m-1}}^{t_m} \left( U_r - U_h^{m+1/2} \right) dt = \int_{t_{m-1}}^{t_m} \left( U_r - u \right) dt.
$$

Summing (59) from $k=1$ to $k=m$, we obtain

$$
\sum_{k=1}^{k=m} \int_{t_{k-1}}^{t_k} \left( U_r - u \right) dt = \mathcal{T} \left( V^m - V_h^m \right)
= \sum_{k=1}^{k=m} \int_{t_{k-1}}^{t_k} \mathcal{T} \left( f(\cdot, \tau^k) - f(\cdot, \tau) \right) dt + \sum_{k=1}^{k=m} \int_{t_{k-1}}^{t_k} \left( U_r - U_h^{k+1/2} \right) dt.
$$

Thus, we have

$$
\left\| \sum_{k=1}^{k=m} \int_{t_{k-1}}^{t_k} \left( U_r - u \right) dt \right\|_1 \leq \left\| \mathcal{T} \left( V^m - V_h^m \right) \right\|_1 + \left\| \sum_{k=1}^{k=m} \int_{t_{k-1}}^{t_k} \mathcal{T} \left( f(\cdot, \tau^k) - f(\cdot, \tau) \right) dt \right\|_1 + \left\| \sum_{k=1}^{k=m} \int_{t_{k-1}}^{t_k} \left( U_r - U_h^{k+1/2} \right) dt \right\|_1.
$$

3.2. Lower Bound. In order to derive the local lower bounds on the error, we will introduce some properties of the bubble functions. For each triangle $K \in T_h$, denote by $\lambda_{K,1}, \lambda_{K,2}, \lambda_{K,3}$ the barycentric coordinates. Define the element-bubble function $\psi_K$ by

$$
\psi_K = 27 \lambda_{K,1} \lambda_{K,2} \lambda_{K,3}, \quad \text{in } K;
$$

$$
\psi_K = 0, \quad \text{in } \Omega \setminus K.
$$

Assume that $K$ and $K'$ share the edge $E \in \mathcal{E}_h$. Let the barycentric coordinates with respect to the end points of $E$ be $\lambda_{E,1}$ and $\lambda_{E,2}$. Define the edge-bubble function $\psi_E$ by

$$
\psi_E = 4 \lambda_{E,1} \lambda_{E,2}, \quad \text{in } \omega_E = K \cup K';
$$

$$
\psi_E = 0, \quad \text{in } \Omega \setminus \omega_E.
$$

For properties of the bubble functions, we have the following lemma [19].
Lemma 5. For each of the elements $K \in T_h$ and $E \in T_E$, functions $\psi_K$ and $\psi_E$ have the following properties:

\[
\begin{align*}
\text{supp } \psi_K & \subset K, \\
\max_{x \in \text{supp } \psi_K} \psi_K & = 1, \\
\int_K \psi_K dx & = \frac{9}{20} |K| \sim h_K^2, \\
\|\nabla \psi_K\|_{0,K} & \leq C h_K^{-1} \|\psi_K\|_{0,K}, \\
\psi_K & \in [0,1],
\end{align*}
\]

By triangle inequality, we have

\[
\|\nabla \psi_K\|_{0,K} \leq \|\nabla \psi_K\|_{0,K} + \|\nabla \psi_K - \nabla \psi_K\|_{0,K}.
\]

\[
\begin{align*}
\sup \psi_E & \subset \omega_E, \\
\max_{x \in \omega_E} \psi_E & = 1, \\
\int_E \psi_E ds & = \frac{2}{3} h_E, \\
\int_{\omega_E} \psi_E dx & = \frac{1}{3} |\omega_E| \sim h_E^2, \\
\|\nabla \psi_E\|_{0,\omega_E} & \leq C h_E^{-1} \|\psi_E\|_{0,\omega_E}, \\
\psi_E & \in [0,1].
\end{align*}
\]

We define the average of $R^n_K$ on $K$ ($\overline{R^n_K}$) and the average of $R^n_E$ on $E$ ($\overline{R^n_E}$) by

\[
\overline{R^n_K} = \frac{1}{|K|} \int_K R^n_K dx, \\
\overline{R^n_E} = \frac{1}{h_E} \int_E R^n_E ds.
\]

Then we have the following local lower bounds.

Theorem 6. For any $K \in T_h$, $E \in T_E$, the following local posteriori lower bounds on the error $u^n - U^n_h$ hold for a positive constant $C$ independent of $h_K$ and $h_E$:

\[
\begin{align*}
& h_K \|R^n_K\|_{0,K} \leq C \left( \|u^n - U^n_h\|_{1,K} + h_K \|u^n - \partial_\nu U^n_h\|_{0,K} \\
& + \|U^{n+1}_h - U^n_h\|_{1,K} + \|U^n_h - U^{n-1}_h\|_{1,K} + 2h_K \|\overline{R^n_K} - \overline{R^n_K}\|_{0,K} \right), \\
& h_E^{1/2} \|R^n_E\|_{0,E} \leq C \left( \|u^n - U^n_h\|_{0,\omega_E} + h_E \|u^n - \partial_\nu U^n_h\|_{0,\omega_E} + \|U^{n+1}_h - U^n_h\|_{1,\omega_E} + h_E \|\overline{R^n_E} - \overline{R^n_E}\|_{0,\omega_E} \right).
\end{align*}
\]

Proof. By triangle inequality, we have

\[
\|\overline{R^n_K}\|_{0,K} \leq \|\overline{R^n_K}\|_{0,K} + \|\overline{R^n_K} - \overline{R^n_K}\|_{0,K}.
\]
Combining (71)–(73), we obtain
\[ h_K \left\| R^n_{K,0} \right\|_{0,K} \leq C \left( \left\| u^n - U_h^{n,1/2} \right\|_{1,K} \right. \]
\[ + h_K \left\| U_h^n - \partial_t U_h^n \right\|_{0,K} + 2h_K \left( \left\| R^n_{K} - \mathcal{R}^n_{K,0} \right\|_{0,K} \right) \]
\[ \leq C \left( \left\| u^n - U_h^n \right\|_{1,K} + h_K \left\| U_h^n - \partial_t U_h^n \right\|_{0,K} \right. \]
\[ + \left. \left\| U_h^{n+1} - U_h^n \right\|_{1,K} + \left\| U_h^n - U_h^{n-1} \right\|_{1,K} \right) \]
\[ + 2h_K \left( \left\| R^n_{K} - \mathcal{R}^n_{K,0} \right\|_{0,K} \right). \] 

(74)

For (69), by triangle inequality, similarly we have
\[ h_E^{1/2} \left\| \mathcal{R}^n_{E,0} \right\|_{0,E} \leq h_E^{1/2} \left\| \mathcal{R}^n_{E,0} \right\|_{0,E} + h_E^{1/2} \left\| \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0} \right\|_{0,E}. \] 

(75)

By Lemma 5 and Green's formulation, we get
\[ \left\| \mathcal{R}^n_{E,0} \right\|^2_{0,E} \sim \left( \mathcal{R}^n_{E,0}, \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} \]
\[ = \left( \mathcal{R}^n_{E,0}, \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} + \left( \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0}, \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} \]
\[ = \left( a(x) \nabla U_h^{n,1/2}, \nabla \left( \psi_E \mathcal{R}^n_{E,0} \right) \right)_{0,E} \]
\[ + \left( \nabla \cdot \left( a(x) \nabla U_h^{n,1/2} \right), \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} \]
\[ + \left( \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0}, \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} \]
\[ = \int_{\omega_E} a(x) \nabla U_h^{n,1/2} \cdot \nabla \left( \psi_E \mathcal{R}^n_{E,0} \right) \ dx \]
\[ - \int_{\omega_E} a(x) \nabla u^n \cdot \nabla \left( \psi_E \mathcal{R}^n_{E,0} \right) \ dx \]
\[ + \int_{\omega_E} a(x) \nabla u^n \cdot \nabla \left( \psi_E \mathcal{R}^n_{E,0} \right) \ dx \]
\[ + \left( \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0}, \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} \]
\[ = \int_{\omega_E} a(x) \left( U_h^{n,1/2} - u^n \right) \cdot \nabla \left( \psi_E \mathcal{R}^n_{E,0} \right) \ dx \]
\[ + \left( \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0}, \psi_E \mathcal{R}^n_{E,0} \right)_{0,E} \].

(76)

Now we will estimate the right-hand terms of (76). By Lemma 5 and the Cauchy-Schwarz inequality, we obtain
\[ |\mathcal{O}_1| \leq C \left\| U_h^{n,1/2} - u^n \right\|_{1,\omega_E} \left\| \nabla \left( \psi_E \mathcal{R}^n_{E,0} \right) \right\|_{0,\omega_E} \]
\[ = C \left\| U_h^{n,1/2} - u^n \right\|_{1,\omega_E} \left\| \psi_E \mathcal{R}^n_{E,0} \right\|_{0,\omega_E} \]
\[ \leq C h_E^{-1} \left\| U_h^{n,1/2} - u^n \right\|_{1,\omega_E} \left\| \psi_E \mathcal{R}^n_{E,0} \right\|_{0,\omega_E}, \] 

(77)

(78)

Combining (77) with (76), we get
\[ \left\| \mathcal{R}^n_{E,0} \right\|_{0,E} \leq C h_E^{-1} \left\| U_h^{n,1/2} - u^n \right\|_{1,\omega_E} + C h_E^{1/2} \left\| \mathcal{R}^n_{E,0} \right\|_{0,\omega_E} \]
\[ + C h_E^{1/2} \left\| U_h^n - \partial_t U_h^n \right\|_{0,\omega_E} \]
\[ + \left\| \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0} \right\|_{0,E} \].

(79)

With (74), we obtain
\[ h_E^{1/2} \left\| \mathcal{R}^n_{E,0} \right\|_{0,E} \leq C \left\| u^n - U_h^{n,1/2} \right\|_{1,\omega_E} + C h_E \left\| U_h^n - \partial_t U_h^n \right\|_{0,\omega_E} \]
\[ + C h_E^{1/2} \left\| U_h^n - \partial_t U_h^n \right\|_{0,\omega_E} \]
\[ + \left\| \mathcal{R}^n_{E} - \mathcal{R}^n_{E,0} \right\|_{0,E} \].

(80)
Table 1: Error estimates for Case 1.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u^N - U_h^N|_0$</th>
<th>Rate</th>
<th>$|u^N - U_h^N|_1$</th>
<th>Rate</th>
<th>$\mathcal{L}^N$</th>
<th>$\mathfrak{R}^N$</th>
<th>$\mathfrak{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^2$</td>
<td>1.3839 e - 02</td>
<td>—</td>
<td>1.6047 e - 01</td>
<td>—</td>
<td>0.4589</td>
<td>21.3382</td>
<td>46.4952</td>
</tr>
<tr>
<td>$1/2^3$</td>
<td>4.0831 e - 03</td>
<td>1.7610</td>
<td>8.2097 e - 02</td>
<td>0.9669</td>
<td>0.4417</td>
<td>21.1967</td>
<td>47.9920</td>
</tr>
<tr>
<td>$1/2^4$</td>
<td>9.3149 e - 04</td>
<td>2.1321</td>
<td>4.1279 e - 02</td>
<td>0.9919</td>
<td>0.4307</td>
<td>21.0618</td>
<td>48.9005</td>
</tr>
<tr>
<td>$1/2^5$</td>
<td>2.0574 e - 04</td>
<td>2.1787</td>
<td>2.0670 e - 02</td>
<td>0.9979</td>
<td>0.4227</td>
<td>20.9680</td>
<td>49.3734</td>
</tr>
<tr>
<td>$1/2^6$</td>
<td>4.6977 e - 05</td>
<td>2.1308</td>
<td>1.0338 e - 02</td>
<td>0.9996</td>
<td>0.4215</td>
<td>20.9138</td>
<td>49.6128</td>
</tr>
</tbody>
</table>

Table 2: Error estimates for Case 2.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$|u^N - U_h^N|_0$</th>
<th>Rate</th>
<th>$|u^N - U_h^N|_1$</th>
<th>Rate</th>
<th>$\mathcal{L}^N$</th>
<th>$\mathfrak{R}^N$</th>
<th>$\mathfrak{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/2^2$</td>
<td>1.2513 e - 01</td>
<td>—</td>
<td>2.3532</td>
<td>—</td>
<td>6.6633</td>
<td>66.7292</td>
<td>10.0144</td>
</tr>
<tr>
<td>$1/2^3$</td>
<td>3.2994 e - 02</td>
<td>1.9232</td>
<td>1.1822</td>
<td>0.9931</td>
<td>6.3537</td>
<td>64.6603</td>
<td>10.1768</td>
</tr>
<tr>
<td>$1/2^4$</td>
<td>8.1195 e - 03</td>
<td>2.0227</td>
<td>5.9245 e - 01</td>
<td>0.9967</td>
<td>6.1806</td>
<td>63.5799</td>
<td>10.2870</td>
</tr>
<tr>
<td>$1/2^5$</td>
<td>1.8443 e - 03</td>
<td>2.1383</td>
<td>2.9641 e - 01</td>
<td>0.9991</td>
<td>6.0904</td>
<td>63.0179</td>
<td>10.3471</td>
</tr>
<tr>
<td>$1/2^6$</td>
<td>4.0436 e - 04</td>
<td>2.1894</td>
<td>1.4822 e - 01</td>
<td>0.9999</td>
<td>6.0444</td>
<td>62.7297</td>
<td>10.3782</td>
</tr>
</tbody>
</table>

Define

\[
\mathfrak{R}^m = \sum_{n=1}^{m} \|u^n - U_h^n\|_1, \\
\mathfrak{R} = \frac{\mathfrak{R}^m}{\mathcal{L}^m}. 
\]

(82)

We present the results of the above cases when $m = N$ at Tables 1 and 2.

From Tables 1 and 2 we can see that the global a posteriori error estimator can predict the exact global error. The error estimator is reliable as evidenced by the ratio $\mathfrak{R}$ listed on the tables. This list shows that the ratio $\mathfrak{R}$ is converging to a constant when the mesh size is decreased by half. This shows that the proposed global a posteriori error estimator is robust for predicting the error in the finite volume element method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (nos. 11301456, 11426193, and 11571297) and Shandong Province Natural Science Foundation (nos. ZR2014AP003 and ZR2014AM003).

References


Submit your manuscripts at http://www.hindawi.com