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Research Article

A Comparative Analysis of Laguerre-Based Approximators to the Grünwald-Letnikov Fractional-Order Difference

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This paper provides a series of new results in both steady-state accuracy and frequency-domain analyses for two Laguerre-based approximators to the Grünwald-Letnikov difference. In a comparative study, the Laguerre-based approximators are found superior to the classical Tustin- and Al-Alaoui-based approximators, which is illustrated in simulation examples.

1. Introduction

Various approximations to a discrete-time fractional difference (FD) have been pursued in order to prevent its possible computational explosion problem and provide high approximation accuracy. Since FD represents in fact (a sort of) an infinite impulse response (IIR) filter, one solution has been to least-squares (LS) fit an impulse/step response of a discrete-time integer-order IIR filter to that of the associated FD [1–3]. However, the problem is to propose a "good" structure of the integer-order filter, possibly involving a low number of parameters. On the other hand, an LS fit of the FIR filter to FD has been analyzed in the frequency domain [4], with the high-order optimal filter providing a good approximation accuracy, at the cost of a remarkable computational effort however. Similar results are reported in other FIR-based approximations to FD [5, 6]. New time-domain modeling concepts for FD have been introduced in [7, 8].

The above introductory reference review is, deliberately, far from completeness. We refer the reader to the excellent surveys of the state of the art in discretization of fractional-order derivatives [9–15], providing a broad spectrum of the discretization machinery. For space saving reasons, we refrain from repeating the discretization principles and technologies covered therein. Rather, we will recall from [16] the main mathematical results on our unique, Laguerre-based approach [16–18] to direct discretization of the Grünwald-Letnikov (GL) fractional-order derivative. The approach advocates the use of the Laguerre filters, rather than, for example, FIR ones. Indeed, the number of FIR components used in, for example, LS-based Pade, Prony, or Shanks discretization schemes [1, 19] is dramatically higher than the number of their Laguerre counterparts. Thus, our Laguerre-based approach is highly competitive in terms of computational efficiency, in addition to a very high approximation accuracy. Also, our discretization approach is computationally superior to the optimization-based competitors of [14]. It is also worth mentioning that another Laguerre-based discretization approach of [20] is related to the Tustin operator which will be shown essentially inferior to our approximation concept.

This paper extends an original concept of the employment of the Laguerre filters in approximation of the Grünwald-Letnikov fractional difference as intimated in [16]. In particular, new effective solutions are offered as a result of time- and frequency-domain analyses of various versions of Laguerre-based fractional differences. An excellent approximator to FD, which is a combination of the classical finite fractional difference (FFD) [8] and finite Laguerre-based difference (FLD) [16], is found superior to the celebrated Al-Alaoui-based approximator. We advocate the contribution of the FFD on the one hand since in the high frequency range it is

identical to the original FD [8, 21, 22]. On the other hand, we advantage approximating the medium/low-frequency "tale" of the FD by means of the Laguerre filter FLD [16, 18].

The remainder of this paper is structured as follows. Section 2 outlines the fundamentals of the Grünwald-Letnikov/Riemann-Liouville fractional-order discrete-time derivative (DTD) comprising the Grünwald-Letnikov fractional-order difference (FD). Also, the finite-length approximation to FD, namely, finite fractional difference (FFD), is recalled. The basics of orthonormal basis functions, in particular Laguerre functions, are presented in Section 3, and their application in the construction of the Laguerre-based difference (LD) and combined fractional/Laguerre-based difference (CFLD) is given in Section 4. Finite approximations of LD and CFLD, called finite Laguerre-based difference (FLD) and finite (combined) fractional/Laguerre-based difference (FFLD), respectively, are shown in Section 5, also comprising, for comparison purposes, the Tustin- and Al-Alaoui-based approximations. This most important Section also includes original analyses of both steady-state errors and frequency-domain behaviors of the FLD/FFLD-based versus Tustin- and Al-Alaoui-based models of DTD. The Section is culminated with an important technical theorem enabling estimation of a sampling interval for the FD-based DTD, guaranteeing the prespecified phase accuracy requirement, which can be projected to the FLD- and FFLD-based models of DTD. Simulation examples of this section demonstrate high performances of the FLD- and, in particular, FFLD-based approximations to DTD as compared with the Tustin/Al-Alaoui-based ones. Conclusions of Section 6 summarize the contributions of this paper.

2. Grünwald-Letnikov Fractional-Order Difference

It is well known [23, 24] that continuous-time fractional-order derivatives of Grünwald-Letnikov and Riemann-Liouville can be discretized at the sampling interval T to obtain the (fractional-order) discrete-time derivative (DTD):

$$\Delta_T^{\alpha} x(t) = \frac{\Delta^{\alpha} x(t)}{T^{\alpha}}, \tag{1}$$

where the Grünwald-Letnikov fractional-order difference (FD) in discrete time t is described by equation

$$\begin{aligned}\Delta^{\alpha} x(t) &= \sum_{j=0}^{t} P_j(\alpha)\, x(t)\, q^{-j} \\ &= x(t) + \sum_{j=1}^{t} P_j(\alpha)\, x(t)\, q^{-j} \quad t = 0, 1, \ldots,\end{aligned} \tag{2}$$

where $\alpha \in (0, 2)$ is the fractional order, q^{-1} is the backward shift operator, and

$$P_j(\alpha) = (-1)^j \beta_j(\alpha) \tag{3}$$

with

$$\beta_j(\alpha) = \binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & j > 0. \end{cases} \tag{4}$$

Remark 1. For brevity, we will proceed with the FD instead of the more general DTD. Whenever substantial, however, we will comment on the effect of T on the results to follow.

In [8, 25], truncated or finite fractional difference (FFD) has (in analogy to FIR) been considered for practical, feasibility reasons, with the convergence to zero of the series $\beta_j(\alpha)$ enabling assuming $\beta_j(\alpha) \approx 0$ for some $j > \bar{J}$, where $\bar{J}$ is the number of backward signal samples used to calculate the fractional difference. We will further proceed with FFD, to be formally defined below.

Definition 2 (see [8]). Let the fractional difference (FD) be defined as in (2) to (4). Then the finite fractional difference (FFD) is defined as

$$\Delta^{\alpha} x(t, J) = x(t) + \sum_{j=1}^{J} P_j(\alpha)\, x(t)\, q^{-j}, \tag{5}$$

where $J = \min(t, \bar{J})$ and $\bar{J}$ is the upper bound for j when $t > \bar{J}$.

The FFD has been analyzed in some papers under the heading of a practical implementation of FD [26–28], or a truncated/finite difference [21, 23, 29], or a short-memory difference [30].

Remark 3. It is well known [8] that, equivalent to (2), FD can be rewritten as the limiting FFD (for $\bar{J} \to \infty$) in the form

$$\begin{aligned}\Delta^{\alpha} x(t) &= x(t) + \sum_{j=1}^{\infty} P_j(\alpha)\, x(t-j) \\ &= x(t) + X_{\mathrm{FD}}(t) \quad t = 0, 1, \ldots,\end{aligned} \tag{6}$$

with $x(l) = 0$ for all $l < 0$.

3. Orthonormal Basis Functions

It is well known that an open-loop stable linear discrete-time IIR system governed by the transfer function

$$G(z) = \sum_{j=1}^{\infty} g_j z^{-j}, \tag{7}$$

where the impulse response $g_j = g(j)$, $j = 1, 2, \ldots$, can be described in the Laurent expansion form [31, 32]

$$G(z) = \sum_{j=1}^{\infty} c_j L_j(z) \tag{8}$$

including a series of orthonormal basis functions (OBF) $L_j(z)$ and the weighting parameters c_j, $j = 1, 2, \ldots$, characterizing the model dynamics.

Various OBF can be used in (8). Two commonly used sets of OBF are simple Laguerre and Kautz functions. These functions are characterized by the "dominant" dynamics of a system, which is given by a single real pole (p) or a pair of complex ones (p, p^*), respectively. In case of discrete-time Laguerre filters to be exploited hereinafter, the orthonormal functions

$$L_j(z) = L_j(z,p) = \frac{k}{z-p}\left[\frac{1-pz}{z-p}\right]^{j-1} \qquad j = 1,2,\ldots, \quad (9)$$

with $k = \sqrt{1-p^2}$ and $p \in (-1,1)$, consist of a first-order low-pass factor and $(j-1)$th-order all-pass filters.

Remark 4. It is important that the factor k need not include the sampling interval T (which can be set to unity) and this is because the FD components $P_j(\alpha)$, $j = 1,2,\ldots$, do not include T.

Remark 5. Depending on the domain context, we will use various arguments in $L_j(\cdot)$, for example, $L_j(z)$ in the z-domain and $L_j(q)$ or $L_j(q^{-1})$ in the time-domain. The same concerns the arguments in $G(\cdot)$.

Remark 6. Our interest in the Laguerre filters also results from the fact that their well-damped behavior fit the nonoscillatory dynamics of DTD (in addition to a low number of Laguerre model parameters involved).

The coefficients c_j, $j = 1,2,\ldots$, can be calculated from the scalar product of $G(z)$ and $L_j(z)$ [31]:

$$c_j = \left\langle G(z), L_j(z)\right\rangle = \frac{1}{2\pi i}\oint_\gamma G^*(z)\, L_j(z)\,\frac{dz}{z}, \qquad (10)$$

where $G^*(z)$ is the complex conjugate of $G(z)$ and γ is the unit circle. Note that $G(z)$ and $L_j(z)$, $j = 1,2,\ldots$, must be analytic in γ. It is also possible to calculate the scalar product in the time-domain

$$c_j = \left\langle g(t), l_j(t)\right\rangle = \sum_{t=1}^{\infty} g(t)\, l_j(t), \qquad (11)$$

where the impulse response of the system $g(t) = G(q^{-1})\delta(t)$, $l_j(t) = L_j(q^{-1})\delta(t)$, $t = 0,1,2,\ldots$, $g(0) = 0$, and $\delta(t)$ is the Kronecker delta.

4. Laguerre-Based Fractional-Order Differences

4.1. Laguerre-Based Difference. Let us firstly define a "sort of" a difference to be referred to as the Laguerre-based difference.

Definition 7 (see [16]). Let c_j and $L_j(z)$, $j = 1,2,\ldots$, be described as in (8) through (10). Then the Laguerre-based difference (LD) is defined as

$$\begin{aligned}\Delta^{\alpha}_{\mathrm{LD}} x(t) &= x(t) + \sum_{j=1}^{\infty} c_j L_j\left(q^{-1}\right) x(t) \\ &= x(t) + X_{\mathrm{LD}}(t) \qquad t = 0,1,\ldots\end{aligned} \qquad (12)$$

with $x(l) = 0$ for all $l < 0$.

Remark 8. Again, whenever substantial we will comment on the effect of the sampling period T when using an LD-based form $\Delta^{\alpha}_{\mathrm{LD}} x(t)/T^{\alpha}$ of DTD (compare Remark 1). This will also hold true for the forthcoming CFLD and its two approximators FLD and FFLD.

Since $X_{\mathrm{FD}}(t)$ in (6) represents a sort of IIR and so does $X_{\mathrm{LD}}(t)$ as in (12), the question arises as to what a relationship between $X_{\mathrm{FD}}(t)$ and $X_{\mathrm{LD}}(t)$ is and, moreover, if and when it is possible to obtain $X_{\mathrm{LD}} = X_{\mathrm{FD}}$.

Now, a fundamental equivalence result in this respect is recalled.

Theorem 9 (see [16]). *Let the FD be defined as in (2) through (4) or, equivalently, as in (6) and let the LD be defined as in Definition 7. Then LD is identical to FD, that is, $X_{LD}(t) \equiv X_{FD}(t)$, if and only if*

$$c_j = \sum_{i=0}^{j-1}\binom{j-1}{i}\frac{k^{2i}(-p)^{j-1-i}}{i!}\left.\frac{d^i C_1(z)}{dz^i}\right|_{z=p} \qquad j = 1,2,\ldots \qquad (13)$$

with $k = \sqrt{1-p^2}$, $p \in (-1,1)\setminus\{0\}$ being the dominant Laguerre pole and

$$C_1(z) = k\frac{(1-z)^{\alpha}-1}{z}. \qquad (14)$$

4.2. Combined Fractional/Laguerre-Based Difference. Let us finally define a combined fractional/Laguerre-based difference, which is a combination of the "classical" FD and our LD.

Definition 10 (see [16]). Let the FD and LD be defined as in (2) and (12), respectively. Then the combined fractional/Laguerre-based difference (CFLD) is defined as

$$\Delta^{\alpha}_{\mathrm{CFLD}} x(t) = x(t) + X_{\mathrm{CFLD}}(t) \qquad t = 0,1,\ldots, \qquad (15)$$

where

$$X_{\mathrm{CFLD}}(t) = \sum_{i=1}^{J} P_i(\alpha)\, x(t)\, q^{-i} + \sum_{j=1}^{\infty} c_j L_j\left(q^{-1}\right) q^{-\bar{J}} x(t) \qquad (16)$$

with the first component at the right-hand side of (16) constitutes the FFD share in the CFLD and the second one is the ($\bar{J}$-delayed) LD share, with $P_j(\alpha)$, $j = 1,\ldots,J$, as in (3) and (4), and $L_j(q^{-1})$ and c_j, $j = 1,2,\ldots$, as in (9) and (10), respectively.

Here is another fundamental equivalence result.

Theorem 11 (see [16]). *Let the Grünwald-Letnikov fractional difference (FD) be defined as in (2) through (4), the Laguerre-based difference (LD) is as in Definition 7 and the combined fractional/Laguerre-based difference (CFLD) is as*

in Definition 10. Then CFLD is equivalent to FD in that $X_{CFLD}(t) \equiv X_{FD}(t)$ if and only if

$$c_j = \sum_{i=0}^{j-1} \binom{j-1}{i} \frac{k^{2i}(-p)^{j-1-i}}{i!} \frac{d^i D_1(z)}{dz^i}\bigg|_{z=p} \qquad j = 1, 2, \ldots \tag{17}$$

with $k = \sqrt{1-p^2}$ and $p \in (-1, 1) \setminus \{0\}$ being the dominant Laguerre pole and

$$D_1(z) = k \frac{(1-z)^\alpha - 1 - \sum_{j=1}^{\bar{J}} P_j(\alpha) z^j}{z^{\bar{J}+1}}. \tag{18}$$

Remark 12. Note that regardless of an actual value of p we have FD $\equiv$ LD $\equiv$ CFLD, in the sense that $X_{\mathrm{FD}}(t) \equiv X_{\mathrm{LD}}(t) \equiv X_{\mathrm{CFLD}}(t)$, $t = 0, 1, \ldots$.

Note that the above-presented fractional-order differences FD, LD, and CFLD may lead to computational explosion. So, in the next section, finite approximations of the above will be considered.

5. Finite Approximations of Fractional-Order Differences

5.1. Finite Fractional Difference. In Section 2, the "classical" finite fractional difference (FFD) has been redefined. In a similar way, we define two finite fractional approximators to LD and CFLD.

5.2. Finite Laguerre-Based Difference. In analogy to the presented finite fractional difference (FFD), the convergence to zero of the series c_j enables assuming $c_j \approx 0$ for some $j > M$, where M is the number of the Laguerre filters used to calculate the finite LD. We will further proceed with the finite Laguerre-based difference (FLD), to be formally defined below.

Definition 13 (see [16]). Let the Laguerre-based discrete-time difference (LD) be defined as in Definition 7. Then the finite Laguerre-based difference (FLD) is defined as

$$\begin{aligned} \Delta^\alpha_{\mathrm{FLD}} x(t) &= x(t) + \sum_{j=1}^{M} c_j L_j\left(q^{-1}\right) x(t) \\ &= x(t) + x_{\mathrm{FLD}}(t) \qquad t = 0, 1, \ldots, \end{aligned} \tag{19}$$

where M is the number of the Laguerre filters used do calculate the difference FLD and c_j, $j = 1, 2, \ldots, M$, are calculated as in (13).

5.3. Finite Fractional/Laguerre-Based Difference. The idea behind combining FFD and FLD comes from *a priori* knowledge about the natures of (1) FFD versus FD in the initial (or high-frequency) part of the model [8] and (2) FLD versus classical FIR in the remaining (or medium/low-frequency) part. In fact, FFD $\equiv$ FD for $t < \bar{J}$ so the "only" problem is to find a "good" $\bar{J}$ and, on the other hand, a "good" number M of the Laguerre filters, which is essentially lower than a number of FIR components, in particular in the medium/low frequency part.

Step by step, we arrive at the most practically important model of FD, being the truncated or finite CFLD.

Definition 14 (see [16]). Let the combined fractional/Laguerre-based difference (CFLD) be defined as in Definition 10. Then the finite (combined) fractional/Laguerre-based difference (FFLD) is defined as

$$\begin{aligned} \Delta^\alpha_{\mathrm{FFLD}} x(t) &= x(t) + \sum_{i=1}^{J} P_i(\alpha) x(t) q^{-i} + \sum_{j=1}^{M} c_j L_j\left(q^{-1}\right) q^{-\bar{J}} x(t) \\ &= x(t) + X_{\mathrm{FFLD}}(t) \qquad t = 0, 1, \ldots, \end{aligned} \tag{20}$$

where M is a number of the Laguerre filters used in the model.

Remark 15. An important problem of selection of the Laguerre pole p for FLD and FFLD has been effectively solved in [16, 17].

Remark 16. It is essential that FLD and, in particular, FFLD have been shown to be computationally very effective, in that surprisingly low numbers of M and $\bar{J}$ are sufficient to provide very high modeling accuracies [16, 17].

5.4. Tustin- and Al-Alaoui-Based Approximations. There are three most popular discretization schemes for fractional-order derivatives, resulting in two Tustin-based and one Al-Alaoui-based approximators [12, 33–35]. Let us recall the so-called Tustin-Muir approximator, mainly in order to rectify some error frequently repeated in the Muir recursion. The Tustin-Muir approximator is

$$\frac{\Delta^\alpha_{\mathrm{Tus}} x(t,n)}{T^\alpha} = \left(\frac{2}{T}\right)^\alpha \frac{A_n\left(q^{-1}, \alpha\right)}{A_n\left(q^{-1}, -\alpha\right)} x(t), \tag{21}$$

where $\alpha \in (0, 1)$, $A_n(q^{-1}, \alpha)$, and $A_n(q^{-1}, -\alpha)$ are the polynomials in q^{-1} of orders n, whose coefficients can be computed in a recursive way:

$$A_n\left(q^{-1}, \alpha\right) = A_{n-1}\left(q^{-1}, \alpha\right) - \gamma_n q^{-n} A_{n-1}(q, \alpha) \tag{22}$$

with

$$\gamma_n = \begin{cases} \dfrac{\alpha}{n} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases} \tag{23}$$

and $A_0(q^{-1}, \alpha) = 0$.

For the Al-Alaoui-based approximator there is

$$\frac{\Delta^\alpha_{\mathrm{Al}} x(t,n)}{T^\alpha} = \left(\frac{8}{7T}\right)^\alpha \frac{P\left(q^{-1}\right)}{Q\left(q^{-1}\right)} x(t), \tag{24}$$

where $\alpha \in (0, 1)$, $P(q^{-1})$, and $Q(q^{-1})$ are the CFE-related polynomials in q^{-1} of, generally, different orders [12], but we will assume equal orders n here.

5.5. Steady-State Error. An important problem encountered in various approximations to FD is an incorrect steady-state gain of the model. This may lead to remarkable steady-state errors in modeling of DTD, the issue being sometimes disregarded, in particular in, for example, the Tustin-based discretization model. Steady-state errors for all the considered models of DTD are characterized below.

Lemma 17. *Let the steady-state error for the FLD-based model of DTD with respect to the DTD one be defined as*

$$\epsilon_{FLD}^{ss}(M) = \epsilon_{FLD}^{ss}(M,p) = \lim_{t\to\infty}\left\{\epsilon_{FLD}(t,M) = \frac{\Delta_{FLD}^{\alpha}x(t)}{T^{\alpha}} - \frac{\Delta^{\alpha}x(t)}{T^{\alpha}}\right\}. \tag{25}$$

Then

$$\varepsilon_{FLD}^{ss} = \frac{1}{T^{\alpha}}\left(1 + \frac{k}{1-p}\sum_{j=1}^{M}c_j\right)x_{ss}, \tag{26}$$

where x_{ss} is the steady-state value of $x(t)$.

Proof. The steady-state value of the outputs $L_j(q^{-1})x(t)$ from the Laguerre filters $L_j(q^{-1})$, $j = 1,\ldots,M$, is given by

$$\lim_{t\to\infty}L_j\left(q^{-1}\right)x(t) = \lim_{q\to 1}L_j\left(q^{-1}\right)x_{ss} = \frac{k}{1-p}x_{ss}. \tag{27}$$

Accounting for (1), Remark 8, and Definition 13 and for the fact that $\lim_{t\to\infty}\Delta^{\alpha}x(t) = 0$, we immediately arrive at (26). □

Lemma 18. *The steady-state error for FFLD-based model of DTD with respect to the DTD one defined as*

$$\varepsilon_{FFLD}^{ss}\left(\bar{J},M\right) = \lim_{t\to\infty}\left\{\epsilon_{FFLD}(t,M) = \frac{\Delta_{FFLD}^{\alpha}x(t)}{T^{\alpha}} - \frac{\Delta^{\alpha}x(t)}{T^{\alpha}}\right\} \tag{28}$$

is

$$\varepsilon_{FFLD}^{ss}\left(\bar{J},M\right) = \frac{1}{T^{\alpha}}\left(1 + \sum_{j=1}^{\bar{J}}P_j(\alpha) + \frac{k}{1-p}\sum_{j=1}^{M}c_j\right)x_{ss}. \tag{29}$$

Proof. It is similar to proof of Lemma 17, with Definition 14 being involved. □

Here we have a nice steady-state accuracy result for the FFLD-based model of DTD.

Corollary 19. *Let the steady-state errors $\epsilon_{FLD}^{ss}(M)$ and $\epsilon_{FFLD}^{ss}(\bar{J}, M)$, $\bar{J} > 1$, be defined as in (25) and (28). Then $\epsilon_{FFLD}^{ss}(\bar{J},M) < \epsilon_{FLD}^{ss}(M)$.*

Proof. The proof is immediate from Lemmas 17 and 18, taking into account that $\sum_{j=1}^{\bar{J}}P_j(\alpha)$ is always negative [8]. □

We are in a position now to recall two important theoretical results for LD and CFLD.

Theorem 20 (see [17, 18]). *Consider LD as in Definition 7, with c_j, $j = 1,2,\ldots$, as in (13). Then*

$$\sum_{j=1}^{\infty}c_j = \frac{p-1}{k}. \tag{30}$$

Remark 21. It is interesting that, with all the coefficients c_j, $j = 1,2,\ldots$, depending on α, their infinite sum as in (30) is, rather surprisingly, independent of α. Of course, the finite sum of those coefficients as in (26) remains dependent on α.

Theorem 22 (see [17, 18]). *Consider CFLD as in Definition 14, with c_j, $j = 1,2,\ldots$, as in (17). Then*

$$\sum_{j=1}^{\infty}c_j = \frac{p-1}{k}\left(1 + \sum_{j=1}^{\bar{J}}P_J(\alpha)\right). \tag{31}$$

Remark 23. For FLD and FFLD, (30) and (31), respectively, are satisfied only approximately due to the finite summations. However, the quality of the FLD and FFLD approximations in the steady state can be assessed from the "closedness" of the right- and left-hand sides of (30) and (31), respectively.

Let us now state a simple steady-state accuracy result for the Tustin-based model of DTD.

Lemma 24. *The steady-state error for the Tustin-based discretization model (21) with respect to DTD defined as*

$$\varepsilon_{Tus}^{ss}(n) = \lim_{t\to\infty}\left\{\epsilon_{Tus}(t,n) = \frac{\Delta_{Tus}^{\alpha}x(t,n)}{T^{\alpha}} - \frac{\Delta^{\alpha}x(t)}{T^{\alpha}}\right\} \tag{32}$$

is

$$\varepsilon_{Tus}^{ss}(n) = \left(\frac{2}{T}\right)^{\alpha}\frac{A_n(1,\alpha)}{A_n(1,-\alpha)}x_{ss}. \tag{33}$$

Proof. It is immediate from (21), with $q^{-1} = 1$. □

Lemma 25. *The steady-state error for the Al-Alaoui-based discretization model (24) with respect to DTD defined as*

$$\varepsilon_{Al}^{ss}(n) = \lim_{t\to\infty}\left\{\epsilon_{Al}(t,n) = \frac{\Delta_{Al}^{\alpha}x(t,n)}{T^{\alpha}} - \frac{\Delta^{\alpha}x(t)}{T^{\alpha}}\right\} \tag{34}$$

is

$$\varepsilon_{Al}^{ss}(n) = \left(\frac{8}{7T}\right)^{\alpha}\frac{P(1)}{Q(1)}x_{ss}. \tag{35}$$

Proof. The proof comes immediately from (24), with $q^{-1} = 1$. □

Remark 26. Note that the steady-state error equations (26), (29), (33), and (35) incorporate the factor $1/T^{\alpha}$ in the same manner. Therefore, in a comparative analysis we can use, for example, $T = 1$.

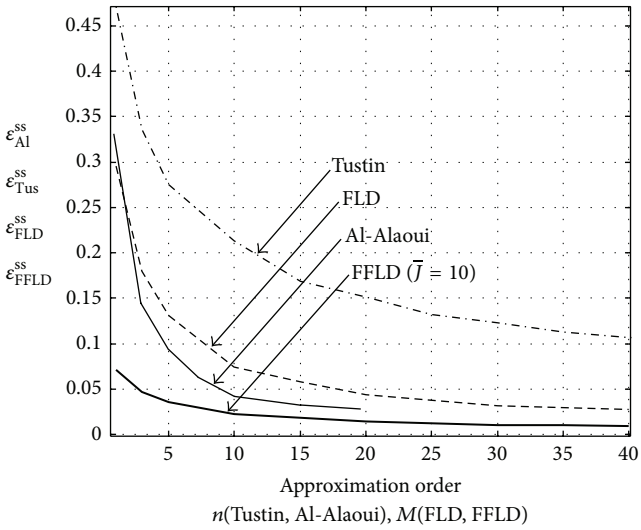

Figure 1: Steady-state errors for FLD-, FFLD-, Tustin-, and Al-Alaoui-based approximations; Example 27.

Example 27. Recall the steady-state errors as in (26), (29), (33), and (35) for the FLD-, FFLD-, Tustin-, and Al-Alaoui-based approximations to DTD, respectively. The error plots presented in Figure 1 are self-explanatory. The FFLD-based model clearly outperforms the three remaining ones, of which the Tustin-based model is definitely inferior, even for very high approximation orders. Also note how low $\bar{J} = 10$ is, which when increased to, for example, 15 or 20 can contribute to further drop of the error. Also note that the Al-Alaoui approximator cannot be used for the order $n > 18$ due to numerical problems, in particular in the Matlab environment.

Remark 28. It is worth mentioning that the steady-state accuracy issue is very important as even low steady-state approximation errors for the fractional difference may be propagated to high modeling errors for a fractional-order dynamical system [8].

Example 29. Consider the FD-based DTD of order $\alpha = 0.5$. The FLD- (with $M = 25$) and FFLD- ($M = 20$ and $\bar{J} = 10$) based models are analyzed versus the Tustin-based approximation of order 25 (which is usually considered very high) and the Al-Alaoui-based approximation of order 17. Figure 2 presents step responses for the DTD and its FLD/FFLD/Tustin/Al-Alaoui-based approximations at the sampling period $T = 1$. The "strange" behavior of the response for the Tustin-based model of DTD is surprising. On the other hand, the responses of DTD and its FLD-, FFLD-, and Al-Alaoui-based approximations are hardly distinguishable, which suggests that a frequency-domain analysis could be welcome here. On the other hand, the mean square prediction errors (MSPE), shown in Table 1, indicate that the time-domain fit of the FFLD-based model of DTD is the best.

Remark 30. The "strange" behavior of the step response for the Tustin-based approximation results from the fact that one pole of the transfer function $A_n(z^{-1}, \alpha)/A_n(z^{-1}, -\alpha)$ is highly negative (in the range of, e.g., -0.9), which itself may raise doubts on the adequacy of the Tustin-based approximation.

Table 1: Mean square prediction error for the analyzed models; Example 29.

Models	MSPE
Al-Alaoui-based Laguerre model	$5.96e-2$
FLD-based Laguerre model	$3.49e-2$
FFLD-based Laguerre model	$1.13e-6$

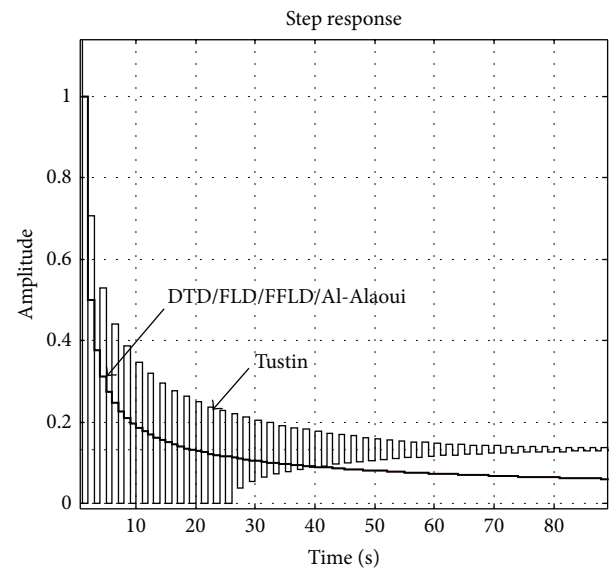

Figure 2: Step responses for DTD and its FLD-, FFLD-, Tustin-, and Al-Alaoui-based approximations; Example 29.

5.6. Frequency-Domain Analysis. Let us start with an instructive simulation example.

Example 31. Consider a fractional-order derivative represented by s^α and its discrete-time approximations as in Example 29. Figure 3 presents Bode plots for the fractional-order derivative and its FLD/FFLD/Tustin-based approximations at the sampling period $T = 1$. In Figure 3, the gray-marked area shows the ± 3 dB error from the actual value of $(i\omega)^\alpha$ in the magnitude spectrum and its equivalent in the phase spectrum $\varphi \in [\alpha(\pi/2)(1/\sqrt{2}), \alpha(\pi/2)(2 - 1/\sqrt{2})]$, respectively, with the error bound fulfilled within the frequency spectrum $\omega \in (\omega^{\min}_{\text{mod}}, \omega^{\max}_{\text{mod}})$ and a specific type of the model subindexed as "mod." Magnitude plots for the FFLD- and FLD-based models are within the error bound over a remarkably wider area of the frequency spectra $\omega \in (9.5e-5, \pi)$ and $\omega \in (7.3e-4, \pi)$, respectively, as compared to the Tustin-based approximation, with $\omega \in (1e-2, 2.1)$. So, in this regard the FFLD and FLD approximations are much more effective than the Tustin-based approach. For lucidity, we refrain from plotting the magnitude characteristic for the Al-Alaoui-based approximator as it is very close to the FLD-based one.

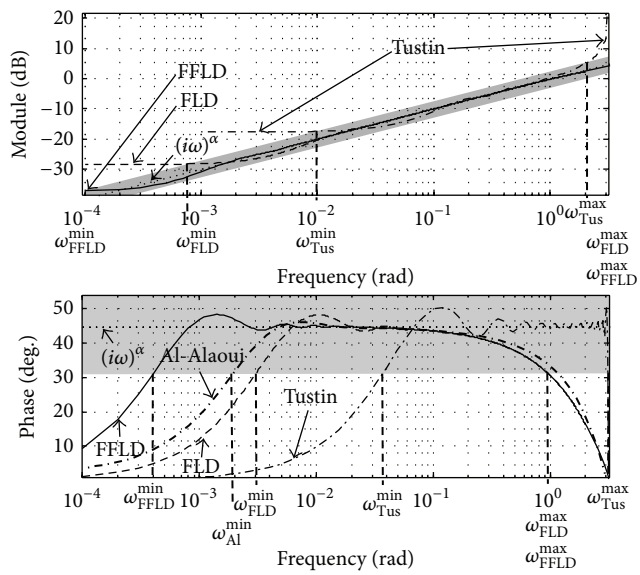

Figure 3: Bode plots for FLD-, FFLD-, and Tustin-based approximations; Example 31.

In case of phase plots, the FFLD- and FLD-based models are within the error bound over a remarkably wider area of the frequency spectra $\omega \in (3.9e-4, 9.8e-1)$ and $\omega \in (3.0e-3, 9.8e-1)$, respectively, as compared to the Tustin-based approximation, with $\omega \in (3.8e-2, 3.1)$. As for the Al-Alaoui-based approximation, the phase plot is slightly better than the FLD-based one, with $\omega \in (2.0e-3, 1.1)$, but remarkably worse than the FFLD-one. (Note: for lucidity, we refrain from marking the $\omega_{\mathrm{Al}}^{\max}$ value as it is very close to $\omega_{\mathrm{FLD}}^{\max}$ and $\omega_{\mathrm{FFLD}}^{\max}$.) Note that the FLD/FFLD/Al-Alaoui-based approximators generate a high phase error for high frequencies (close to the sampling frequency). This phase error is a result of the backward difference based discretization scheme for the continuous-time derivative.

Finally, on the basis of both magnitude and phase plots we obtain adequacy ranges for the FFLD-based model: $\omega \in (3.9e-4, 9.8e-1)$, Al-Alaoui-based model: $\omega \in (2.0e-3, 1.1)$, FLD-based model: $\omega \in (3.0e-3, 9.8e-1)$, and Tustin-based model: $\omega \in (3.8e-2, 2)$. So, the Tustin-based approximation has two times higher upper frequency limit as compared to the FFLD/FLD/Al-Alaoui-based approximations. However, taking into account that the FFLD/FLD/Al-Alaoui-based models include the sampling period in their denominators only, we can easily left/right-shift the frequency spectrum by changing the sampling period T. For our FFLD- and FLD-based models we obtain the adequacy ranges $\omega \in ((3.9e-4)/T, (9.8e-1)/T)$ and $\omega \in ((3.0e-3)/T, (9.8e-1)/T)$, respectively, whereas for the Al-Alaoui model we have $\omega \in ((2.0e-3)/T, 1.1/T)$. Therefore, using the sampling period $T = 0.5$ for the FFLD/FLD/Al-Alaoui-based models we obtain the adequacy ranges $\omega \in (7.8e-4, 1.96)$ for FFLD, $\omega \in (6.0e-3, 1.96)$ for FLD, and $\omega \in (4.0e-3, 2.2)$ for the Al-Alaoui model, with the upper frequency bounds being similar to the Tustin-based approximation for $T = 1$. (Note: for lucidity, we refrain from showing both magnitude and phase plots for the FD-based DTD model, with the former one being identical to that for $(i\omega)^\alpha$ and the latter one being very close to that for $(i\omega)^\alpha$ at the low and medium frequency ranges and very close to that for the FLD/FFLD-based approximation for high frequencies.)

Example 31 illustrates that possible high argument errors for the FLD/FFLD-based models in the high frequency range are related to the backward difference argument error which depends on the sampling period T. Now, the argument error analysis deserves a special attention. Here we present a new method to calculate this error.

Theorem 32 (main result). *Consider an α-order continuous-time derivative and its Laplace transform s^α, $\alpha \in (0,2)$. Discretize the derivative using the backward difference scheme to obtain the FD-based DTD as in (1) and (2). The frequency-domain argument error $\varepsilon_{ph}(\omega)$ for DTD with respect to $(i\omega)^\alpha$ defined as*

$$\varepsilon_{ph}(\omega) = \arg\left[\Delta_T^\alpha\left(e^{i\omega T}\right)\right] - \arg\left[(i\omega)^\alpha\right] \tag{36}$$

is equal to

$$\varepsilon_{ph}(\omega) = -\frac{\alpha\omega T}{2}. \tag{37}$$

Proof. It is well known that DTD can be described in the z-domain as

$$\Delta_T^\alpha(z) = \frac{1}{T^\alpha}\left(1 - z^{-1}\right)^\alpha. \tag{38}$$

Accounting for T in the frequency-domain form $z = e^{i\omega T}$ we obtain

$$\begin{aligned}\Delta_T^\alpha\left(e^{i\omega T}\right) &= \frac{1}{T^\alpha}\left(1 - e^{-i\omega T}\right)^\alpha \\ &= e^{-i(\alpha\omega T/2)}\left(e^{i(\omega T/2)} - e^{-i(\omega T/2)}\right)^\alpha \\ &= e^{-i(\alpha\omega T/2)}\left(2\,\mathrm{Im}\left(e^{i(\omega T/2)}\right)\right)^\alpha\end{aligned} \tag{39}$$

with the latter manipulation using the fact that $e^{i(\omega T/2)}$ is the conjugate to $e^{-i(\omega T/2)}$. Finally, the argument of $\Delta_T^\alpha(e^{i\omega T})$ is

$$\arg\left(\Delta_T^\alpha\left(e^{i\omega T}\right)\right) = \frac{\alpha\pi}{2} - \frac{\alpha\omega T}{2}. \tag{40}$$

Now, recalling the definition in (36) and taking into account that $\arg((i\omega)^\alpha) = \alpha\pi/2$ we arrive at (37). □

5.6.1. Discussion. On the basis of Theorem 32, when we assume the maximum value of modulus of the argument error $\varepsilon_{\mathrm{ph}}^{\max} = \max_\omega|\varepsilon_{\mathrm{ph}}(\omega)|$ and the upper bound for frequency range $\omega_{\max}$ we can immediately select such a sampling period T which can guarantee that $|\varepsilon_{\mathrm{ph}}(\omega)| < \varepsilon_{\mathrm{ph}}^{\max}$ for $\omega \in (0, \omega_{\max})$; namely,

$$T < \frac{2\varepsilon_{\mathrm{ph}}^{\max}}{\alpha\omega_{\max}}. \tag{41}$$

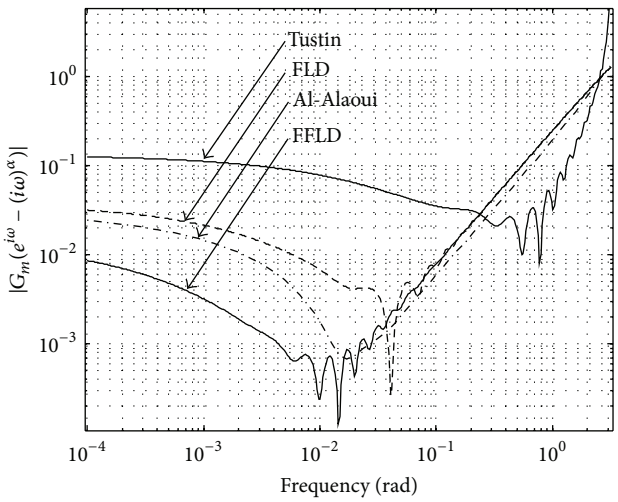

Figure 4: Approximation error for FFLD-, FLD-, Tustin-, and Al-Alaoui-based approximations; Example 34.

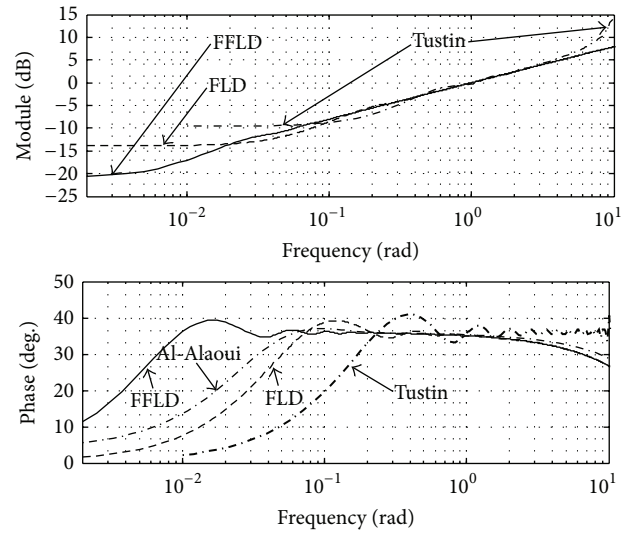

Figure 5: Bode plots for FLD-, FFLD-, Al-Alaoui-, and Tustin-based approximations; Example 35.

It is worth mentioning that the error $|\varepsilon_{\text{ph}}(\omega)|$ can be quite high for high frequencies; for example, in the highest frequency range ($\omega T = \pi$) we obtain $|\varepsilon_{\text{ph}}(\omega)| = \alpha\pi/2$. On the other hand, in Example 31 it has been presented that, for high frequency ranges, the results generated by the FLD/FFLD/Al-Alaoui-based approximations are very similar to DTD. Therefore, we can write that for high frequencies we have $|\arg(\Delta^\alpha_{\text{FLD/FFLD/Al}}(e^{i\omega T})/T^\alpha) - \arg(\Delta^\alpha_T(e^{i\omega T}))| \ll \varepsilon_{\text{ph}}(\omega)$ and we can use (37) and (41) to estimate the sampling period T for the FLD/FFLD/Al-Alaoui-based approximations.

Remark 33. It is time now to recall the "noncasual compensator" method for elimination of the phase error in the Al-Alaoui-based approximator [12], which could also be used for the FLD/FFLD-based ones. However, with such an "artificial" phase rectification, we dismiss the original time-domain interpretation (and applications) of the fractional-order derivative/difference. Therefore we claim that our "frequency shifting" method, based on selection of the sampling period T, is more practically oriented.

Example 34. Consider the fractional-order derivative and its discrete-time approximations as in Example 31. Figure 4 presents the approximation error defined as $|G_m(e^{i\omega}) - (i\omega)^\alpha|$ for the four particular models, where $m = 1$ denotes the FLD-based model ($\Delta^\alpha_{\text{FLD}}(i\omega)/T^\alpha$), $m = 2$ the FFLD-based model ($\Delta^\alpha_{\text{FFLD}}(i\omega)/T^\alpha$), $m = 3$ the Tustin-based model ($\Delta^\alpha_{\text{Tus}}(i\omega)/T^\alpha$), and $m = 4$ the Al-Alaoui-based model ($\Delta^\alpha_{\text{Al}}(i\omega)/T^\alpha$). It can be seen from Figure 4 that the Tustin-based approach shows a better performance for high frequencies, except for very high frequencies when ωT tends to π. In medium/low frequency ranges, FLD-, Al-Alaoui-, and, particularly, FFLD-based models provide much better results than the Tustin-based approach. This can be illustrated by the aggregate relative error defined as $\sum_{j=1}^N |G_m(e^{i\omega_j}) - (i\omega_j)^\alpha|/\omega_j^\alpha$, which, for a set of $N = 100$ selected values of ω_j, is equal to 13.323 for the FFLD-based, 29.998 for Al-Alaoui-based, 40.577 for FLD-based, and 173.35 for Tustin-based approximations.

Example 35. Consider the fractional-order derivative of order $\alpha = 0.4$. Assume that for $\omega_{\max} = 10$ we should have the phase error $|\varepsilon_{\text{ph}}(\omega)| < \pi/18$ [Rad] (or 10 degrees). On the basis of (41) we have $T < 0.08727$. So if we assume $T = 0.08$ in the FFLD/FLD/Al-Alaoui-based approximations, we obtain $|\varepsilon_{\text{ph}}(\omega)| < 10^0$. Figure 5 presents Bode plots for the FFLD-based ($\bar{J} = 10$, $M = 20$), FLD-based ($M = 25$), and Al-Alaoui-based ($n = 17$) models for $T = 0.08$ versus the Tustin-based approximation ($n = 25$) using the sampling period $T = 0.3$. Since the magnitude plots for the FLD- and Al-Alaoui-based models are very close to each other, we show only phase plots for the Al-Alaoui-based approximation.

Remark 36. The above examples are only an illustrative selection from a plethora of simulation runs, all of them confirming the above-presented results.

Remark 37. Clearly, the FFLD-based approximation outperforms the three other ones, also in that it covers the fractional-order range $\alpha \in (0, 2)$, in contrast to the Tustin- and Al-Alaoui-based models.

6. Conclusion

This paper has presented a bunch of original results on modeling of fractional-order discrete-time derivative (DTD) by means of its two Laguerre-based approximators. The FLD- and, in particular, FFLD-based approximators have been shown to substantially outperform another popular approximator, namely, the Tustin-based one but also, in case of FFLD, the Al-Alaoui-based one. New results on steady-state accuracy and, in particular, frequency-domain phase analyses, supported with simulations examples, confirm the usefulness

of the considered Laguerre-based approximators, in particular the FFLD one.

Abbreviations

CFE: Continuous fraction expansion
CFLD: Combined fractional/Laguerre-based difference
DTD: Fractional-order discrete-time derivative
FD: Grünwald-Letnikov fractional-order difference
FFD: Finite fractional difference
FFLD: Finite (combined) fractional/Laguerre-based difference
FLD: Finite Laguerre-based difference
LD: Laguerre-based difference
OBF: Orthonormal basis functions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] R. S. Barbosa and J. A. T. Machado, "Implementation of discrete-time fractional- order controllers based on LS approximations," *Acta Polytechnica Hungarica*, vol. 3, no. 4, pp. 5–22, 2006.

[2] Y. Chen, B. Vinagre, and I. Podlubny, "A new discretization methodfor fractional order differentiators via continued fraction expansion," in *Proceedings of ASME Design Engineering Technical Conferences (DETC '03)*, vol. 340, pp. 349–362, Chicago, Ill, USA, 2003.

[3] B. M. Vinagre, I. Podlubny, A. Hernandez, and V. Feliu, "Some approximations of fractional order operators used in control theory and applications," *Fractional Calculus & Applied Analysis*, vol. 3, no. 3, pp. 945–950, 2000.

[4] P. J. Kootsookos and R. C. Williamson, "FIR approximation of fractional sample delay systems," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 43, no. 3, pp. 269–271, 1996.

[5] C. X. Jiang, J. E. Carletta, and T. T. Hartley, "Implementation of fractional-order operators on field programmable gate arrays," in *Advances in Fractional Calculus*, pp. 333–346, Springer, Dordrecht, The Netherlands, 2007.

[6] C.-C. Tseng, S.-C. Pei, and S.-C. Hsia, "Computation of fractional derivatives using Fourier transform and digital FIR differentiator," *Signal Processing*, vol. 80, no. 1, pp. 151–159, 2000.

[7] K. J. Latawiec, R. Stanisławski, W. P. Hunek, and M. Łukaniszyn, "Adaptive finite fractional difference with a time-varying forgetting factor," in *Proceedings of the 17th International Conference on Methods and Models in Automation and Robotics (MMAR '12)*, pp. 64–69, Miedzyzdroje, Poland, August 2012.

[8] R. Stanisławski and K. J. Latawiec, "Normalized finite fractional differences: computational and accuracy breakthroughs," *International Journal of Applied Mathematics and Computer Science*, vol. 22, no. 4, pp. 907–919, 2012.

[9] M. A. Al-Alaoui, "Al-Alaoui operator and the new transformation polynomials for discretization of analogue systems," *Electrical Engineering*, vol. 90, no. 6, pp. 455–467, 2008.

[10] M. A. Al-Alaoui, "Simulation and discretization of fractional-order systems," in *Proceedings of the International Conference on Genetics and Evlutionary Methods*, pp. 249–255, Las Vegas, Nev, USA, 2009.

[11] M. A. Al-Alaoui, "Class of digital integrators and differentiators," *IET Signal Processing*, vol. 5, no. 2, pp. 251–260, 2011.

[12] Y. Q. Chen and K. L. Moore, "Discretization schemes for fractional-order differentiators and integrators," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 3, pp. 363–367, 2002.

[13] Y. Chen, B. M. Vinagre, and I. Podlubny, "Continued fraction expansion approaches to discretizing fractional order derivatives—an expository review," *Nonlinear Dynamics*, vol. 38, no. 1–4, pp. 155–170, 2004.

[14] M. Gupta and R. Yadav, "New improved fractional order differentiator models based on optimized digital differentiators," *The Scientific World Journal*, vol. 2014, Article ID 741395, 11 pages, 2014.

[15] B. T. Krishna, "Studies on fractional order differentiators and integrators: a survey," *Signal Processing*, vol. 91, no. 3, pp. 386–426, 2011.

[16] R. Stanisławski, "New Laguerre filter approximators to the Grünwald-LETnikov fractional difference," *Mathematical Problems in Engineering*, vol. 2012, Article ID 732917, 21 pages, 2012.

[17] R. Stanisławski, *Advances in Modeling of Fractional Difference Systems—New Accuracy, Stability and Computational Results*, vol. 243 of *Studia i Monografie*, Opole University of Technology Press, Opole, Poland, 2013.

[18] R. Stanislawski, K. J. Latawiec, and M. Lukaniszyn, "The steady-state error issue in Laguerre-based modeling of the GL fractional difference," in *Proceedings of the IEEE 22nd International Symposium on Industrial Electronics (ISIE '13)*, pp. 1–6, Taipei, Taiwan, May 2013.

[19] Y. Ferdi, "Computation of fractional order derivative and integral via power series expansion and signal modeling," *Nonlinear Dynamics*, vol. 46, no. 1-2, pp. 1–15, 2006.

[20] G. Maione, "On the Laguerre rational approximation to fractional discrete derivative and integral operators," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1579–1585, 2013.

[21] Z. Liao, "A new definition of fractional derivatives based on truncated left-handed Grunwald-Letnikov formula with $0 < \alpha < 1$ and mediancorrection," *Abstract and Applied Analysis*, vol. 2014, Article ID 914386, 9 pages, 2014.

[22] R. Stanisławski, W. P. Hunek, and K. J. Latawiec, "Finite approximations of a discrete-time fractional derivative," in *Proceedings of the 16th International Conference on Methods and Models in Automation and Robotics (MMAR '11)*, pp. 142–145, Miedzyzdroje, Poland, August 2011.

[23] C. Monje, Y. Chen, B. Vinagre, D. Xue, and V. Feliu, *Fractional-Order Systems and Controls*, Springer, London, UK, 2010.

[24] P. Ostalczyk, "Equivalent descriptions of a discrete-time fractional-order linear system and its stability domains," *International Journal of Applied Mathematics and Computer Science*, vol. 22, no. 3, pp. 533–538, 2012.

[25] R. Stanislawski, "Identification of open-loop stable linear systems using fractional orthonormal basis functions," in *Proceedings of the 14th International Conference on Methods and Models in Automation and Robotics*, pp. 935–985, Miedzyzdroje, Poland, 2009.

[26] M. Buslowicz, "Simple analytic conditions for stability of fractional discrete-time linear systems with diagonal state matrix,"

Bulletin of the Polish Academy of Sciences: Technical Sciences, vol. 60, no. 4, pp. 809–814, 2012.

[27] M. Buslowicz and T. Kaczorek, "Simple conditions for practical stability of positive fractional discrete-time linear systems," *International Journal of Applied Mathematics and Computer Science*, vol. 19, no. 2, pp. 263–269, 2009.

[28] T. Kaczorek, "Practical stability of positive fractional discrete-time linear systems," *Bulletin of the Polish Academy of Sciences: Technical Sciences*, vol. 56, no. 4, pp. 313–317, 2008.

[29] A. Dzieliński and D. Sierociuk, "Stability of discrete fractional order state-space systems," *Journal of Vibration and Control*, vol. 14, no. 9-10, pp. 1543–1556, 2008.

[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, Orlando, Fla, USA, 1999.

[31] P. S. C. Heuberger, P. M. J. van den Hof, and B. Wahlberg, *Modelling and Identification with Rational Orthogonal Basis Functions*, Springer, London, UK, 2005.

[32] P. M. van den Hof, P. S. Heuberger, and J. Bokor, "System identification with generalized orthonormal basis functions," *Automatica*, vol. 31, no. 12, pp. 1821–1834, 1995.

[33] Y. Q. Chen and K. L. Vinagre, "A new IIR-type digital fractional order differentiator," *Signal Processing*, vol. 83, no. 11, pp. 2359–2365, 2003.

[34] M. Siami, M. Saleh Tavazoei, and M. Haeri, "Stability preservation analysis in direct discretization of fractional order transfer functions," *Signal Processing*, vol. 91, no. 3, pp. 508–512, 2011.

[35] B. M. Vinagre, Y. Q. Chen, and I. Petráš, "Two direct Tustin discretization methods for fractional-order differentiator/integrator," *Journal of the Franklin Institute. Engineering and Applied Mathematics*, vol. 340, no. 5, pp. 349–362, 2003.