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Research Article

Fast Sampling Control of Singularly Perturbed Systems with Actuator Saturation and L_2 Disturbance

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We will consider the problem of fast sampling control for singularly perturbed systems subject to actuator saturation and L_2 disturbance. A sufficient condition for the existence of a state feedback controller is proposed. Under this controller, the boundedness of the trajectories in the presence of L_2 disturbances is guaranteed for any singular perturbation parameter less than or equal to a predefined upper bound. To improve the capacity of disturbance tolerance and disturbance rejection, two convex optimization problems are formulated. Finally, a numerical example is presented to demonstrate the effectiveness of the main results of this paper.

1. Introduction

Many practical physical systems consist of subsystems operating on different time scales. Applying the normal control methods to these systems usually lead to ill-conditioned numerical problems. To overcome the numerical problem, singular perturbation theory was introduced to the field of control system and widely used in practice [1]. In this framework, a multiple time-scale system is modeled by a singularly perturbed system (SPS) with a small positive parameter such that the degree of separation between fast and slow modes can be determined. For example, in a power system, the flux linkages of the rotor windings are fast variables while the emf behind transient reactance, the generator's rotor angle in radians, and the actual rotor speed are the slow ones. Thus the power system can be modeled as a SPS, where the torque of the winding represents the perturbation parameter ε [2].

Since most of the modern control systems are implemented by computer, sampling control of SPSs has been widely investigated. There are three sampling modes for SPSs: multirate, slow, and fast sampling mode. Under multirate sampling mode, the slow and fast states are measured at different sampling rate. In [3], a multirate sampling model predictive control method is proposed for large-scale nonlinear uncertain systems. In [4], a multirate sampling composite controller is designed. Slow sampling control is usually under the assumption that the fast subsystem is stable [5, 6]. Slow sampling discrete-time SPS is considered in [7, 8]. If the fast subsystem is not stable, fast sampling control is necessary [9, 10]. Fast sampling control of SPSs with disturbance is considered in [10–12]. In [11], H_∞ controller design method together with a stability bound optimization method is proposed, and a less conservative method is proposed in [12]. But the above achievements do not take into account the actuator saturation, which is common in practice. It is known that actuator saturation may force the systems to work in the open-loop and thus destroy stability of control systems [13]. Thus many research efforts have been devoted to analysis and design of control systems with actuator saturation [14–19]. Recently, SPSs with actuator saturation are considered. Besides basin of attraction, stability bound is also an important stability index for SPSs with actuator saturation. In [20–24], state feedback controller is designed and the basin of attraction is estimated. The obtained results guarantee the existence of the stability bound but can not present an estimate of the bound. Many results have been proposed for estimating or enlarging the stability bound of the SPSs without actuator saturation [25–32]. In [33], continuous-time SPS with actuator saturation is considered and a state feedback controller is designed to achieve a desired stability

bound while the basin of attraction is optimized. To the best knowledge of the authors, the fast sampling control problem of the SPSs with actuator saturation and disturbance has not been considered.

This paper focuses on fast sampling control of SPSs subject to actuator saturation and L_2 disturbance. First, a state feedback controller design method is proposed such that the trajectories of the closed-loop SPSs starting from a bounded set remain bounded for any allowable singular perturbation parameter and L_2 disturbance. Then, a method to enlarge the capacity of disturbance tolerance is proposed in terms of linear matrix inequalities (LMIs). Furthermore, a convex optimization problem is formulated to optimize the disturbance rejection. Finally, an example is given to show the effectiveness of the proposed results.

The rest of this paper is organized as follows: Section 2 provides the problems under consideration. Section 3 gives the main results of this paper. In Section 4, an example is presented to demonstrate the proposed approaches. Section 5 makes a conclusion of the paper.

Notations. The superscript T stands for matrix transpose. For a matrix M, the notation M^{-T} denotes the transpose of the inverse matrix of M and $M_{(i)}$ denotes ith row of M. Let $Q \in R^{n\times n}$ be a positive definite matrix. An ellipsoid $\Omega(Q,\rho)$ is defined as $\Omega(Q,\rho) \triangleq \{\eta \in R^n \mid \eta^T Q\eta \le \rho\}$.

2. Problem Formulation

The fast sampling model of SPSs with actuator saturation and L_2 disturbance is described by the following compact form:

$$\begin{aligned} x(k+1) &= A_\varepsilon x(k) + B_\varepsilon \operatorname{sat}(u(k)) + E_\varepsilon w(k), \\ z(k) &= Cx(k), \end{aligned} \tag{1}$$

where

$$\begin{aligned} x(k) &= \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in R^n, \\ A_\varepsilon &= \begin{bmatrix} I+\varepsilon A_{11} & \varepsilon A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\ B_\varepsilon &= \begin{bmatrix} \varepsilon B_1 \\ B_2 \end{bmatrix}, \\ E_\varepsilon &= \begin{bmatrix} \varepsilon E_1 \\ E_2 \end{bmatrix}, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \end{aligned} \tag{2}$$

ε represents the singular perturbation parameter and is assumed to be available for controller design in this paper. $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are the state variables, $u(k)$ is the control input, $z(k)$ is the output of the system, A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2 and E_1, E_2 are constant matrices of appropriate dimensions and $\operatorname{sat}(\cdot)$ is a componentwise saturation map $R^m \mapsto R^m$ defined by $\operatorname{sat}(u_j) = \operatorname{sign}(u_j)\min\{1, |u_j|\}$, $j = 1,2,\ldots,m$, and $w(k)$ is the L_2 disturbance which belongs to

$$W_\alpha^2 := \left\{ w : R_+ \longrightarrow R^q : \sum_{k=0}^{+\infty} w^T(k)\, w(k) \le \alpha \right\}, \tag{3}$$

where α is a positive number.

Under the state feedback controller

$$u(k) = K_\varepsilon x(k), \tag{4}$$

the closed-loop system can be described by

$$\begin{aligned} x(k+1) &= A_\varepsilon x(k) + B_\varepsilon \operatorname{sat}\left(K_\varepsilon x(k)\right) + E_\varepsilon w(k), \\ z(k) &= Cx(k). \end{aligned} \tag{5}$$

The problems under consideration are as follows.

Problem 1. Given $\varepsilon_0 > 0$ and $\alpha^* > 0$, design a state feedback controller (4), such that, for any $\varepsilon \in (0,\varepsilon_0]$ and $w \in W_{\alpha^*}^2$, all the trajectories of the closed-loop system starting from inside in $\Omega(P^{-1}(\varepsilon), 1)$ will remain inside of $\Omega(P^{-1}(\varepsilon), 1+\eta\alpha^*)$ with $\eta > 0$ and $P(\varepsilon) > 0$ to be determined.

Problem 2. Given $\varepsilon_0 > 0$, design a state feedback controller (4) to maximize α^*, such that for any $\varepsilon \in (0,\varepsilon_0]$ and $w \in W_{\alpha^*}^2$ all the trajectories of the closed-loop system starting from the origin are bounded.

Problem 3. Given $\varepsilon_0 > 0$ and $\alpha^* > 0$, design a state feedback controller (4) to minimize $\gamma > 0$, such that, for any $\varepsilon \in (0,\varepsilon_0]$ and $w \in W_{\alpha^*}^2$, the restricted L_2 gain from output to disturbance of the closed-loop system with zero initial condition is less than γ under the control of (4).

Remark 4. The upper bound ε_0 characterizes the robustness of the system performance with respect to ε. The disturbance tolerance bound α^* describes the largest disturbance that can be tolerated. The disturbance rejection γ means the largest ratio between the L_2 norms of the output and the disturbance.

There are two common tools to handle the saturation nonlinearities, one is the sector bound approach [34], and the other is the convex hull approach [13, 18]. The latter is adopted in this paper since it has been shown to be less conservative than the former [18]. The related preliminaries are recalled in the following.

For a given matrix $K \in R^{m\times n}$, denote the ith row of matrix K as k_i and define

$$\mathscr{L}(K) = \left\{\eta \in R^n : \left|k_i\eta\right| \le 1,\ i \in [1,m]\right\}. \tag{6}$$

Let $\mathscr{D}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m elements in $\mathscr{D}$. Suppose these elements of $\mathscr{D}$ are labeled as D_i, $i \in [1, 2^m]$. Denote $D_i^- = I - D_i$. Clearly, $D_i^- \in \mathscr{D}$ if $D_i \in \mathscr{D}$.

Lemma 5 (see [13]). *Let $K, H \in R^{m\times n}$. Then, for any $x \in \mathscr{L}(H)$, it holds that*

$$\operatorname{sat}(Kx) \in \operatorname{co}\left\{D_i Kx + D_i^- Hx,\ i \in \left[1, 2^m\right]\right\}, \tag{7}$$

where co *stands for the convex hull.*

3. Main Results

3.1. Controller Design. This subsection will present a solution to Problem 1.

Theorem 6. *Given $\varepsilon_0 > 0$ and $\alpha^* > 0$, if there exist symmetric matrices $P_{11} \in R^{n_1*n_1}$, $P_{22} \in R^{n_2*n_2}$ and matrices $P_{12} \in R^{n_1*n_2}$, $Z_1 \in R^{m*n_1}$, $Z_2 \in R^{m*n_2}$, $Y_1 \in R^{m*n_1}$, $Y_2 \in R^{m*n_2}$, as well as a positive scalar η satisfying*

$$\begin{bmatrix} P_{22} & \phi_1^T & \phi_2^T & 0 \\ \star & -\phi_3-\phi_3^T & P_{12}-\phi_4^T & E_1 \\ \star & \star & P_{22} & E_2 \\ \star & \star & \star & \eta I \end{bmatrix} > 0, \quad i \in [1, 2^m], \tag{8}$$

$$\begin{bmatrix} \varepsilon_0 P_{11} & \varepsilon_0 P_{12} & \varepsilon_0 \phi_3^T & \varepsilon_0 \phi_4^T & 0 \\ \star & P_{22} & \varepsilon_0 P_{12}^T A_{11}^T + \phi_1^T & \varepsilon_0 P_{12}^T A_{21}^T + \phi_2^T & 0 \\ \star & \star & -\phi_3-\phi_3^T & P_{12}-\phi_4^T & E_1 \\ \star & \star & \star & P_{22} & E_2 \\ \star & \star & \star & \star & \eta I \end{bmatrix} > 0, \quad i \in [1, 2^m], \tag{9}$$

$$\begin{bmatrix} P_{22} & Y_{2(r)}^T \\ Y_{2(r)} & \dfrac{1}{1+\eta\alpha^*} \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m, \tag{10}$$

$$\begin{bmatrix} \varepsilon_0 P_{11} & \varepsilon_0 P_{12} & \varepsilon_0 Y_{1(r)}^T \\ \varepsilon_0 P_{12}^T & P_{22} & Y_{2(r)}^T \\ \varepsilon_0 Y_{1(r)}^T & Y_{2(r)} & \dfrac{1}{1+\eta\alpha^*} \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m, \tag{11}$$

$$\begin{aligned} \phi_1 &= A_{12}P_{22} + B_1 D_i Z_2 + B_1 D_i^- Y_2, \\ \phi_2 &= A_{22}P_{22} + B_2 D_i Z_2 + B_2 D_i^- Y_2, \\ \phi_3 &= A_{11}P_{11} + A_{12}P_{12}^T + B_1 D_i Z_1 + B_1 D_i^- Y_1, \\ \phi_4 &= A_{21}P_{11} + B_2 D_i Z_1 + B_2 D_i^- Y_1 + A_{22}P_{12}^T, \\ & \qquad i \in [1, 2^m], \end{aligned} \tag{12}$$

then, for any $\varepsilon \in (0, \varepsilon_0]$ and $w \in W_{\alpha^*}^2$, all of the trajectories of the closed-loop system (5) starting within $\Omega(P^{-1}(\varepsilon), 1)$ will remain inside of $\Omega(P^{-1}(\varepsilon), 1+\eta\alpha^*)$ with $P(\varepsilon) = \begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & P_{22} \end{bmatrix} > 0$.

And the state feedback controller gain matrix is given by

$$K_\varepsilon = [Z_1 \quad Z_2] \begin{bmatrix} P_{11} & \varepsilon P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}^{-1}. \tag{13}$$

Proof. From (10) and (11), it follows that, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} & \varepsilon Y_{1(r)}^T \\ \varepsilon P_{12}^T & P_{22} & Y_{2(r)}^T \\ \varepsilon Y_{1(r)}^T & Y_{2(r)} & \dfrac{1}{1+\eta\alpha^*} \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m. \tag{14}$$

Let $Z_\varepsilon = [\varepsilon Z_1 \quad Z_2]$, $Y_\varepsilon = [\varepsilon Y_1 \quad Y_2]$. Then (14) is equivalent to

$$\begin{bmatrix} P(\varepsilon) & Y_{\varepsilon(r)}^T \\ Y_{\varepsilon(r)} & \dfrac{1}{1+\eta\alpha^*} \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m, \tag{15}$$

which implies that

$$\begin{bmatrix} P^{-1}(\varepsilon) & H_{\varepsilon(r)}^T \\ H_{\varepsilon(r)} & \dfrac{1}{1+\eta\alpha^*} \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m, \tag{16}$$

where $H_\varepsilon = Y_\varepsilon P^{-1}(\varepsilon)$.

By Schur complement, it follows from (16) that

$$P^{-1}(\varepsilon) > H_{\varepsilon(r)}^T (1+\eta\alpha^*) H_{\varepsilon(r)}, \quad r = 1, 2, \ldots, m, \tag{17}$$

which implies that $\Omega(P^{-1}(\varepsilon), 1+\eta\alpha^*) \subseteq \mathscr{L}(H_\varepsilon)$, $\forall \varepsilon \in (0, \varepsilon_0]$. Then by Lemma 5, we have

$$\begin{aligned} & A_\varepsilon x(k) + B_\varepsilon \operatorname{sat}(K_\varepsilon x(k)) \\ & \quad \in \operatorname{co}\{A_\varepsilon x(k) + B_\varepsilon (D_i K_\varepsilon + D_i^- H_\varepsilon) x(k)\}, \\ & \qquad \forall \varepsilon \in (0, \varepsilon_0], \ i \in [1, 2^m]. \end{aligned} \tag{18}$$

From (8) and (9), it follows that

$$\begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} & \varepsilon \phi_3^T & \varepsilon \phi_4^T & 0 \\ \star & P_{22} & \varepsilon P_{12}^T A_{11}^T + \phi_1^T & \varepsilon P_{12}^T A_{21}^T + \phi_2^T & 0 \\ \star & \star & -\phi_3-\phi_3^T & P_{12}-\phi_4^T & E_1 \\ \star & \star & \star & P_{22} & E_2 \\ \star & \star & \star & \star & \eta I \end{bmatrix} > 0, \tag{19}$$

$$\forall \varepsilon \in (0, \varepsilon_0], \ i \in [1, 2^m].$$

Before and after multiplying (19) by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & \varepsilon I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{20}$$

and its transpose, respectively, we have

$$\begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} & \varepsilon^2\phi_3^T & \varepsilon\phi_4^T & 0 \\ \star & P_{22} & \varepsilon^2 P_{12}^T A_{11}^T + \varepsilon\phi_1^T & \varepsilon P_{12}^T A_{21}^T + \phi_2^T & 0 \\ \star & \star & \varepsilon^2\left(-\phi_3 - \phi_3^T\right) & \varepsilon P_{12} - \varepsilon\phi_4^T & \varepsilon E_1 \\ \star & \star & \star & P_{22} & E_2 \\ \star & \star & \star & \star & \eta I \end{bmatrix} > 0, \quad \forall\varepsilon \in (0,\varepsilon_0],\ i \in [1,2^m]. \tag{21}$$

Before and after multiplying (21) by the matrix

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{22}$$

and its transpose, respectively, we have

$$\begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} & \varepsilon P_{11} + \varepsilon^2\phi_3^T & \varepsilon\phi_4^T & 0 \\ \star & P_{22} & \varepsilon P_{12}^T + \varepsilon^2 P_{12}^T A_{11}^T + \varepsilon\phi_1^T & \varepsilon P_{12}^T A_{21}^T + \phi_2^T & 0 \\ \star & \star & \varepsilon P_{11} & \varepsilon P_{12} & \varepsilon E_1 \\ \star & \star & \star & P_{22} & E_2 \\ \star & \star & \star & \star & \eta I \end{bmatrix} > 0, \quad \forall\varepsilon \in (0,\varepsilon_0],\ i \in [1,2^m]. \tag{23}$$

Taking into account the definitions of A_ε, B_ε, E_ε, and $P(\varepsilon)$, we can rewrite (23) as

$$\begin{bmatrix} P(\varepsilon) & \left(A_\varepsilon P(\varepsilon) + B_\varepsilon D_i Z_\varepsilon + B_\varepsilon D_i^- Y_\varepsilon\right)^T & 0 \\ \star & P(\varepsilon) & E_\varepsilon \\ \star & \star & \eta I \end{bmatrix} > 0, \quad \forall\varepsilon \in (0,\varepsilon_0],\ i \in [1,2^m]. \tag{24}$$

Define

$$K_\varepsilon = Z_\varepsilon P^{-1}(\varepsilon). \tag{25}$$

Before and after multiplying (24) by diag($P^{-1}(\varepsilon)$, $P^{-1}(\varepsilon)$, I), we have

$$\begin{bmatrix} P^{-1}(\varepsilon) & \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right)^T P^{-1}(\varepsilon) & 0 \\ \star & P^{-1}(\varepsilon) & P^{-1}(\varepsilon) E_\varepsilon \\ \star & \star & \eta I \end{bmatrix} > 0, \quad \forall\varepsilon \in (0,\varepsilon_0],\ i \in [1,2^m]. \tag{26}$$

By Schur complement, inequality (26) is equivalent to

$$\begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_2^T & \Xi_3 - \eta I \end{bmatrix} < 0, \tag{27}$$

where

$$\begin{aligned} \Xi_1 &= \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right)^T P^{-1}(\varepsilon) \\ &\quad \cdot \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right) - P^{-1}(\varepsilon), \\ \Xi_2 &= \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right)^T P^{-1}(\varepsilon) E_\varepsilon, \\ \Xi_3 &= E_\varepsilon^T P^{-1}(\varepsilon) E_\varepsilon, \\ &\qquad \forall\varepsilon \in (0,\varepsilon_0],\ i \in [1,2^m]. \end{aligned} \tag{28}$$

Define an ε-dependent Lyapunov function:

$$V(x) = x^T P^{-1}(\varepsilon) x. \tag{29}$$

Calculating the difference of $V(k)$ along the trajectories of the closed-loop system (5), and using (18), we have

$$\begin{aligned} \Delta V(x) &= V(x(k+1)) - V(x(k)) = \left(A_\varepsilon x(k) \right. \\ &\quad \left. + B_\varepsilon \operatorname{sat}(u) + E_\varepsilon w(k)\right)^T P^{-1}(\varepsilon)\left(A_\varepsilon x(k) + B_\varepsilon \operatorname{sat}(u)\right. \\ &\quad \left. + E_\varepsilon w(k)\right) - x^T(k) P^{-1}(\varepsilon) x(k) \\ &\le \max_{i\in[1,2^m]} \left\{\left(A_\varepsilon x(k) + B_\varepsilon\left(D_i K_\varepsilon + D_i^- H_\varepsilon\right) x(k)\right.\right. \\ &\quad + E_\varepsilon w(k)\Big)^T P^{-1}(\varepsilon)\left(A_\varepsilon x(k) + B_\varepsilon\left(D_i K_\varepsilon + D_i^- H_\varepsilon\right)\right. \\ &\quad \left.\left. \cdot x(k) + E_\varepsilon w(k)\right) - x^T(k) P^{-1}(\varepsilon) x(k)\right\} \\ &= \max_{i\in[1,2^m]} \left\{x^T(k)\left[\left(A_\varepsilon + B_\varepsilon\left(D_i K_\varepsilon + D_i^{-1} H_\varepsilon\right)\right)^T\right.\right. \\ &\quad \left. \cdot P^{-1}(\varepsilon)\left(A_\varepsilon + B_\varepsilon\left(D_i K_\varepsilon + D_i^{-1} H_\varepsilon\right)\right) - P^{-1}(\varepsilon)\right] \\ &\quad \cdot x(k) + x^T(k)\left[A_\varepsilon + B_\varepsilon\left(D_i K_\varepsilon + D_i^{-1} H_\varepsilon\right)\right]^T \\ &\quad \cdot P^{-1}(\varepsilon) E_\varepsilon w(k) + w^T(k) E_\varepsilon^T P^{-1}(\varepsilon)\left[A_\varepsilon\right. \\ &\quad \left. + B_\varepsilon\left(D_i K_\varepsilon + D_i^{-1} H_\varepsilon\right)\right] x(k) + w^T(k) E_\varepsilon^T P^{-1}(\varepsilon) \\ &\quad \left. \cdot E_\varepsilon w(k)\right\}, \\ &\forall x(k) \in \Omega\left(P^{-1}(\varepsilon), 1 + \eta\alpha^*\right),\ \varepsilon \in (0,\varepsilon_0],\ w \in W_{\alpha^*}^2. \end{aligned} \tag{30}$$

It follows that

$$\begin{aligned} \Delta V(x) - \eta w^T(k) w(k) &= \xi^T(k)\begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_2^T & \Xi_3 - \eta I \end{bmatrix}\xi(k) \\ &\le 0, \\ \forall x(k) &\in \left(\Omega\left(P^{-1}(\varepsilon), 1 + \eta\alpha^*\right)\right),\ \varepsilon \in (0,\varepsilon_0],\ w \in W_{\alpha^*}^2, \end{aligned} \tag{31}$$

with

$$\xi(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \tag{32}$$

Then we have

$$\begin{aligned}&\Delta V(x) \le \eta w^T(k) w(k),\\&\forall x(k) \in \Omega\left(P^{-1}(\varepsilon), 1+\eta\alpha^*\right),\ \varepsilon \in (0,\varepsilon_0],\ w \in W^2_{\alpha^*}.\end{aligned} \tag{33}$$

Summing up both sides of (33) from 0 to m, we can get

$$\begin{aligned}V(x(m+1)) &\le V(x(0)) + \eta\sum_{k=0}^{m} w^T(k) w(k)\\&\le V(x(0)) + \alpha\eta,\\&\forall x(k) \in \Omega\left(P^{-1}(\varepsilon), 1+\eta\alpha^*\right),\ \varepsilon \in (0,\varepsilon_0],\ w \in W^2_{\alpha^*},\end{aligned} \tag{34}$$

which shows that when $x(0) \in \Omega(P^{-1}(\varepsilon), 1)$, we can get $V(x(m+1)) \le 1+\eta\alpha^*$, that is, $x(m+1) \in \Omega(P^{-1}(\varepsilon), 1+\eta\alpha^*)$.

And, according to (25), the controller gain matrix is

$$\begin{aligned}K_\varepsilon = Z_\varepsilon P^{-1}(\varepsilon) &= \begin{bmatrix}\varepsilon Z_1 & Z_2\end{bmatrix}\begin{bmatrix}\varepsilon P_{11} & \varepsilon P_{12}\\ \varepsilon P_{12}^T & P_{22}\end{bmatrix}^{-1}\\&= \begin{bmatrix}Z_1 & Z_2\end{bmatrix}\begin{bmatrix}P_{11} & \varepsilon P_{12}\\ P_{12}^T & P_{22}\end{bmatrix}^{-1}.\end{aligned} \tag{35}$$

□

Remark 7. According to (33), when the disturbance $w(k) = 0$, it holds that $\Delta V(x) < 0$, $\forall x \ne 0$, which means that the closed-loop system (5) is locally asymptotically stable. In this case, the ellipsoid $\Omega(P^{-1}(\varepsilon), 1)$ is an estimation of the basin of attraction of the closed-loop system. In addition, when ε is small enough, an ε-independent gain matrix can be computed by

$$\begin{aligned}K &= \lim_{\varepsilon\to 0^+}\begin{bmatrix}Z_1 & Z_2\end{bmatrix}\begin{bmatrix}P_{11} & \varepsilon P_{12}\\ P_{12}^T & P_{22}\end{bmatrix}^{-1}\\&= \begin{bmatrix}Z_1 & Z_2\end{bmatrix}\begin{bmatrix}P_{11} & 0\\ P_{12}^T & P_{22}\end{bmatrix}^{-1}\\&= \left[\left(Z_1 - Z_2P_{22}^{-1}P_{12}^T\right)P_{11}^{-1} \quad Z_2P_{22}^T\right].\end{aligned} \tag{36}$$

3.2. Disturbance Tolerance. The ability of the closed-loop system to tolerate the disturbance is characterized by α^*. Based on Theorem 6, we have the following corollary which can be used to maximize α^*.

Corollary 8. *Given $\varepsilon_0 > 0$ and $\alpha > 0$, if there exist symmetric matrices $P_{11} \in R^{n_1*n_1}$, $P_{22} \in R^{n_2*n_2}$ and matrices $P_{12} \in R^{n_1*n_2}$, $Z_1 \in R^{m*n_1}$, $Z_2 \in R^{m*n_2}$, $Y_1 \in R^{m*n_1}$, $Y_2 \in R^{m*n_2}$ satisfying*

$$\begin{bmatrix}P_{22} & \phi_1^T & \phi_2^T & 0\\ \star & -\phi_3-\phi_3^T & P_{12}-\phi_4^T & E_1\\ \star & \star & P_{22} & E_2\\ \star & \star & \star & I\end{bmatrix} > 0, \quad i \in [1, 2^m], \tag{37}$$

$$\begin{bmatrix}\varepsilon_0P_{11} & \varepsilon_0P_{12} & \varepsilon_0\phi_3^T & \varepsilon_0\phi_4^T & 0\\ \star & P_{22} & \varepsilon_0P_{12}^TA_{11}^T+\phi_1^T & \varepsilon_0P_{12}^TA_{21}^T+\phi_2^T & 0\\ \star & \star & -\phi_3-\phi_3^T & P_{12}-\phi_4^T & E_1\\ \star & \star & \star & P_{22} & E_2\\ \star & \star & \star & \star & I\end{bmatrix} > 0, \quad i \in [1, 2^m], \tag{38}$$

$$\begin{bmatrix}P_{22} & Y_{2(r)}\\ Y_{2(r)}^T & \dfrac{1}{\alpha}\end{bmatrix} > 0, \tag{39}$$

$$\begin{bmatrix}\varepsilon_0P_{11} & \varepsilon_0P_{12} & \varepsilon_0Y_{1(r)}\\ \varepsilon_0P_{12}^T & P_{22} & Y_{2(r)}\\ \varepsilon_0Y_{1(r)}^T & Y_{2(r)}^T & \dfrac{1}{\alpha}\end{bmatrix} > 0, \tag{40}$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are defined in Theorem 6, then all the trajectories of the closed-loop system (5) starting from the origin will still remain inside of $\Omega(P^{-1}(\varepsilon), \alpha)$. And the state feedback controller gain matrix is given by

$$K_\varepsilon = \begin{bmatrix}Z_1 & Z_2\end{bmatrix}\begin{bmatrix}P_{11} & \varepsilon P_{12}\\ P_{12}^T & P_{22}\end{bmatrix}^{-1}. \tag{41}$$

Proof. According to (39) and (40), for any $\varepsilon \in (0, \varepsilon_0]$, we have

$$\begin{bmatrix}\varepsilon P_{11} & \varepsilon P_{12} & \varepsilon Y_{1(r)}\\ \varepsilon P_{12}^T & P_{22} & Y_{2(r)}\\ \varepsilon Y_{1(r)}^T & Y_{2(r)}^T & \dfrac{1}{\alpha}\end{bmatrix} > 0, \tag{42}$$

which is equivalent to

$$\begin{bmatrix}P^{-1}(\varepsilon) & H_{\varepsilon(r)}^T\\ H_{\varepsilon(r)} & \dfrac{1}{\alpha}\end{bmatrix} > 0, \quad r = 1, 2, \ldots, m. \tag{43}$$

By Schur complement, it follows from (43) that

$$P^{-1}(\varepsilon) > H_{\varepsilon(r)}^T\alpha H_{\varepsilon(r)}, \quad r = 1, 2, \ldots, m; \tag{44}$$

thus $\Omega(P^{-1}(\varepsilon), \alpha) \subseteq L(H_\varepsilon)$, $\forall\varepsilon \in (0, \varepsilon_0]$.

Similarly to (34) in the proof for Theorem 6, for the trajectories start from the origin, we have

$$\begin{aligned}V(x(m+1)) &\le V(0) + \sum_{k=0}^{m} w^T(k) w(k) \le V(0) + \alpha\\&= \alpha.\end{aligned} \tag{45}$$

This complete the proof. □

From Corollary 8, the bigger α means the better disturbance tolerance ability. To get the best disturbance tolerance ability we formulate the following optimization problem:

$$\begin{aligned}\max_{P_{11},P_{12},P_{22},Z_1,Z_2,Y_1,Y_2} \quad & \alpha\\ \text{s.t.} \quad & (37), (38), (39), (40),\end{aligned} \tag{46}$$

where $\Omega(S, 1)$ represents the initial condition set.

Let $\mu = 1/\alpha$. Then inequalities (39) and (40) can be rewritten as

$$\begin{bmatrix} P_{22} & Y_{2(r)}^T \\ Y_{2(r)} & \mu \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m,$$
$$\begin{bmatrix} \varepsilon_0 P_{11} & \varepsilon_0 P_{12} & \varepsilon_0 Y_{1(r)}^T \\ \varepsilon_0 P_{12}^T & P_{22} & Y_{2(r)}^T \\ \varepsilon_0 Y_{1(r)}^T & Y_{2(r)} & \mu \end{bmatrix} > 0, \quad r = 1, 2, \ldots, m. \tag{47}$$

Then the optimization problem (46) can be converted into

$$\min_{P_{11}, P_{12}, P_{22}, Z_1, Z_2, Y_1, Y_2} \quad \mu$$
$$\text{s.t.} \quad (37), (38), (47). \tag{48}$$

Solving the optimization problem (48) yields the minimal value μ^*. Then the largest disturbance that can be tolerated by the closed-loop system at zero initial condition is bounded by $\alpha^* = 1/\mu^*$. Therefore, Problem 2 can be solved by the optimization problem (48).

3.3. Disturbance Rejection. Problem 3 will be considered in this subsection.

Theorem 9. *Given $\varepsilon_0 > 0$, $\alpha^* > 0$, and $\gamma > 0$, if there exist symmetric matrices $P_{11} \in R^{n_1 * n_1}$, $P_{22} \in R^{n_2 * n_2}$ and matrices $P_{12} \in R^{n_1 * n_2}$, $Z_1 \in R^{m * n_1}$, $Z_2 \in R^{m * n_2}$, $Y_1 \in R^{m * n_1}$, $Y_2 \in R^{m * n_2}$ satisfying*

$$\begin{bmatrix} P_{22} & \phi_1^T & \phi_2^T & 0 & P_{22} C_2^T \\ \star & -\phi_3 - \phi_3^T & P_{12} - \phi_4^T & E_1 & -\phi_5^T \\ \star & \star & P_{22} & E_2 & 0 \\ \star & \star & \star & \gamma^2 I & 0 \\ \star & \star & \star & \star & I \end{bmatrix} > 0, \tag{49}$$

$$\begin{bmatrix} \varepsilon_0 P_{11} & \varepsilon_0 P_{12} & \varepsilon_0 \phi_3^T & \varepsilon_0 \phi_4^T & 0 & \varepsilon_0 \phi_5^T \\ \star & P_{22} & \varepsilon_0 P_{12}^T A_{11}^T + \phi_1^T & \varepsilon_0 P_{12}^T A_{21}^T + \phi_2^T & 0 & \varepsilon_0 P_{12}^T C_1^T + P_{22} C_2^T \\ \star & \star & -\phi_3 - \phi_3^T & P_{12} - \phi_4^T & E_1 & -\phi_5^T \\ \star & \star & \star & P_{22} & E_2 & 0 \\ \star & \star & \star & \star & \gamma^2 I & 0 \\ \star & \star & \star & \star & \star & I \end{bmatrix} > 0, \quad i \in [1, 2^m], \tag{50}$$

$$\begin{bmatrix} P_{22} & Y_2 \\ Y_2^T & \dfrac{1}{\alpha^*} \end{bmatrix} > 0, \tag{51}$$

$$\begin{bmatrix} \varepsilon_0 P_{11} & \varepsilon_0 P_{12} & \varepsilon_0 Y_1 \\ \varepsilon_0 P_{12}^T & P_{22} & Y_2 \\ \varepsilon_0 Y_1^T & Y_2^T & \dfrac{1}{\alpha^*} \end{bmatrix} > 0, \tag{52}$$

where

$$\begin{aligned} \phi_1 &= A_{12} P_{22} + B_1 D_i Z_2 + B_1 D_i^- Y_2, \\ \phi_2 &= A_{22} P_{22} + B_2 D_i Z_2 + B_2 D_i^- Y_2, \\ \phi_3 &= A_{11} P_{11} + A_{12} P_{12}^T + B_1 D_i Z_1 + B_1 D_i^- Y_1, \\ \phi_4 &= A_{21} P_{11} + B_2 D_i Z_1 + B_2 D_i^- Y_1 + A_{22} P_{12}^T, \\ & \qquad i \in [1, 2^m], \\ \phi_5 &= C_1 P_{11} + C_2 P_{12}, \end{aligned} \tag{53}$$

then the L_2 gain from w to z of the closed-loop system (5) with $x(0) = 0$ is less than γ. And the state feedback controller gain matrix is given by

$$K_\varepsilon = [Z_1 \;\; Z_2] \begin{bmatrix} P_{11} & \varepsilon P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}^{-1}. \tag{54}$$

Proof. Let $P(\varepsilon) = \begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} \\ \varepsilon P_{12}^T & P_{22} \end{bmatrix} > 0$, $Z_\varepsilon = [\varepsilon Z_1 \;\; Z_2]$, $Y_\varepsilon = [\varepsilon Y_1 \;\; Y_2]$, $H_\varepsilon = Y_\varepsilon P^{-1}(\varepsilon)$, $K_\varepsilon = Z_\varepsilon P^{-1}(\varepsilon)$.

Similarly to the proof for Theorem 6, it follows from (49) and (50) that

$$\begin{bmatrix} P(\varepsilon) & \left(A_\varepsilon P(\varepsilon) + B_\varepsilon D_i Z_\varepsilon + B_\varepsilon D_i^- Y_\varepsilon\right)^T & 0 & P(\varepsilon) C^T \\ \star & P(\varepsilon) & E_\varepsilon & 0 \\ \star & \star & \gamma^2 I & 0 \\ \star & \star & \star & I \end{bmatrix} > 0, \quad \forall \varepsilon \in (0, \varepsilon_0], \ i \in [1, 2^m]. \tag{55}$$

Applying the Schur complement formula to (55), we have

$$\begin{bmatrix} \Xi_1 + C^T C & \Xi_2 \\ \Xi_2^T & \Xi_3 - \gamma^2 I \end{bmatrix} < 0, \tag{56}$$

where

$$\begin{aligned} \Xi_1 &= \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right)^T P^{-1}(\varepsilon) \\ &\quad \cdot \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right) - P^{-1}(\varepsilon), \\ \Xi_2 &= \left(A_\varepsilon + B_\varepsilon D_i K_\varepsilon + B_\varepsilon D_i^- H_\varepsilon\right)^T P^{-1}(\varepsilon) E_\varepsilon, \\ \Xi_3 &= E_\varepsilon^T P^{-1}(\varepsilon) E_\varepsilon, \\ &\qquad \forall \varepsilon \in (0, \varepsilon_0], \ i \in [1, 2^m]. \end{aligned} \tag{57}$$

From (51) and (52), for any $\varepsilon \in (0, \varepsilon_0]$, it follows that

$$\begin{bmatrix} \varepsilon P_{11} & \varepsilon P_{12} & \varepsilon Y_1 \\ \varepsilon P_{12}^T & P_{22} & Y_2 \\ \varepsilon Y_1^T & Y_2^T & \dfrac{1}{\alpha^*} \end{bmatrix} > 0. \tag{58}$$

From the proof for Theorem 6, we can get

$$P^{-1}(\varepsilon) > H_{\varepsilon(r)}^T \alpha H_{\varepsilon(r)}, \quad r = 1, 2, \ldots, m, \tag{59}$$

which shows that $\Omega(P^{-1}(\varepsilon), \alpha^*) \subseteq \mathscr{L}(H_\varepsilon)$, $\forall \varepsilon \in (0, \varepsilon_0]$.

Define an ε-dependent Lyapunov function

$$V(x) = x^T P^{-1}(\varepsilon) x. \tag{60}$$

Similarly to proof for Theorem 6, it follows from (56) that

$$\Delta V(x(k)) + z^T(k) z(k) - \gamma^2 w^T(k) w(k) < 0, \quad \forall x \in \Omega\left(P^{-1}(\varepsilon), \alpha^*\right). \tag{61}$$

Then, summing up left and right of (61), respectively, with $x(0) = 0$, yields that

$$V(x(m+1)) - 0 + \sum_{k=0}^{m} \left(z^T(k) z(k) - \gamma^2 w^T(k) w(k)\right) < 0, \tag{62}$$

which implies

$$\sum_{k=0}^{m} z^T(k) z(k) < \gamma^2 \sum_{k=0}^{j} w^T(k) w(k). \tag{63}$$

Thus the L_2 gain from w to z of the closed-loop system with $x(0) = 0$ is less than γ. This completes the proof. □

By Theorem 9, the minimal L_2 gain can be obtained by solving the following optimization problem:

$$\begin{aligned} \min_{P_{11}, P_{12}, P_{22}, Z_1, Z_2, Y_1, Y_2} \quad & \gamma^2 \\ \text{s.t.} \quad & (49), (50), (51), (52). \end{aligned} \tag{64}$$

Remark 10. As mentioned in Section 1, discretization of a continuous-time SPS can lead to different discrete-time models depending on the sampling rate. Since the structure of fast sampling models is different from that of slow sampling models, it is not easy to generalize the proposed results to slow sampling control of SPSs, as will be considered in our future work.

4. Examples

This section will illustrate the proposed results by an example.

Consider an inverted pendulum system controlled by DC motor via a gear train. The model, which was first established in [35], is described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{g}{l} \sin x_1(t) + \frac{N K_m}{m l^2} x_3(t), \\ L_a \dot{x}_3(t) &= -K_b N x_2(t) - R_a x_3(t) + u(t) + w(t), \end{aligned} \tag{65}$$

where $x_1(t) = \theta_p(t)$ denotes the the angle (rad) of the pendulum from the vertical upward, $x_2(t) = \dot{\theta}_p(t)$, $x_3(t) = I_a(t)$ denotes the current of the motor, $u(t)$ is the control input voltage, $w(t)$ is the disturbance, K_m is the motor torque constant, K_b is the back emf constant, N is the gear ratio, and L_a is the inductance which is usually a small positive constant. The parameters for the plant are as follows: $g = 9.8\,\mathrm{m/s^2}$, $N = 10$, $l = 1\,\mathrm{m}$, $m = 1\,\mathrm{kg}$, $K_m = 0.1\,\mathrm{Nm/A}$, $K_b = 0.1\,\mathrm{Vs/rad}$, $R_a = 1\,\Omega$, and $L_a = 0.05\,\mathrm{H}$ and the input voltage is required to satisfy $|u| \le 1$. Note that L_a represents the singular perturbation parameter of the system. Substituting the parameters into (65) and linearizing the equations, we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= 9.8 x_1(t) + x_3(t), \\ \varepsilon \dot{x}_3(t) &= -x_2(t) - x_3(t) + u(t) + w(t), \end{aligned} \tag{66}$$

where $\varepsilon = L_a$.

The equilibrium point of system (66), that is, $x_e = [0 \ 0 \ 0]^T$, corresponds to the upright rest position of the inverted pendulum. We will design a controller to balance the pendulum around its upright rest position.

According to [36], we choose the sampling period as $T_f = \alpha_f \varepsilon$, where $\alpha_f = 0.1$, $\varepsilon = 0.05$. Then the fast sampling discrete-time model of system (66) is in the form of (1) with

$$\begin{aligned} A_{11} &= \begin{bmatrix} 0 & 0.1 \\ 0.98 & -0.0048 \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} 0 \\ 0.0952 \end{bmatrix}, \\ A_{22} &= 0.9048, \\ B_1 &= \begin{bmatrix} 0 \\ 0.0048 \end{bmatrix}, \\ B_2 &= 0.0952, \\ E_1 &= \begin{bmatrix} 0 \\ 0.0048 \end{bmatrix}, \\ E_2 &= 0.0952, \\ C_1 &= \begin{bmatrix} 1 & 1 \end{bmatrix}, \\ C_2 &= 1. \end{aligned} \tag{67}$$

Solving the LMIs of Theorem 6 with $\varepsilon_0 = 0.05$, $\alpha^* = 0.2$, and $\eta = 1$, we have

$$\begin{aligned} P_{11} &= \begin{bmatrix} 23.4260 & -52.6506 \\ -52.6506 & 157.2814 \end{bmatrix}, \\ P_{12} &= \begin{bmatrix} -92.2040 \\ 68.0912 \end{bmatrix}, \\ P_{22} &= 92.9218, \\ Z_1 &= \begin{bmatrix} -152.3819 & 187.0950 \end{bmatrix}, \\ Z_2 &= -491.8613, \\ Y_1 &= \begin{bmatrix} 0.1739 & -0.1772 \end{bmatrix}, \\ Y_2 &= -0.1620. \end{aligned} \tag{68}$$

Choosing $\varepsilon = 0.05 \in (0, \varepsilon_0]$, then we have

$$\begin{aligned} P(\varepsilon) &= \begin{bmatrix} 1.1713 & -2.6325 & -4.6102 \\ -2.6325 & 7.8641 & 3.4046 \\ -4.6102 & 3.4046 & 92.9218 \end{bmatrix}, \\ Z_\varepsilon &= \begin{bmatrix} -7.6191 & 9.3548 & -491.8613 \end{bmatrix}, \\ Y_\varepsilon &= \begin{bmatrix} 0.0087 & -0.0089 & -0.1620 \end{bmatrix}. \end{aligned} \tag{69}$$

Then we can calculate the state feedback controller gain:

$$K_\varepsilon = \begin{bmatrix} -147.2017 & -43.3206 & -11.0093 \end{bmatrix}. \tag{70}$$

Solving the optimization problem (48) with $\varepsilon_0 = 0.05$, then we get

$$\begin{aligned} P_{11} &= \begin{bmatrix} 0.0097 & -0.0277 \\ -0.0277 & 0.1043 \end{bmatrix}, \\ P_{12} &= \begin{bmatrix} -0.0122 \\ -0.0551 \end{bmatrix}, \\ P_{22} &= 0.0710, \\ Z_1 &= \begin{bmatrix} -0.0748 & 0.0910 \end{bmatrix}, \\ Z_2 &= -0.0492, \\ Y_1 &= \begin{bmatrix} -0.0442 & -0.0299 \end{bmatrix}, \\ Y_2 &= 0.0133. \end{aligned} \tag{71}$$

Choosing $\varepsilon = 0.05 \in (0, \varepsilon_0]$, we have

$$\begin{aligned} P(\varepsilon) &= \begin{bmatrix} 0.0005 & -0.0014 & -0.0006 \\ -0.0014 & 0.0052 & -0.0028 \\ -0.0006 & -0.0028 & 0.0710 \end{bmatrix}, \\ Z_\varepsilon &= \begin{bmatrix} -0.0037 & 0.0045 & -0.0492 \end{bmatrix}, \\ Y_\varepsilon &= \begin{bmatrix} -0.0022 & -0.0015 & 0.0133 \end{bmatrix}, \\ K_\varepsilon &= \begin{bmatrix} -37.7656 & -9.9051 & -1.4045 \end{bmatrix}, \end{aligned} \tag{72}$$

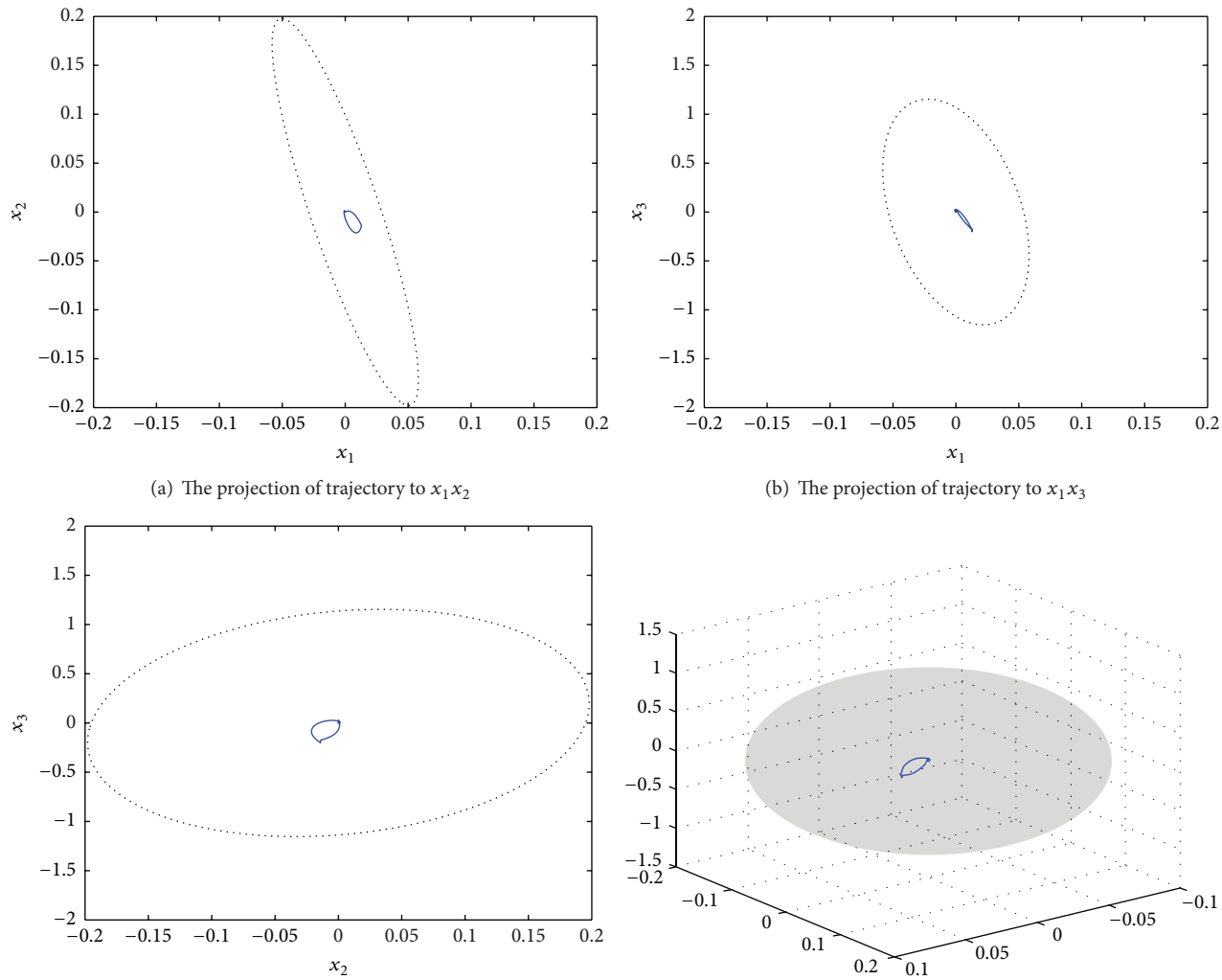
and $\mu = 0.0641$, which means the capacity of disturbance tolerance of the system is $\alpha^* = 15.5997$.

To improve the ability of disturbance rejection, solving the optimization problem (64) with $\varepsilon_0 = 0.05$, $\alpha^* = 15$, we have

$$\begin{aligned} P_{11} &= \begin{bmatrix} 0.0045 & -0.0133 \\ -0.0133 & 0.0522 \end{bmatrix}, \\ P_{12} &= \begin{bmatrix} -0.0338 \\ 0.0513 \end{bmatrix}, \\ P_{22} &= 0.0888, \\ Z_1 &= \begin{bmatrix} -0.0479 & 0.0363 \end{bmatrix}, \\ Z_2 &= -0.6921, \\ Y_1 &= \begin{bmatrix} -0.0238 & -0.0405 \end{bmatrix}, \\ Y_2 &= -0.0004. \end{aligned} \tag{73}$$

Choosing $\varepsilon = 0.05 \in (0, \varepsilon_0]$, we have

$$P(\varepsilon) = \begin{bmatrix} 0.0002 & -0.0007 & -0.0017 \\ -0.0007 & 0.0026 & 0.0026 \\ -0.0017 & 0.0026 & 0.0888 \end{bmatrix},$$



(a) The projection of trajectory to x_1x_2

(b) The projection of trajectory to x_1x_3

(c) The projection of trajectory to x_2x_3

(d) The trajectory starting from the origin

Figure 1: The ellipsoid $\Omega(P^{-1}(\varepsilon), \alpha)$ and the trajectory starting from the origin.

$$Z_\varepsilon = [-0.0024 \quad 0.0018 \quad -0.6921],$$
$$Y_\varepsilon = [-0.0012 \quad -0.0020 \quad -0.0004],$$
$$K_\varepsilon = [-234.9799 \quad -48.4215 \quad -10.8701], \tag{74}$$

and $\gamma = 1.0369$. Thus the L_2 gain from w to z of the closed-loop system with $x(0) = 0$ is less than 1.0369.

Define a piecewise function as follows:

$$v(k) = \begin{cases} 0.3 & \text{if } 1 \le k \le 100 \\ 0 & \text{if } k > 100. \end{cases} \tag{75}$$

Given $\alpha^* = 15 < 15.5997$ and $\varepsilon_0 = 0.05$, it is easy to see that the disturbance $w(k) = v(k) \in W^2_{\alpha^*}$. As shown in Figure 1, under controller (4), with

$$K_\varepsilon = [-234.9799 \quad -48.4215 \quad -10.8701] \tag{76}$$

the trajectory of the closed-loop system with the disturbance $\omega(k) = v(k)$ starting from the origin is bounded and converges to the origin when the disturbance disappears.

5. Conclusion

This paper investigated the problem of fast sampling control for singularly perturbed systems subject to actuator saturation and L_2 disturbance. A state feedback controller method was proposed such that all the trajectories of the closed-loop system starting from a bounded set will remain bounded for any singular perturbation parameter less than or equal to a predefined upper bound. Convex optimization problems were formulated to optimize the ability of disturbance tolerance and disturbance rejection, respectively. The presented example has illustrated the significance and validity of the proposed approaches.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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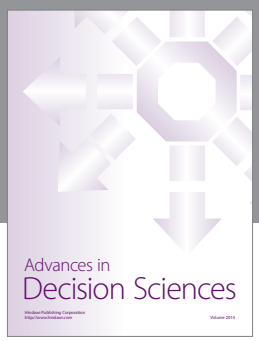

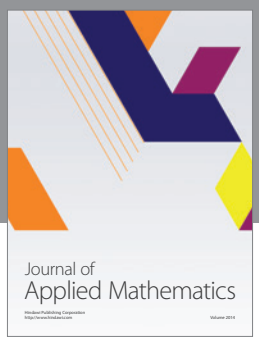

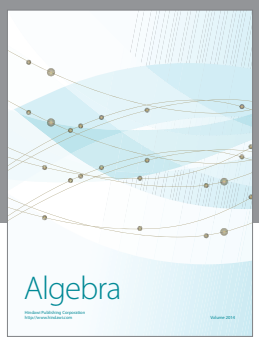

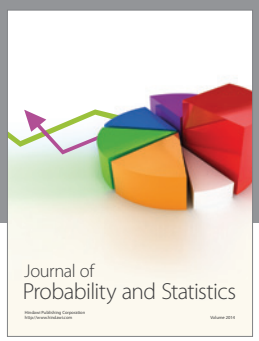

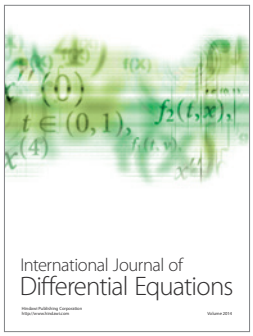

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