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Research Article

Smoothing Analysis of Distributive Red-Black Jacobi Relaxation for Solving 2D Stokes Flow by Multigrid Method

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Smoothing analysis process of distributive red-black Jacobi relaxation in multigrid method for solving 2D Stokes flow is mainly investigated on the nonstaggered grid by using local Fourier analysis (LFA). For multigrid relaxation, the nonstaggered discretizing scheme of Stokes flow is generally stabilized by adding an artificial pressure term. Therefore, an important problem is how to determine the zone of parameter in adding artificial pressure term in order to make stabilization of the algorithm for multigrid relaxation. To end that, a distributive red-black Jacobi relaxation technique for the 2D Stokes flow is established. According to the $2h$-harmonics invariant subspaces in LFA, the Fourier representation of the distributive red-black Jacobi relaxation for discretizing Stokes flow is given by the form of square matrix, whose eigenvalues are meanwhile analytically computed. Based on optimal one-stage relaxation, a mathematical relation of the parameter in artificial pressure term between the optimal relaxation parameter and related smoothing factor is well yielded. The analysis results show that the numerical schemes for solving 2D Stokes flow by multigrid method on the distributive red-black Jacobi relaxation have a specified convergence parameter zone of the added artificial pressure term.

1. Introduction

Multigrid methods [1–7] are generally considered as one of the fastest numerical methods which have an optimally computational complexity for solving partial differential equations (PDEs), especially for 3D steady incompressible Newtonian flow governed by Navier-Stokes equations.

In multigrid methods, smoothing relaxations play an important role. Several multigrid relaxation methods were developed for solving PDEs, which are roughly classified into two categories, collective and decoupled relaxations [8]. The collective relaxations are considered as a straightforward generalization of the scalar case [2]. The early decoupled relaxation is on a distributive Gauss-Seidel relaxation [9]. Gradually, it is generalized to an incomplete LU factorization relaxation [10]. Recently, Stokes system with distributive Gauss-Seidel relaxation based on the least squares commutator has been researched [11]. Much of the relaxations for Stokes system is seen in [12, 13].

For multigrid methods, LFA is a very useful tool to design efficient algorithms and to predict convergence factors for solving PDEs with high order accuracy [1–7]. Distributive relaxation for poroelasticity equations is optimized by LFA [14]. Using LFA, textbook efficiency multigrid solver for compressible Navier-Stokes equations is designed [15]. All-at-once multigrid approach for optimality systems with LFA is discussed in detail, and an analytical expression of the convergence factors is given by using symbolic computation [16–18].

The smoothing analysis of the distributive relaxations for solving 2D Stokes flow is investigated with LFA. As we know, the discretizing Stokes flow in computational domain is not stable by means of standard central differencing on nonstaggered grid. Thus, in order to overcome the stability

problem, an artificial pressure term is generally added by the method in [1, 2]. The optimal one-stage relaxation parameter and related smoothing factor of the distributive relaxation with the red-black Jacobi point relaxation need to be developed. In deriving an explicit formulation of the smoothing factor for the multigrid method, the symbolic operation process is carried out by using the MATLAB and Mathematica software, especially, by the cylindrical algebraic decomposition (CAD) function in the Mathematica build-in command [19].

2. Discretizing Stokes Flow and LFA

2.1. Discrete Stokes Flow. 3D steady incompressible Newtonian flow governed by Navier-Stokes equations is given as

$$\begin{aligned} -\Delta\vec{u} + \vec{\nabla} p &= \vec{f} \quad (x,y,z) \in \Omega, \\ \vec{\nabla}\cdot\vec{u} &= 0 \quad (x,y,z) \in \Omega, \\ \vec{u} &= \vec{g} \quad (x,y,z) \in \partial\Omega, \end{aligned} \tag{1}$$

where $\vec{u} = (u(x,y,z), v(x,y,z), w(x,y,z))$ is the velocity field, $p = p(x,y,z)$ represents the pressure, $\vec{f} = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ is the external force field, $(x,y,z) \in \Omega \subseteq \mathbb{R}^3$, and $\partial\Omega$ is the Dirichlet boundary of the computing domain. From (1), 2D Stokes operator is written as

$$L = \begin{pmatrix} -\Delta & 0 & \partial_x \\ 0 & -\Delta & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \tag{2}$$

on nonstaggered grid

$$G_h = \left\{\vec{x} = (x,y) := (k_1 h, k_2 h) \mid (k_1,k_2) \in \mathbb{Z}^2\right\}. \tag{3}$$

Discretizing Stokes operator (2) by means of standard central differencing is given as

$$L_h' = \begin{pmatrix} -\Delta_h & 0 & \partial_x^h \\ 0 & -\Delta_h & \partial_y^h \\ \partial_x^h & \partial_y^h & 0 \end{pmatrix}, \tag{4}$$

where h denotes the uniform mesh size and $-\Delta_h$, ∂_x^h, and ∂_y^h are the second-order difference operator with the following discrete stencils:

$$\begin{aligned} -\Delta_h &= \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h, \\ \partial_x^h &= \frac{1}{2h}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}_h, \qquad \partial_y^h = \frac{1}{2h}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_h. \end{aligned} \tag{5}$$

From [1], the above nonstaggered schemes (4) are not stable. Stabilization may be achieved by adding an artificial elliptic pressure term $-ch^2\Delta_h$ to the continuity equation in (2) [1, 2, 6]. With discrete operator in (5) and parameter c, the discrete Stokes operator is changed to

$$L_h = \begin{pmatrix} -\Delta_h & 0 & \partial_x^h \\ 0 & -\Delta_h & \partial_y^h \\ \partial_x^h & \partial_y^h & -ch^2\Delta_h \end{pmatrix}. \tag{6}$$

2.2. Elements of LFA in Multigrid. In LFA, a current approximation and its corresponding error and residual are represented by a linear combination of certain exponential functions, for example, Fourier modes, which form a unitary basis in space on a bounded infinite grid functions [1–7].

From [1, 2], on nonstaggered grid (3), a unitary basis of the Fourier modes is defined by

$$\varphi_h\left(\vec{\theta}, \vec{x}\right) := \exp\left(\frac{i\vec{\theta}\cdot\vec{x}}{h}\right), \tag{7}$$

in which $\vec{\theta} = (\theta_1, \theta_2) \in \Theta := (-\pi, \pi]^2$ is called Fourier frequency, $\vec{x} \in G_h$, and complex unit $i = \sqrt{-1}$. Thus, a Fourier space is yielded as

$$F\left(\vec{\theta}\right) := \operatorname{span}\left\{\varphi_h\left(\vec{\theta}, \vec{x}\right) \mid \vec{\theta} \in \Theta\right\}. \tag{8}$$

From [1–7], applying (3) and (7), for 2D scalar discrete operator D_h with discrete stencil

$$D_h = \left[l_{\vec{k}}\right]_h, \tag{9}$$

where $l_{\vec{k}} \in \mathbb{R}$ and $\vec{k} \in J \subset \mathbb{Z}^2$ containing $(0,0)$; the Fourier mode of (9) is defined by

$$\widetilde{D}_h\left(\vec{\theta}\right) := \sum_{\vec{k}\in J} l_{\vec{k}} \exp\left(i\vec{\theta}\cdot\vec{k}\right), \tag{10}$$

with $\vec{\theta}\cdot\vec{k} = \theta_1 k_1 + \theta_2 k_2$, subjected to

$$D_h\varphi_h\left(\vec{\theta}, \vec{x}\right) = \widetilde{D}_h\left(\vec{\theta}\right)\varphi_h\left(\vec{\theta}, \vec{x}\right). \tag{11}$$

A main idea of LFA is to analyze relaxation properties in multigrid for (6) by evaluating their effects on the Fourier components. From [2, 14, 16], if standard coarsening in 2D is selected, each low frequency $\vec{\theta} = \vec{\theta}^{00} \in \Theta_{\text{low}}^{2h} = (-\pi/2, \pi/2]^2$ is coupled with three high frequencies $\{\vec{\theta}^{11}, \vec{\theta}^{10}, \vec{\theta}^{01}\} \in \Theta_{\text{high}}^{2h}$ in the transition from G_h to G_{2h}, where $\Theta_{\text{high}}^{2h} = \Theta \backslash \Theta_{\text{low}}^{2h}$, and

$$\vec{\theta}^{\vec{\alpha}} = \vec{\theta} - \left(\alpha_1 \operatorname{sign}(\theta_1), \alpha_2 \operatorname{sign}(\theta_2)\right)\pi, \tag{12}$$

where $\vec{\alpha} \in \Lambda = \{00, 11, 10, 01\}$ and $\vec{\alpha} = (\alpha_1, \alpha_2)$ are denoted by $(\alpha_1, \alpha_2) := \alpha_1\alpha_2$. In this paper, standard coarsening is to

be assumed. Then, the Fourier space (8) is subdivided into $2h$-harmonics subspaces

$$F_{2h}\left(\vec{\theta}\right) := \operatorname{span}\left\{\varphi_h\left(\vec{\theta}^{00},\vec{x}\right),\varphi_h\left(\vec{\theta}^{11},\vec{x}\right),\varphi_h\left(\vec{\theta}^{10},\vec{x}\right),\varphi_h\left(\vec{\theta}^{01},\vec{x}\right)\right\}. \tag{13}$$

3. Smoothing Process

3.1. Distributive Relaxation of System (6). From [1, 2, 7], a distributive operator for the discrete system (6) is constructed as

$$C_h = \begin{pmatrix} I_h & 0 & -\partial_x^h \\ 0 & I_h & -\partial_y^h \\ 0 & 0 & -\Delta_h \end{pmatrix}, \tag{14}$$

where I_h is the unit operator with discrete stencil $[1]_h$. From (14), the discrete system (6) is transformed as

$$L_h C_h = \begin{pmatrix} -\Delta_h & 0 & 0 \\ 0 & -\Delta_h & 0 \\ \partial_x^h & \partial_y^h & ch^2\Delta_h^2 - \Delta_{2h} \end{pmatrix}, \tag{15}$$

where the discrete stencils of Δ_h^2 and $-\Delta_{2h}$ are

$$\Delta_h^2 = \frac{1}{h^4}\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix}_h,$$
$$-\Delta_{2h} = \frac{1}{4h^2}\begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}_h = \frac{1}{4h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_{2h}. \tag{16}$$

From (9)–(11), the Fourier modes of the scalar discrete operators of (16) are

$$\tilde{\Delta}_h^2\left(\vec{\theta}\right) = \left(-\tilde{\Delta}_h\left(\vec{\theta}\right)\right)^2,$$
$$-\tilde{\Delta}_{2h}\left(\vec{\theta}\right) = -\left[\tilde{\partial}_x^h\left(\vec{\theta}\right)\right]^2 - \left[\tilde{\partial}_y^h\left(\vec{\theta}\right)\right]^2, \tag{17}$$

where

$$-\tilde{\Delta}_h\left(\vec{\theta}\right) = \frac{1}{h^2}\left(4 - \exp\left(-i\theta_1\right) - \exp\left(i\theta_1\right) - \exp\left(-i\theta_2\right) - \exp\left(i\theta_2\right)\right) = \frac{1}{h^2}\left(4 - 2\cos\theta_1 - 2\cos\theta_2\right), \tag{18}$$

$$\tilde{\partial}_x^h\left(\vec{\theta}\right) = \frac{1}{2h}\left(\exp\left(i\theta_1\right) - \exp\left(-i\theta_1\right)\right) = \frac{1}{h} i\sin\theta_1, \tag{19}$$

$$\tilde{\partial}_y^h\left(\vec{\theta}\right) = \frac{1}{2h}\left(\exp\left(i\theta_2\right) - \exp\left(-i\theta_2\right)\right) = \frac{1}{h} i\sin\theta_2. \tag{20}$$

3.2. Optimal One-Stage Relaxation. For the discrete scalar operator of (15), standard coarsening and an ideal coarse grid correction operator [2] are applied as

$$Q_h^{2h}\big|_{F_{2h}(\vec{\theta})} =: \widehat{Q}_h^{2h} = \operatorname{diag}(0,1,1,1) \in \mathbb{C}^{4\times 4}, \tag{21}$$

where $\widehat{Q}_h^{2h}$ is the Fourier representation of the operator Q_h^{2h} with subspace (13), which suppresses the low frequency error components and makes the high frequency components unchanged. Then, from [2], the smoothing factor for discrete operator (9) is defined by

$$\rho\left(n, D_h\right) = \sup_{\vec{\theta}\in\Theta_{\text{low}}}\left(\rho\left(\widehat{Q}_h^{2h}\left(\widehat{S}_h\left(\vec{\theta},\omega\right)\right)^n\right)\right)^{1/n}. \tag{22}$$

It implies that the asymptotic error reduction of the high frequency error components is given by n sweeps of the relaxation method, where $\widehat{S}_h(\vec{\theta},\omega)$ is the Fourier representation of the relaxation operator $S_h(\omega)$ on subspace (13) and ω is the relaxation parameter.

From [2, 14], a good smoothing factor is obtained by using one-stage parameter ω in the relaxation operator $S_h(\omega)$; the optimal smoothing factor and related smoothing parameter are given by

$$\omega_{\text{opt}} = \frac{2}{2 - S_{\max} - S_{\min}},$$
$$\rho_{\text{opt}} = \frac{S_{\max} - S_{\min}}{2 - S_{\max} - S_{\min}}, \tag{23}$$

where $S_{\max}$ and $S_{\min}$ are the max and min eigenvalues of the product matrix $\widehat{Q}_h^{2h}\widehat{S}_h(\vec{\theta},1)$ with the relaxation parameter $\omega = 1$ for $\theta \in \Theta_{\text{low}}^{2h}$. From [2, 19], the smoothing factor of (6) with the distributive relaxation (14) is determined by the diagonal blocks of the transformed system (15), which is given by

$$\rho\left(n, L_h\right) = \max\left\{\rho\left(n, -\Delta_h\right), \rho\left(n, ch^2\Delta_h^2 - \Delta_{2h}\right)\right\}. \tag{24}$$

3.3. Optimal Smoothing for Stokes Flow. The red-black Jacobi point relaxation S_h^{RB} is applied to (15) to discuss the optimal

smoothing problems for Stokes flow. From [1, 2, 14], the operator S_h^{RB} makes the $2h$-harmonics subspace (13) invariant; that is,

$$S_h^{RB}\big|_{F_{2h}(\vec{\theta})} =: \widehat{S}_h^{RB}\left(\vec{\theta}\right) \in \mathbb{C}^{4\times 4}, \tag{25}$$

where $\widehat{S}_h^{RB}(\vec{\theta})$ is the Fourier representation of $S_h^{RB}(\omega)$ with relaxation parameter $\omega = 1$ and is given as

$$\begin{aligned}
&\widehat{S}_h^{RB}\left(\vec{\theta},1\right)\\
&\quad= \widehat{S}_h^{RB}\left(\vec{\theta}\right)\\
&\quad= \widehat{S}_h^{B}\left(\vec{\theta}\right)\widehat{S}_h^{R}\left(\vec{\theta}\right)\\
&\quad= \frac{1}{2}\begin{pmatrix} A_{00}+1 & -A_{11}+1 & 0 & 0\\ -A_{00}-1 & A_{11}+1 & 0 & 0\\ 0 & 0 & A_{10}+1 & -A_{01}+1\\ 0 & 0 & -A_{10}+1 & A_{01}+1\end{pmatrix}\\
&\qquad\cdot\frac{1}{2}\begin{pmatrix} A_{00}+1 & A_{11}-1 & 0 & 0\\ A_{00}-1 & A_{11}+1 & 0 & 0\\ 0 & 0 & A_{10}+1 & A_{01}-1\\ 0 & 0 & A_{10}-1 & A_{01}+1\end{pmatrix},
\end{aligned} \tag{26}$$

in which

$$A_{\vec{\alpha}} = 1 - \frac{\widetilde{D}_h\left(\vec{\theta}^{\vec{\alpha}}\right)}{\widetilde{D}_h^0\left(\vec{\theta}^{\vec{\alpha}}\right)} \tag{27}$$

denotes the Fourier mode of the point Jacobi relaxation for the discrete operator (9) on subspace (13) and $\widetilde{D}_h^0(\vec{\theta}^{\vec{\alpha}})$ is the Fourier mode of the discrete operator with the stencil $[l_{(0,0)}]_h$ in (9). For the sake of convenient discussion in the following, two variables are introduced as

$$\begin{aligned}
s_1 &= \sin^2\frac{\theta_1^0}{2} = \sin^2\frac{\theta_1}{2},\\
s_2 &= \sin^2\frac{\theta_2^0}{2} = \sin^2\frac{\theta_2}{2}.
\end{aligned} \tag{28}$$

Thus, $\vec{\theta} = (\theta_1,\theta_2) \in \Theta_{\text{low}}^{2h} = (-\pi/2,\pi/2]^2$ is transformed to $\vec{s} = (s_1,s_2) \in S_{\text{low}} = [0,1/2]^2$.

Theorem 1. *For the Poisson operator $-\Delta_h$, the optimal one-stage relaxation parameter and related smoothing factor of the red-black Jacobi point relaxation are stated as*

$$\omega_{opt} = \frac{16}{15}, \qquad \rho_{opt} = \frac{1}{5}. \tag{29}$$

Proof. For the red-black Jacobi point relaxation for the Poisson operator $D_h = -\Delta_h$, substituting (12), (18), and (28) into (26) and (27), and from (5), the product of (21) and (25) is written as

$$\begin{aligned}
\widehat{Q}_h^{2h}\widehat{S}_h\left(\vec{\theta},1\right) &= \widehat{Q}_h^{2h}\widehat{S}_h\left(\vec{\theta}\right)\\
&= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0\\ (s_1+s_2)(1-s_1-s_2) & (s_1+s_2)(s_1+s_2-1) & 0 & 0\\ 0 & 0 & (s_1-s_2)(s_1-s_2+1) & (s_2-s_1)(s_1-s_2+1)\\ 0 & 0 & (s_1-s_2)(s_2-s_1+1) & (s_2-s_1)(s_2-s_1+1)\end{pmatrix}.
\end{aligned} \tag{30}$$

Thus, eigenvalues of (30) are obtained as

$$\begin{aligned}
\lambda_1 &= 0, \qquad \lambda_2 = (s_1-s_2)^2,\\
\lambda_3 &= 0, \qquad \lambda_4 = \frac{(s_1+s_2)(s_1+s_2-1)}{2}.
\end{aligned} \tag{31}$$

From (31), the max and min eigenvalues of (30) are yielded as

$$\begin{aligned}
S_{\max} &= \max_{(s_1,s_2)\in[0,1/2]^2}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} = \max_{(s_1,s_2)\in[0,1/2]^2}\lambda_2 = \frac{1}{4},\\
S_{\min} &= \min_{(s_1,s_2)\in[0,1/2]^2}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} = \min_{(s_1,s_2)\in[0,1/2]^2}\lambda_4 = -\frac{1}{8}.
\end{aligned} \tag{32}$$

From (23) and (32), (29) is obtained. Theorem 1 holds. □

Next, $\rho(n, ch^2\Delta_h^2 - \Delta_{2h})$ for the red-black Jacobi point relaxation need to be computed. Meanwhile, the smoothing factor of distributive relaxation (15) is given as follows.

Theorem 2. *For the discrete operator $ch^2\Delta_h^2 - \Delta_{2h}$ with $c > 0$, the optimal one-stage relaxation parameter and related*

smoothing factor of the red-black Jacobi point relaxation are given by

$$\omega_{opt} = \begin{cases} \dfrac{1+20c}{1+16c} & 0 < c \le \dfrac{1}{32} \\ \dfrac{2(1+20c)^2}{1+56c+1744c^2} & \dfrac{1}{32} < c \le \dfrac{1}{12}, \end{cases}$$
$$\rho_{opt} = \begin{cases} \dfrac{1}{1+16c} & 0 < c \le \dfrac{1}{32} \\ \dfrac{1+24c+1104c^2}{1+56c+1744c^2} & \dfrac{1}{32} < c \le \dfrac{1}{12}. \end{cases} \tag{33}$$

Proof. For the discrete operator

$$D_h = ch^2\Delta_h^2 - \Delta_{2h}, \tag{34}$$

from (17)–(20), the Fourier mode of (34) is given by

$$\widetilde{D}_h\left(\overrightarrow{\theta}\right) = \frac{1}{h^2}\left[4c\left(2-\cos\theta_1-\cos\theta_2\right)^2 + \sin^2\theta_1 + \sin^2\theta_2\right]. \tag{35}$$

Thus, when the red-black point relaxation is applied to (34), from (16), substituting (12), (28), and (35) into (26) and (27), the product of (21) and (25) is

$$\widehat{Q}_h^{2h}\widehat{S}_h^{RB}\left(\overrightarrow{\theta},1\right) = \widehat{Q}_h^{2h}\widehat{S}_h^{RB}\left(\overrightarrow{\theta}\right) = \frac{1}{4}\operatorname{diag}\left(R_{11},R_{22}\right), \tag{36}$$

where both R_{11} and R_{22} are 2×2 square matrices, whose expressions are below:

$$\begin{aligned} R_{11} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_{00}^B+1 & -A_{11}^B+1 \\ -A_{00}^B-1 & A_{11}^B+1 \end{pmatrix} \cdot \begin{pmatrix} A_{00}^R+1 & A_{11}^R-1 \\ A_{00}^R-1 & A_{11}^R+1 \end{pmatrix} \\ &= \frac{4}{(1+20c)^2} \\ &\quad \cdot \begin{pmatrix} 0 & 0 \\ \begin{matrix} 1+4\left[-1-36c+16c\left(s_1+s_2\right)\right] \\ \cdot\left[s_1-s_1^2+s_2-s_2^2+4c\left(s_1+s_2\right)^2\right] \end{matrix} & 64c\left(-1+s_1+s_2\right)\left[s_1-s_1^2+s_2-s_2^2+4c\left(-2+s_1+s_2\right)^2\right] \end{pmatrix}, \\ R_{22} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_{10}^B+1 & -A_{01}^B+1 \\ -A_{10}^B+1 & A_{01}^B+1 \end{pmatrix} \cdot \begin{pmatrix} A_{10}^R+1 & A_{01}^R-1 \\ A_{10}^R-1 & A_{01}^R+1 \end{pmatrix} \\ &= \frac{4}{(1+20c)^2} \\ &\quad \cdot \begin{pmatrix} \begin{bmatrix} 1+24c+80c^2-(48c+4)\left(s_1+s_2\right) \\ +384c^2\left(s_1-s_2\right)+\left(192c^2+4\right)\left(s_1^2+s_2^2\right) \\ -\left(256c^2-64c\right)\left(s_1^3-s_2^3\right)-64cs_2 \\ +128cs_2^2+32cs_1s_2\left(2s_2-2s_1-1\right)\left(1-12c\right) \end{bmatrix} & -64c\left(s_1-s_2\right)\begin{bmatrix} 4c\left(s_1-s_2\right)^2+8c\left(s_1-s_2\right) \\ +4c+s_1-s_1^2+s_2-s_2^2 \end{bmatrix} \\ 64c\left(s_1-s_2\right)\begin{bmatrix} 4c\left(s_1-s_2\right)^2-8c\left(s_1-s_2\right) \\ +4c+s_1-s_1^2+s_2-s_2^2 \end{bmatrix} & \begin{bmatrix} 1+24c+80c^2-(48c+4)\left(s_1+s_2\right) \\ -384c^2\left(s_1-s_2\right)+\left(4+192c^2\right)\left(s_1^2+s_2^2\right) \\ +\left(256c^2-64c\right)\left(s_1^3-s_2^3\right)-64cs_1 \\ +128cs_1^2+32cs_1s_2\left(2s_1-2s_2+1\right)\left(1-12c\right) \end{bmatrix} \end{pmatrix}. \end{aligned} \tag{37}$$

Thus, the eigenvalues of matrix (36) are obtained as

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_2 &= \frac{64c}{(1+20c)^2}\left(-1+s_1+s_2\right)\left[s_1-s_1^2+s_2-s_2^2+4c\left(-2+s_1+s_2\right)^2\right], \end{aligned} \tag{38}$$

$$\lambda_{3,4} = \frac{1}{(1+20c)^2}\left[\begin{matrix} 1+24c+80c^2-(4+80c)\left(s_1+s_2\right) \\ +\left(4+64c+192c^2\right)\left(s_1^2+s_2^2\right)+\left(32c-384c^2\right)s_1s_2 \\ \pm 32c\left(s_1-s_2\right)\sqrt{\begin{matrix} 1+80c^2+24c+\left(-64c^2+64c+4\right)\left(s_1^2+s_2^2\right) \\ -(80c+4)\left(s_1+s_2\right)+\left(128c^2+32c\right)s_1s_2 \end{matrix}} \end{matrix}\right]. \tag{39}$$

By using the MATLAB and Mathematica software with cylindrical algebraic decomposition function [19], for $\vec{s} = (s_1, s_2) \in (0, 1/2)^2$, there is no extreme value for (39); when $0 < c \leq 1/32$, one of extreme values of (38) is obtained as

$$s_1^* = \frac{\sqrt{64c^2 + 3} + 40c - 3}{48c - 6}, \quad s_2^* = \frac{\sqrt{64c^2 + 3} + 40c - 3}{48c - 6}. \tag{40}$$

Thus, for $\vec{s} \in S_{\text{low}} = [0, 1/2]^2$, besides (40), the possible extreme values of the eigenvalues of matrix (36) are placed on the boundary of S_{low}. From $-1 \leq \lambda_k \leq 1$ with $k = 1, \ldots, 4$, then $0 < c \leq 1/12$. Noting that (40) exists with $0 < c \leq 1/32$. From (38)–(40), when $0 < c \leq 1/12$, for $\vec{s} \in S_{\text{low}}$, the max and min eigenvalues of (36) are yielded as

$$S_{\max} = \lambda_{3,4}(0, 0) = \frac{1 + 4c}{1 + 20c},$$
$$S_{\min} = \begin{cases} \lambda_{3,4}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{4c - 1}{1 + 20c} & 0 < c \leq \frac{1}{32} \\ \lambda_2(0, 0) = -\left(\frac{32c}{20c + 1}\right)^2 & \frac{1}{32} < c \leq \frac{1}{12}. \end{cases} \tag{41}$$

Substituting (41) into (23), (33) is obtained. Theorem 2 holds. □

From (33), $1/2 \leq \rho_{\text{opt}}(ch^2\Delta_h^2 - \Delta_{2h}) < 1$ holds with $0 < c \leq 1/12$. Therefore, from Theorems 1 and 2, when $0 < c \leq 1/12$, the smoothing factor of (6) with the distributive relaxation (14) is as

$$\begin{aligned} \frac{1}{2} &\leq \rho_{\text{opt}}(L_h) \\ &= \max\left\{\rho_{\text{opt}}(-\Delta_h), \rho_{\text{opt}}\left(ch^2\Delta_h^2 - \Delta_{2h}\right)\right\} \\ &= \rho_{\text{opt}}\left(ch^2\Delta_h^2 - \Delta_{2h}\right) < 1. \end{aligned} \tag{42}$$

4. Conclusions

The smoothing analysis process of the distributive red-black Jacobi point relaxation for solving 2D Stokes flow is analytically presented. Applying (28), the Fourier modes with the trigonometric functions for the discrete operator and relaxation are mapped to rational functions. So, it is possible to apply the cylindrical algebraic decomposition function in the Mathematica software to realize complex smoothing analysis, and the computation process is simplified. The analytical expressions of the smoothing factor for the distributive red-black Jacobi point relaxation are obtained, which is an upper bound for the smoothing rates and is independent of the mesh size with the parameter c. Obviously, it is valuable to understand numerical experiments in multigrid method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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