Research Article

Robust Stability and Stabilization of Interval Uncertain Descriptor Fractional-Order Systems with the Fractional-Order α : The $1 \leq \alpha < 2$ Case

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Stability and stabilization of fractional-order interval system is investigated. By adding parameters to linear matrix inequalities, necessary and sufficient conditions for stability and stabilization of the system are obtained. The results on stability check for uncertain FO-LTI systems with interval coefficients of dimension n only need to solve one $4n$ -by- $4n$ LMI. Numerical examples are presented to shown the effectiveness of our results.

1. Introduction

During the last two decades, the study of fractional-order control systems has received more and more attention. As a generalization of the traditional calculus, the fractional calculus have found many applications in viscoelastic systems, robotics, finance, and so on ([1–4], etc.). Studying on fractional-order calculus has become an active research field.

Stability analysis is a basic problem in control theory. For Caputo fractional derivative-based linear systems, the stability results were formulated with the fractional-order α belonging to $0 < \alpha < 1$ and $1 \leq \alpha < 2$, in [5–7] and so on. In [8], the stability issue of interval fractional-order linear time-invariant (FO-LTI) systems was first presented and discussed. The stability of single-input single-output FO-LTI systems was further discussed based on an experimentally verified Kharitonov-like procedure [9].

Robust stability analysis was carried out for FO-LTI interval system with fractional commensurate order $1 \leq \alpha < 2$ based on the maximum eigenvalue of a Hermitian matrix by applying Lyapunov inequality in [10] and then further discussed in [7].

The uncertain FO-LTI systems have been wildly studied. In [11], the robust stability problem was discussed based on the ranges of the corresponding interval eigenvalues by

applying the matrix perturbation theory. In [12], a new and effectively robust stability checking method was first proposed for FO-LTI interval uncertain systems in terms of LMIs, and an analytical design of the stabilizing controllers for fractional-order dynamic interval systems was given. Note that the above-mentioned results on stability check for uncertain FO-LTI systems with interval coefficients of dimension n need to solve one $(n^2 + 2n) \times (n^2 + 2n)$ LMI. Therefore, it is valuable to seek some simple necessary and sufficient conditions for checking robust stability of uncertain FO-LTI systems with interval coefficients. With the above motivation, based on the results of [12], the robust stability and stabilization problems of uncertain FO-LTI interval systems with the fractional-order α belonging to $1 \leq \alpha < 2$ are further investigated in this paper.

This paper is organized as follows: in Section 2, we present some preliminaries results on the fractional derivative, the linear algebra and the matrix theory. In Section 3, we study the problems of the stability and stabilization of uncertain FO-LTI systems with interval coefficients in terms of LMIs. In Section 4, numerical examples are presented to illustrate our proposed results. Finally, Section 5 concludes this work.

Notations. Throughout this paper, $\mathbf{R}^{m \times n}$ stands for the set of m by n matrices with real entries. The symbols M^T , $\text{Sym}\{M\}$,

and I_n stand for the transpose of M , the expression $M^T + M$, and the identity matrix of order n , respectively. The symbol $e_n \in \mathbf{R}^n$ is used to denote the row vector with the k th element being 1, $i = 1, \dots, n$; that is,

$$e_n = (1, 1, \dots, 1) \in \mathbf{R}^n. \quad (1)$$

The symbol $\otimes$ is the Kronecker product of two matrices and

$$(A \otimes C)^T = A^T \otimes C^T. \quad (2)$$

The symbol $\cdot$ will be used in some matrix expressions to indicate a symmetric structure; that is, if matrices $H_1 = H_1^H \in \mathbf{R}^{m \times m}$ and $H_2 = H_2^H \in \mathbf{R}^{n \times n}$ were given, then

$$\begin{pmatrix} H_1 & \cdot \\ L & H_2 \end{pmatrix} = \begin{pmatrix} H_1 & L^T \\ L & H_2 \end{pmatrix}. \quad (3)$$

Let $N = \text{diag}(a_1, a_2, \dots, a_n)$; consider the symbol

$$N^{1/2} = \text{diag}(\sqrt{a_1}, \sqrt{a_2}, \dots, \sqrt{a_n}), \quad (4)$$

in which $a_i \geq 0$, $i = 1, \dots, n$.

2. Preliminaries

Throughout the paper, only the Caputo definition is used. The following Caputo definition is adopted for fractional derivatives of order α of function $f(t)$ [13]:

$$D^\alpha f(t) = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}(\tau)}{(t - \tau)^{(\alpha - m + 1)}} d\tau, \quad (5)$$

$$m - 1 \leq \alpha < m$$

with $m \in \mathbf{Z}$, $\alpha \in \mathbf{R}^+$, where $\Gamma(\cdot)$ is the Gamma function:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \quad (6)$$

Consider the following FO-LTI interval system:

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad (7)$$

where α is the fractional commensurate order, $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^p$ stand for the state vector and control input, respectively, and the system matrices A and B are interval uncertain in the sense that

$$\begin{aligned} A &\in A_I = [\underline{A}, \bar{A}] \\ &= \{(a_{ij})_{n \times n} : \underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}, \quad 1 \leq i, j \leq n\}, \\ B &\in B_I = [\underline{B}, \bar{B}] \\ &= \{(b_{ij})_{n \times p} : \underline{b}_{ij} \leq b_{ij} \leq \bar{b}_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq p\}, \end{aligned} \quad (8)$$

where $\underline{A}$, $\bar{A}$, $\underline{B}$, and $\bar{B}$ are given matrices.

To take into account the stability [14], we introduce the following definition.

Definition 1. The fractional-order interval system (7) is said to be asymptotically stabilizable via linear state-feedback control if there exists a state-feedback controller $u(t) = Kx(t)$ such that the closed-loop system

$$D^\alpha f(t) = (A + BK)x(t) \quad (9)$$

is asymptotically stable.

Denote

$$\begin{aligned} A_0 &= \frac{1}{2}(\underline{A} + \bar{A}), \\ \Delta A &= \frac{1}{2}(\bar{A} - \underline{A}) = (\gamma_{ij})_{n \times n}, \\ B_0 &= \frac{1}{2}(\underline{B} + \bar{B}), \\ \Delta B &= \frac{1}{2}(\bar{B} - \underline{B}) = (\beta_{ij})_{n \times p}. \end{aligned} \quad (10)$$

To handle the interval uncertain, the following notations are introduced:

$$\begin{aligned} D_A &= \text{diag}(\gamma_{11}, \dots, \gamma_{1n}, \dots, \gamma_{n1}, \dots, \gamma_{nn})_{n^2 \times n^2}, \\ H_A &= \{\text{diag}(\delta_{11}, \dots, \delta_{1n}, \dots, \delta_{n1}, \dots, \delta_{nn}) \in \mathbf{R}^{n^2 \times n^2}, \\ &\quad |\delta_{ij}| \leq 1, \quad i, j = 1, \dots, n\}, \\ D_B &= \text{diag}(\beta_{11}, \dots, \beta_{1p}, \dots, \beta_{n1}, \dots, \beta_{np})_{(np) \times (np)}, \\ H_B &= \{\text{diag}(\eta_{11}, \dots, \eta_{1p}, \dots, \eta_{n1}, \dots, \eta_{np}) \\ &\quad \in \mathbf{R}^{(np) \times (np)}, \quad |\eta_{ij}| \leq 1, \quad i = 1, \dots, n, \quad j = 1, \dots, p\}. \end{aligned} \quad (11)$$

Lemma 2. Let $t_{ij} > 0$, $i, j = 1, \dots, n$:

$$\begin{aligned} T_{A1} &= \text{diag}(t_{11}, \dots, t_{1n}, \dots, t_{n1}, \dots, t_{nn})_{n^2 \times n^2}, \\ T_{A2} &= T_{A1}^{-1}, \\ \tau_{ij} &> 0, \quad i = 1, \dots, n, \quad j = 1, \dots, p, \end{aligned} \quad (12)$$

$$\begin{aligned} T_{B1} &= \text{diag}(\tau_{11}, \dots, \tau_{1p}, \dots, \tau_{n1}, \dots, \tau_{np})_{(np) \times (np)}, \\ T_{B2} &= T_{B1}^{-1}, \\ A_J &= \{A = A_0 \\ &\quad + (I_n \otimes e_n) T_{A1} D_A F_A T_{A2} (e_n^T \otimes I_n), \quad F_A \in H_A\}, \\ B_J &= \{B = B_0 + (I_n \otimes e_p) T_{B1} D_B F_B T_{B2} (e_n^T \otimes I_p), \quad F_B \\ &\quad \in H_B\}. \end{aligned} \quad (13)$$

Then $A_J = A_I$, $B_J = B_I$.

Proof. Since T_{A1} , F_A , D_A , and T_{A2} are all diagonal, it is easy to check that

$$T_{A1}D_A F_A T_{A2} = D_A F_A,$$

$$I_n \otimes e_n = \begin{pmatrix} e_n & 0 & \cdots & 0 \\ 0 & e_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_n \end{pmatrix},$$

$$T_{A1}D_A F_A T_{A2} (e_n^T \otimes E_n) = D_A F_A \begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix} \quad (15)$$

$$= \begin{pmatrix} \text{diag}(\delta_{11}\gamma_{11}, \dots, \delta_{1n}\gamma_{1n}) \\ \text{diag}(\delta_{21}\gamma_{21}, \dots, \delta_{2n}\gamma_{2n}) \\ \vdots \\ \text{diag}(\delta_{n1}\gamma_{n1}, \dots, \delta_{nn}\gamma_{nn}) \end{pmatrix},$$

$$e_n \text{diag}(\delta_{i1}\gamma_{i1}, \dots, \delta_{in}\gamma_{in})$$

$$= (\delta_{i1}\gamma_{i1} \quad \delta_{i2}\gamma_{i2} \quad \cdots \quad \delta_{in}\gamma_{in})_{1 \times n}, \quad i = 1, \dots, n.$$

It follows that

$$\begin{aligned} & (I_n \otimes e_n) D_A F_A (e_n^T \otimes E_n) \\ &= \begin{pmatrix} e_n & 0 & \cdots & 0 \\ 0 & e_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_n \end{pmatrix} \begin{pmatrix} \text{diag}(\delta_{11}\gamma_{11}, \dots, \delta_{1n}\gamma_{1n}) \\ \text{diag}(\delta_{21}\gamma_{21}, \dots, \delta_{2n}\gamma_{2n}) \\ \vdots \\ \text{diag}(\delta_{n1}\gamma_{n1}, \dots, \delta_{nn}\gamma_{nn}) \end{pmatrix} \\ &= \begin{pmatrix} e_n \text{diag}(\delta_{11}\gamma_{11}, \dots, \delta_{1n}\gamma_{1n}) \\ e_n \text{diag}(\delta_{21}\gamma_{21}, \dots, \delta_{2n}\gamma_{2n}) \\ \vdots \\ e_n \text{diag}(\delta_{n1}\gamma_{n1}, \dots, \delta_{nn}\gamma_{nn}) \end{pmatrix} \\ &= \begin{pmatrix} \delta_{11}\gamma_{11} & \delta_{12}\gamma_{12} & \cdots & \delta_{1n}\gamma_{1n} \\ \delta_{21}\gamma_{21} & \delta_{22}\gamma_{22} & \cdots & \delta_{2n}\gamma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1}\gamma_{n1} & \delta_{n2}\gamma_{n2} & \cdots & \delta_{nn}\gamma_{nn} \end{pmatrix}. \end{aligned} \quad (16)$$

Thus, $A_I \subseteq A_J$.

Noting that the above proof is reversible, it is easy to know that $A_J \subseteq A_I$. Therefore, $A_J = A_I$.

In the same way, we have $B_J = B_I$. $\square$

Lemma 3 (see [7, 15]). Let $A \in \mathbf{R}^{n \times n}$ be a deterministic real matrix without uncertainty. Then, a necessary and sufficient condition for the asymptotical stability of $D^\alpha x(t) = Ax(t)$ is

$$|\arg(\text{spec}(A))| > \frac{\pi}{2}\alpha, \quad (17)$$

where $\text{spec}(A)$ is the spectrum of all eigenvalues of A .

Lemma 4 (see [16]). Let $A \in \mathbf{R}^{n \times n}$ be a real matrix. Then $|\arg(\text{spec}(A))| > (\pi/2)\alpha$ where $1 \leq \alpha < 2$, if and only if there exists $P > 0$ such that

$$\begin{pmatrix} (AP + PA^T) \sin \theta & \cdot \\ (PA^T - AP) \cos \theta & (AP + PA^T) \sin \theta \end{pmatrix} < 0, \quad (18)$$

where $\theta = \pi - \alpha(\pi/2)$.

Lemma 5 (see [17]). For any matrices X and Y with appropriate dimensions, we have

$$X^T Y + Y^T X \leq \varepsilon X^T X + \varepsilon^{-1} Y^T Y, \quad \forall \varepsilon > 0. \quad (19)$$

Lemma 6 (see [18]). Let X , Y , and Z be real matrices of suitable dimensions. Then, for any $x \in \mathbf{R}^n$,

$$\begin{aligned} & \max \left\{ (x^T X F Y x)^2 : F^T F \leq I \right\} \\ &= (x^T X X^T x) (x^T Y^T Y x). \end{aligned} \quad (20)$$

Lemma 7 (see [18]). Let X , Y , and Z be $n \times n$ symmetric matrices such that $X \geq 0$, $Y < 0$, and $Z \geq 0$. Furthermore, assume that $(x^T Y x)^2 - 4(x^T X x)(x^T Z x) > 0$ for all nonzero $x \in \mathbf{R}^n$. Then, there exists a constant $\lambda > 0$ such that

$$\lambda^2 X + \lambda Y + Z < 0. \quad (21)$$

3. Main Results

In this section, by adding parameters into linear matrix inequalities, necessary and sufficient conditions for stability and stabilization of the system are obtained. Those results are generalization of the main theorems in [12].

Theorem 8. Let $1 \leq \alpha < 2$. The uncertain FO-LTI interval system (7) with controller $u(t)$ is asymptotically stable if and

only if there exist some symmetric positive definite matrix $P \in \mathbf{R}^{n \times n}$ and a real scalar constant $\varepsilon > 0$ such that

$$\Gamma = \begin{pmatrix} \Gamma_{11} & \cdot & \cdot & \cdot \\ \Gamma_{21} & \Gamma_{22} & \cdot & \cdot \\ \Omega_2 P & 0 & -\varepsilon I_n & \cdot \\ 0 & \Omega_2 P & 0 & -\varepsilon I_n \end{pmatrix} < 0, \quad (22)$$

where

$$\begin{aligned} \theta &= \pi - \alpha \left(\frac{\pi}{2} \right), \\ \Gamma_{11} &= \Gamma_{22} = (A_0 P + P A_0^T) \sin \theta + \varepsilon \Omega_1, \\ \Gamma_{21} &= (P A_0^T - A_0 P) \cos \theta, \\ \Omega_1 &= \text{diag} \left(\sum_{i=1}^n t_{1i}^2 \gamma_{1i}^2, \dots, \sum_{i=1}^n t_{ni}^2 \gamma_{ni}^2 \right)_{n \times n}, \\ \Omega_2 &= \text{diag} \left(\sqrt{\sum_{i=1}^n t_{i1}^{-2}}, \dots, \sqrt{\sum_{i=1}^n t_{in}^{-2}} \right)_{n \times n}. \end{aligned} \quad (23)$$

Proof. It is easy to check that

$$\begin{aligned} & (I_n \otimes e_n) T_{A1} D_A ((I_n \otimes e_n) T_{A1} D_A)^T \\ &= \text{diag} \left(\sum_{i=1}^n t_{1i}^2 \gamma_{1i}^2, \dots, \sum_{i=1}^n t_{ni}^2 \gamma_{ni}^2 \right). \end{aligned} \quad (24)$$

Denote $\Pi = T_{A2}(e_n^T \otimes I_n)$; then we see that

$$\begin{aligned} \Pi &= \text{diag} (t_{11}^{-1}, \dots, t_{1n}^{-1}, \dots, t_{n1}^{-1}, \dots, t_{nm}^{-1}) \begin{pmatrix} I_n \\ \vdots \\ I_n \end{pmatrix} \\ &= \begin{pmatrix} \text{diag} (t_{11}^{-1}, \dots, t_{1n}^{-1}) \\ \vdots \\ \text{diag} (t_{n1}^{-1}, \dots, t_{nm}^{-1}) \end{pmatrix}, \end{aligned} \quad (25)$$

$$\begin{aligned} \Pi^T \Pi &= \text{diag} \left(\sum_{i=1}^n t_{i1}^{-2}, \dots, \sum_{i=1}^n t_{in}^{-2} \right)_{n \times n}, \\ \begin{pmatrix} \Pi P & 0 \\ 0 & \Pi P \end{pmatrix}^T \begin{pmatrix} \Pi P & 0 \\ 0 & \Pi P \end{pmatrix} &= \begin{pmatrix} P \Pi^T \Pi P & 0 \\ 0 & P \Pi^T \Pi P \end{pmatrix} \\ &= \begin{pmatrix} P (\Pi^T \Pi)^{1/2} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & P (\Pi^T \Pi)^{1/2} (\Pi^T \Pi)^{1/2} P \end{pmatrix} \\ &= \begin{pmatrix} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & (\Pi^T \Pi)^{1/2} P \end{pmatrix}^T \\ &= \begin{pmatrix} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & (\Pi^T \Pi)^{1/2} P \end{pmatrix}, \end{aligned} \quad (26)$$

where $P \in \mathbf{R}^{n \times n}$ is an arbitrary positive definite matrix. By applying (13), we have

$$\begin{aligned} & \begin{pmatrix} (AP + PA^T) \sin \theta & \cdot \\ (PA^T - AP) \cos \theta & (AP + PA^T) \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} (A_0 P + P A_0^T) \sin \theta & \cdot \\ (P A_0^T - A_0 P) \cos \theta & (A_0 P + P A_0^T) \sin \theta \end{pmatrix} \\ &+ \text{Sym} \left\{ \begin{pmatrix} (I_n \otimes e_n) T_{A1} D_A F_A T_{A2} (e_n^T \otimes I_n) P \sin \theta & (I_n \otimes e_n) T_{A1} D_A F_A T_{A2} (e_n^T \otimes I_n) P \cos \theta \\ -(I_n \otimes e_n) T_{A1} D_A F_A T_{A2} (e_n^T \otimes I_n) P \cos \theta & (I_n \otimes e_n) T_{A1} D_A F_A T_{A2} (e_n^T \otimes I_n) P \sin \theta \end{pmatrix} \right\} \\ &= \begin{pmatrix} (A_0 P + P A_0^T) \sin \theta & \cdot \\ (P A_0^T - A_0 P) \cos \theta & (A_0 P + P A_0^T) \sin \theta \end{pmatrix} \\ &+ \text{Sym} \left\{ \begin{pmatrix} (I_n \otimes e_n) T_{A1} D_A \sin \theta & (I_n \otimes e_n) T_{A1} D_A \cos \theta \\ -(I_n \otimes e_n) T_{A1} D_A \cos \theta & (I_n \otimes e_n) T_{A1} D_A \sin \theta \end{pmatrix} \begin{pmatrix} F_A & 0 \\ 0 & F_A \end{pmatrix} \begin{pmatrix} T_{A2} (e_n^T \otimes I_n) P & 0 \\ 0 & T_{A2} (e_n^T \otimes I_n) P \end{pmatrix} \right\} \\ &= Z + \text{Sym} \{ M \widehat{F_A} N \}, \end{aligned} \quad (27)$$

in which

$$\begin{aligned}
 Z &= \begin{pmatrix} (A_0 P + P A_0^T) \sin \theta & \cdot \\ (P A_0^T - A_0 P) \cos \theta & (A_0 P + P A_0^T) \sin \theta \end{pmatrix}, \\
 M &= \begin{pmatrix} (I_n \otimes e_n) T_{A1} D_A \sin \theta & (I_n \otimes e_n) T_{A1} D_A \cos \theta \\ -(I_n \otimes e_n) T_{A1} D_A \cos \theta & (I_n \otimes e_n) T_{A1} D_A \sin \theta \end{pmatrix}, \\
 \widehat{F}_A &= \begin{pmatrix} F_A & 0 \\ 0 & F_A \end{pmatrix}, \\
 N &= \begin{pmatrix} T_{A2} (e_n^T \otimes I_n) P & 0 \\ 0 & T_{A2} (e_n^T \otimes I_n) P \end{pmatrix}.
 \end{aligned} \quad (28)$$

Sufficiency. Suppose that there exists a symmetric positive definite matrix $P \in \mathbf{R}^{n \times n}$ such that (22) holds. By applying (26), (27), and Lemma 5, we have

$$\begin{aligned}
 Z + \text{Sym} \{M \widehat{F}_A N\} &\leq Z + \varepsilon M M^T + \varepsilon^{-1} N^T N \\
 &= \begin{pmatrix} \Gamma_{11} & \cdot \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} + \frac{1}{\varepsilon} \begin{pmatrix} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & (\Pi^T \Pi)^{1/2} P \end{pmatrix}^T \\
 &\quad \cdot \begin{pmatrix} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & (\Pi^T \Pi)^{1/2} P \end{pmatrix} < 0.
 \end{aligned} \quad (29)$$

By using the Schur complement of (22), one obtains

$$\begin{pmatrix} (AP + P A^T) \sin \theta & \cdot \\ (P A^T - AP) \cos \theta & (AP + P A^T) \sin \theta \end{pmatrix} < 0. \quad (30)$$

It follows from Lemma 4 that $|\arg(\text{spec}(A))| > \pi/2$. Therefore, by Lemma 3, the uncertain FO-LTI interval system (7) is asymptotically stable.

Necessity. Suppose that the uncertain FO-LTI interval system (7) is asymptotically stable. Then, $|\arg(\text{spec}(A))| > \pi/2$. It follows from Lemma 4 that there exists a symmetric positive definite matrix P such that

$$\begin{pmatrix} (AP + P A^T) \sin \theta & \cdot \\ (P A^T - AP) \cos \theta & (AP + P A^T) \sin \theta \end{pmatrix} < 0. \quad (31)$$

By using Lemma 2 and after some calculations, one can obtain from (27) that

$$Z + \text{Sym} \{M \widehat{F}_A N\} < 0. \quad (32)$$

Therefore, for all $\varsigma_1, \varsigma_2 \in \mathbf{R}^n$, $\varsigma = (\varsigma_1^T \ \varsigma_2^T)^T \neq 0$,

$$\varsigma^T Z \varsigma < -2 \varsigma^T M \widehat{F}_A N \varsigma; \quad (33)$$

that is,

$$\varsigma^T Z \varsigma < -2 \max \left\{ \varsigma^T M \widehat{F}_A N \varsigma \mid \widehat{F}_A^T \widehat{F}_A \leq I \right\} \leq 0. \quad (34)$$

Consequently, given any $\varsigma \in \mathbf{R}^{2n}$ and $\varsigma \neq 0$, we have

$$(\varsigma^T Z \varsigma)^2 > 4 \max \left\{ \varsigma^T M \widehat{F}_A N \varsigma \mid \widehat{F}_A^T \widehat{F}_A \leq I \right\}. \quad (35)$$

Applying Lemma 6, we obtain

$$(\varsigma^T Z \varsigma)^2 > 4 (\varsigma^T M M^T \varsigma) (\varsigma^T N^T N \varsigma). \quad (36)$$

It follows by Lemma 7 that there exists a constant $\varepsilon > 0$ such that

$$Z + \varepsilon^{-1} N^T N + \varepsilon M M^T < 0. \quad (37)$$

So we derive that

$$\begin{aligned}
 Z + \varepsilon M M^T + \varepsilon^{-1} N^T N &= \begin{pmatrix} \Gamma_{11} & \cdot \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \\
 &\quad + \frac{1}{\varepsilon} \begin{pmatrix} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & (\Pi^T \Pi)^{1/2} P \end{pmatrix}^T \\
 &\quad \cdot \begin{pmatrix} (\Pi^T \Pi)^{1/2} P & 0 \\ 0 & (\Pi^T \Pi)^{1/2} P \end{pmatrix} < 0.
 \end{aligned} \quad (38)$$

Applying the well-known Schur complement yields (22). This ends the proof. $\square$

Next, let us establish a stabilization result.

Theorem 9. Let $1 \leq \alpha < 2$. The uncertain FO-LTI interval system (7) is asymptotically stable if and only if there are a matrix $X \in \mathbf{R}^{p \times n}$, a symmetric positive definite matrix $P \in \mathbf{R}^{n \times n}$, and two real scalars $\varepsilon_i > 0$ ($i = 1, 2$), such that

$$\begin{aligned}
 \Gamma^1 &= \begin{pmatrix} \Gamma_{11}^1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Gamma_{21}^1 & \Gamma_{22}^1 & \cdot & \cdot & \cdot & \cdot \\ \Omega_2 P & 0 & -\varepsilon_1 I_n & \cdot & \cdot & \cdot \\ 0 & \Omega_2 P & 0 & -\varepsilon_1 I_n & \cdot & \cdot \\ \Omega_4 X & 0 & 0 & 0 & -\varepsilon_2 I_p & \cdot \\ 0 & \Omega_4 X & 0 & 0 & 0 & -\varepsilon_2 I_p \end{pmatrix} \\
 &< 0,
 \end{aligned} \quad (39)$$

where

$$\begin{aligned}
 \theta &= \pi - \alpha \left(\frac{\pi}{2} \right), \\
 \Gamma_{11}^1 &= \Gamma_{22}^1 \\
 &= (A_0 P + P A_0^T + B_0 X + (B_0 X)^T) \sin \theta + \varepsilon_1 \Omega_1 \\
 &\quad + \varepsilon_2 \Omega_3, \\
 \Gamma_{21}^1 &= (P A_0^T - A_0 P + (B_0 X)^T - B_0 X) \cos \theta, \\
 \Omega_1 &= \text{diag} \left(\sum_{i=1}^n t_{1i}^2 \gamma_{1i}^2, \dots, \sum_{i=1}^n t_{ni}^2 \gamma_{ni}^2 \right)_{n \times n}, \\
 \Omega_2 &= \text{diag} \left(\sqrt{\sum_{i=1}^n t_{i1}^{-2}}, \dots, \sqrt{\sum_{i=1}^n t_{in}^{-2}} \right)_{n \times n}, \\
 \Omega_3 &= \text{diag} \left(\sum_{i=1}^p \tau_{1i}^2 \eta_{1i}^2, \dots, \sum_{i=1}^p \tau_{ni}^2 \eta_{ni}^2 \right)_{p \times p}, \\
 \Omega_4 &= \text{diag} \left(\sqrt{\sum_{i=1}^n \tau_{i1}^{-2}}, \dots, \sqrt{\sum_{i=1}^n \tau_{ip}^{-2}} \right)_{p \times p}.
 \end{aligned} \tag{40}$$

Moreover, the robustly asymptotically stabilizing state-feedback gain matrix is given by

$$K = X P^{-1}. \tag{41}$$

Remark 10. When one takes $T_{A1} = I_{n^2}$ and $T_{A2} = I_{np}$, it is easy to obtain equivalence of Theorem 8 and [12, Theorem 1] and Theorem 9 and [12, Theorem 2], respectively.

4. Numerical Examples

Example 1 (see [12]). Consider the robust stability of the following uncertain FO-LTI interval system:

$$D^\alpha x(t) = Ax(t), \tag{42}$$

where $\alpha = 1.6$ and $A \in A_I = [\underline{A}, \bar{A}]$:

$$\begin{aligned}
 \underline{A} &= \begin{pmatrix} -1.5 & 0.5 & 1.0 \\ 0.4 & -2.5 & 0.9 \\ 0.2 & -1.3 & -2.0 \end{pmatrix}, \\
 \bar{A} &= \begin{pmatrix} -1.05 & 0.65 & 1.3 \\ 0.7 & -2.1 & 1.3 \\ 0.35 & -1.05 & -1.75 \end{pmatrix}.
 \end{aligned} \tag{43}$$

Taking $T_{A1}^2 = 0.001 * I_9$, a feasible solution of (22) is as follows:

$$\begin{aligned}
 P &= \begin{pmatrix} 0.1709 & 0.0299 & -0.0501 \\ 0.0299 & 0.1070 & 0.0050 \\ -0.0501 & 0.0050 & 0.0872 \end{pmatrix}, \\
 \varepsilon &= 4.8484.
 \end{aligned} \tag{44}$$

Let

$$A = \begin{pmatrix} -1.25 & 0.55 & 1.15 \\ 0.55 & -2.25 & 1 \\ 0.27 & -1.18 & -1.85 \end{pmatrix} \tag{45}$$

and let the initial conditions be $x_1(0) = 2$, $x_2(0) = -1$, and $x_3(0) = 1$. Time response of the state variables is depicted in Figure 1.

In the following example, we have shown the effectiveness of our results by choosing different parameters.

Example 2 (see [12]). Consider the robust stability of the following uncertain FO-LTI interval system:

$$D^\alpha x(t) = Ax(t) + Bu(t), \tag{46}$$

where $\alpha = 1.8$ and $A \in A_I = [\underline{A}, \bar{A}]$, $B \in B_I = [\underline{B}, \bar{B}]$ with

$$\begin{aligned}
 \underline{A} &= \begin{pmatrix} -1.95 & 0.35 & 0.7 \\ -1.3 & -3.9 & 0.7 \\ -0.65 & -1.95 & -3.25 \end{pmatrix}, \\
 \underline{B} &= \begin{pmatrix} 0.4 \\ 0.5 \\ 0.9 \end{pmatrix}, \\
 \bar{A} &= \begin{pmatrix} -1.05 & 0.65 & 1.3 \\ -0.7 & -2.1 & 1.3 \\ -0.35 & -1.05 & -1.75 \end{pmatrix}, \\
 \bar{B} &= \begin{pmatrix} 0.5 \\ 0.6 \\ 1 \end{pmatrix}.
 \end{aligned} \tag{47}$$

(I) Taking $T_{A1}^2 = 0.1 \times I_9$ and $T_{B1}^2 = 0.1 \times I_3$, a feasible solution of (39) is as follows:

$$\begin{aligned}
 P &= \begin{pmatrix} 19.0307 & -4.8726 & 3.6969 \\ -4.8726 & 9.9174 & -4.4783 \\ 3.6969 & -4.4783 & 9.8415 \end{pmatrix}, \\
 \varepsilon_1 &= 23.2647, \\
 X &= (4.4393 \quad 0.1546 \quad -27.5631), \\
 \varepsilon_2 &= 23.4538.
 \end{aligned} \tag{48}$$

Finally, the asymptotically stabilizing state-feedback gain matrix is obtained as

$$K = (0.5980 \quad -1.3300 \quad -3.6306). \tag{49}$$

FIGURE 1: Time response of the state variables.

FIGURE 2: Time response of the state variables when K is as in (49).

(II) Taking $T_{A1}^2 = I_9$ and $T_{B1}^2 = I_3$, we have

$$P = 10^{-10} \times \begin{pmatrix} 0.0509 & -0.0334 & 0.0506 \\ -0.0334 & 0.1002 & 0.0405 \\ 0.0506 & 0.0405 & 0.3195 \end{pmatrix},$$

$$\varepsilon_1 = 8.5956 \times 10^{-10},$$

$$X = 10^{-9} \times (-0.2978 \quad -0.3742 \quad -0.7021),$$

$$\varepsilon_2 = 2.7808 \times 10^{-10},$$

$$K = (-116.5937 \quad -78.8292 \quad 6.4958). \quad (50)$$

(III) Taking $T_{A1}^2 = 4 \times I_9$ and $T_{B1}^2 = 4 \times I_3$, we have

$$P = 10^{-11} \times \begin{pmatrix} 0.0374 & -0.0091 & 0.0146 \\ -0.0091 & 0.0664 & 0.0264 \\ 0.0146 & 0.0264 & 0.1502 \end{pmatrix},$$

$$\varepsilon_1 = -2.7962 \times 10^{-12},$$

$$X = 10^{-10} \times (-0.2460 \quad -0.3091 \quad -0.5579),$$

$$\varepsilon_2 = 8.3122 \times 10^{-11},$$

$$K = (-68.4565 \quad -47.1044 \quad -22.2054). \quad (51)$$

Let

$$A = \begin{pmatrix} -1.5 & 0.45 & 1 \\ -1 & -3 & 1 \\ -0.5 & -1.5 & -2 \end{pmatrix} \in A_I,$$

$$B = \begin{pmatrix} 0.45 \\ 0.55 \\ 0.95 \end{pmatrix} \in B_I, \quad (54)$$

FIGURE 3: Time response of the state variables when K is as in (51).

and let the initial conditions be $x_1(0) = 2$, $x_2(0) = -1$, and $x_3(0) = 1$, and let K be as in (49), (51), and (53), respectively. Time response of the state variables is depicted in Figures 2–4, respectively.

Remark 11. Applying [12, Theorem 2 (28)] to Example 2 gives the same results as in Example 2 (II).

5. Conclusion

In this paper, the robust asymptotical stability of fractional-order interval systems with the fractional-order α belonging to $1 \leq \alpha < 2$ has been studied. The results on stability check for uncertain FO-LTI systems with interval coefficients of dimension n only need to solve one $4n$ -by- $4n$ LMI. LMI stability conditions for fractional systems are proposed. Numerical examples have shown the effectiveness of our results. To the best of our knowledge, the idea of introducing free parameters is used for the first time to derive an analytical design of the stabilizing controllers for fractional-order

FIGURE 4: Time response of the state variables when K is as in (53).

dynamic interval systems. Working towards relaxing the requirements for the knowledge of system uncertainties and applying the proposed control methods to fractional-order nonlinear systems while maintaining the simplicity of the controller design are our further investigation directions.

Conflict of Interests

The authors declare that they have no competing interests.

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References

- [1] E. Ahmed and A. S. Elgazzar, "On fractional order differential equations model for nonlocal epidemics," *Physica A*, vol. 379, no. 2, pp. 607–614, 2007.
- [2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, New York, NY, USA, 2006.
- [3] Y. Ma, J.-G. Lu, W. Chen, and Y. Chen, "Robust stability bounds of uncertain fractional-order systems," *Fractional Calculus and Applied Analysis*, vol. 17, no. 1, pp. 136–153, 2014.
- [4] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.
- [5] I. N'Doye, M. Darouach, M. Zasadzinski, and N.-E. Radhy, "Robust stabilization of uncertain descriptor fractional-order systems," *Automatica*, vol. 49, no. 6, pp. 1907–1913, 2013.
- [6] M. Moze, J. Sabatier, and A. Oustaloup, "LMI characterization of fractional systems stability," in *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, pp. 419–434, Springer, New York, NY, USA, 2007.
- [7] M. Moze and J. Sabatier, "LMI tools for stability analysis of fractional systems," in *Proceedings of the International Design Engineering Technical Conferences & Computers and Information in Engineering Conference*, Long Beach, Calif, USA, 2005.
- [8] I. Petráš, Y. Q. Chen, and B. M. Vinagre, "Robust stability test for interval fractional order linear systems," in *Unsolved Problems in the Mathematics of Systems and Control*, Princeton University Press, Princeton, NJ, USA, 2004.
- [9] I. Petráš, Y. Q. Chen, B. M. Vinagre, and I. Podlubny, "Stability of linear time invariant systems with interval fractional orders and interval coefficients," in *Proceedings of the 2nd IEEE International Conference on Computational Cybernetics (ICCC '04)*, pp. 341–346, IEEE, Vienna, Austria, 2004.
- [10] H.-S. Ahn, Y. Q. Chen, and I. Podlubny, "Robust stability test of a class of linear time-invariant interval fractional-order system using Lyapunov inequality," *Applied Mathematics and Computation*, vol. 187, no. 1, pp. 27–34, 2007.
- [11] Y. Q. Chen, H.-S. Ahn, and I. Podlubny, "Robust stability check of fractional order linear time invariant systems with interval uncertainties," *Signal Processing*, vol. 86, no. 10, pp. 2611–2618, 2006.
- [12] J.-G. Lu and G. Chen, "Robust stability and stabilization of fractional-order interval systems: an LMI approach," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1294–1299, 2009.
- [13] I. Podlubny, "Geometric and physical interpretation of fractional integration and fractional differentiation," *Fractional Calculus and Applied Analysis*, vol. 5, no. 4, pp. 367–386, 2002.
- [14] D. Matignon and B. d'Andréa-Novet, *Observer-Based Controllers for Fractional Differential Systems*, SIAM, San Diego, Calif, USA, 1997.
- [15] D. Matignon, "Stability result for fractional differential equations with applications to control processing," in *Computational Engineering in Systems Applications*, pp. 963–968, 1997.
- [16] M. Chilali, P. Gahinet, and P. Apkarian, "Robust pole placement in LMI regions," *IEEE Transactions on Automatic Control*, vol. 44, no. 12, pp. 2257–2270, 1999.
- [17] P. P. Khargonekar, I. R. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: quadratic stabilizability and H_∞ control theory," *IEEE Transactions on Automatic Control*, vol. 35, no. 3, pp. 356–361, 1990.
- [18] I. R. Petersen, "A stabilization algorithm for a class of uncertain linear systems," *Systems & Control Letters*, vol. 8, no. 4, pp. 351–357, 1987.