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Research Article

Lucas Polynomial Approach for System of High-Order Linear Differential Equations and Residual Error Estimation

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An approximation method based on Lucas polynomials is presented for the solution of the system of high-order linear differential equations with variable coefficients under the mixed conditions. This method transforms the system of ordinary differential equations (ODEs) to the linear algebraic equations system by expanding the approximate solutions in terms of the Lucas polynomials with unknown coefficients and by using the matrix operations and collocation points. In addition, the error analysis based on residual function is developed for present method. To demonstrate the efficiency and accuracy of the method, numerical examples are given with the help of computer programmes written in *Maple* and *Matlab*.

1. Introduction

The systems of differential equations with variable coefficients have been encountered in many scientific and technological problems. Some of these differential equation systems do not have analytic solutions, so numerical methods are required. The systems of linear differential equations have been solved by many mathematicians and engineers by using the various methods such as variational iteration method [1], the differential transform method [2–5], the Adomian decomposition method [6, 7] and the linearizability criteria [8, 9], finite difference method [10], and Adomian-Pade technique [11].

Taylor, Chebyshev, Legendre, Berstein, Hermite, Laguerre, and Bessel matrix methods are used for solving differential and integral equations, integrodifferential-difference equations, and their systems in [12–20]. In this paper, by means of the above-mentioned methods and the Lucas polynomials, we have developed a new method called Lucas collocation method to solve the system of high-order linear differential equations with variable coefficients in the form

$$L\left[y_i(x)\right] = \sum_{n=0}^{m}\sum_{j=1}^{k} p_{i,j}^{n}(x)\, y_j^{(n)}(x) = g_i(x), \quad (i = 1, 2, \ldots, k,\ 0 \le a \le x \le b) \tag{1}$$

under the mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j} y_j^{(n)}(a) + b_{i,n}^{j} y_j^{(n)}(b)\right] = c_{j,i}, \quad \left(i = 0, 1, \ldots, m-1,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m-1\right), \tag{2}$$

where $y_j^{(0)}(x) = y_j(x)$ is an unknown function, $p_{i,j}^{n}(x)$ and $g_i(x)$ are the known continuous functions defined on interval $[a, b]$, and coefficients $a_{i,n}^{j}$, $b_{i,n}^{j}$ and $c_{j,i}$ are the real constants.

In addition, by improving the present method with the help of the residual error function used in [21–25], we obtain the corrected approximate solutions of the system (1) expressed in the truncated Lucas series form

$$y_{j,N,M}(x) = y_{j,N}(x) + e_{j,N,M}(x), \quad j = 1, 2, \ldots, k, \tag{3}$$

where

$$y_j \cong y_{j,N}(x) = \sum_{n=0}^{N} a_{j,n} L_n(x) \tag{4}$$

is the Lucas polynomial solution and

$$e_{j,N,M}(x) = \sum_{n=0}^{M} a_{j,n}^{*} L_n(x), \quad (M > N) \tag{5}$$

is the solution of the error problem obtained with the aid of the residual error function. Here $a_{j,n}$, $(n = 0, 1, 2, \ldots, N)$ and $a^*_{j,n}$, $(n = 0, 1, 2, \ldots, N)$ are the unknown Lucas coefficients, and $L_n(x)$, $n = 0, 1, \ldots, N$ are the Lucas polynomials defined by

$$L_0(x) = 2;$$
$$L_n(x) = \sum_{k=0}^{[n/2]} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}, \quad (n \geq 1), \tag{6}$$
$$\left[\frac{n}{2}\right] = \begin{cases} \dfrac{n}{2}, & n \text{ even} \\ \dfrac{(n-1)}{2}, & n \text{ odd} \end{cases}$$

[26–28]. The purpose of this study is to improve the approximate solutions for high-order systems of ODEs by means of the residual error function and to give an efficient and useful error estimation via the error problem.

In order to find solutions of the system (1), with the mixed conditions (2), we can use the collocation points defined by

$$x_i = a + \frac{b-a}{N} i, \quad i = 0, 1, \ldots, N, \ 0 \leq a \leq x \leq b. \tag{7}$$

2. Fundamental Matrix Relations

The Lucas polynomials $L_n(x)$ can be written in the matrix form as

$$\mathbf{L}(x) = \mathbf{X}(x)\mathbf{D}^T, \tag{8}$$

where

$$\mathbf{L}(x) = \left[L_0(x) \ \ L_1(x) \ \ L_2(x) \ \ \cdots \ \ L_N(x)\right],$$
$$\mathbf{X}(x) = \left[1 \ \ x \ \ x^2 \ \ \cdots \ \ x^N\right], \tag{9}$$

and if N is odd,

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0 \\ \frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{n-1}{(n-1)/2}\begin{pmatrix} \frac{(n-1)}{2} \\ \frac{(n-1)}{2} \end{pmatrix} & 0 & \frac{n-1}{(n+1)/2}\begin{pmatrix} \frac{(n+1)}{2} \\ \frac{(n-3)}{2} \end{pmatrix} & 0 & \frac{n-1}{(n+3)/2}\begin{pmatrix} \frac{(n+3)}{2} \\ \frac{(n-5)}{2} \end{pmatrix} & & 0 \\ 0 & \frac{n}{(n+1)/2}\begin{pmatrix} \frac{(n+1)}{2} \\ \frac{(n-1)}{2} \end{pmatrix} & 0 & \frac{n}{(n+3)/2}\begin{pmatrix} \frac{(n+3)}{2} \\ \frac{(n-3)}{2} \end{pmatrix} & 0 & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}. \tag{10}$$

If N is even,

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \frac{3}{3}\binom{3}{0} & 0 & \cdots & 0 \\ \frac{4}{2}\binom{2}{2} & 0 & \frac{4}{3}\binom{3}{1} & 0 & \frac{4}{4}\binom{4}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{n-1}{n/2}\begin{pmatrix} \frac{n}{2} \\ \frac{(n-2)}{2} \end{pmatrix} & 0 & \frac{n-1}{(n+2)/2}\begin{pmatrix} \frac{(n+2)}{2} \\ \frac{(n-4)}{2} \end{pmatrix} & 0 & \cdots & 0 \\ \frac{n}{n/2}\begin{pmatrix} \frac{n}{2} \\ \frac{n}{2} \end{pmatrix} & 0 & \frac{n}{(n+2)/2}\begin{pmatrix} \frac{(n+2)}{2} \\ \frac{(n-2)}{2} \end{pmatrix} & 0 & \frac{n}{(n+4)/2}\begin{pmatrix} \frac{(n+4)}{2} \\ \frac{(n-4)}{2} \end{pmatrix} & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}. \tag{11}$$

We can write the approximate solutions $y_{j,N}(x)$ given by (4) in the matrix form

$$y_{j,N}(x) = \mathbf{L}(x)\mathbf{A}_j, \quad j = 1, 2, \ldots, k, \tag{12}$$

where

$$\mathbf{A}_j = \left[a_{j,0} \;\; a_{j,1} \;\; a_{j,2} \;\; \cdots \;\; a_{j,N}\right]^T. \tag{13}$$

From (8) and (12), we obtain the matrix relation

$$y_{j,N}(x) = \mathbf{X}(x)\mathbf{D}^T\mathbf{A}_j. \tag{14}$$

Also, the relation between the matrix $\mathbf{X}(x)$ and its derivatives $\mathbf{X}^{(k)}(x)$ is

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k, \tag{15}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{16}$$

and $\mathbf{B}^0 = \mathbf{I}_{(N+1)\times(N+1)}$ is the unit matrix.

By using the relations (14) and (15), we obtain the following relations:

$$\begin{gathered} y_{j,N}^{(i)}(x) = \mathbf{X}(x)\mathbf{B}^i\mathbf{D}^T\mathbf{A}_j, \\ i = 0, 1, \ldots, m, \quad j = 1, 2, \ldots, k. \end{gathered} \tag{17}$$

Hence, we can write the matrix relations as

$$\mathbf{Y}^{(i)}(x) = \overline{\mathbf{X}}(x)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}, \quad i = 0, 1, \ldots, m, \tag{18}$$

where

$$\begin{gathered} \mathbf{Y}^{(i)}(x) = \begin{bmatrix} y_{1,N}^{(i)}(x) \\ y_{2,N}^{(i)}(x) \\ \vdots \\ y_{k,N}^{(i)}(x) \end{bmatrix}, \\ \overline{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x) \end{bmatrix}, \\ \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}, \\ \overline{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 & \cdots & 0 \\ 0 & \mathbf{D}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^T \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix}. \end{gathered} \tag{19}$$

3. Method for Solution

Firstly, we can write the system (1) in the matrix form

$$\sum_{i=0}^{m}\mathbf{P}_i(x)\mathbf{Y}^{(i)}(x) = \mathbf{G}(x), \tag{20}$$

where

$$\begin{gathered} \mathbf{P}_i(x) = \begin{bmatrix} p_{1,1}^i(x) & p_{1,2}^i(x) & \cdots & p_{1,k}^i(x) \\ p_{2,1}^i(x) & p_{2,2}^i(x) & \cdots & p_{2,k}^i(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{k,1}^i(x) & p_{k,2}^i(x) & \cdots & p_{k,k}^i(x) \end{bmatrix}, \\ \mathbf{Y}^{(i)}(x) = \begin{bmatrix} y_{1,N}^{(i)}(x) \\ y_{2,N}^{(i)}(x) \\ \vdots \\ y_{k,N}^{(i)}(x) \end{bmatrix}, \qquad \mathbf{G}(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_k(x) \end{bmatrix}. \end{gathered} \tag{21}$$

By substituting the collocation points (7) into (20) we obtain the system of matrix equations

$$\sum_{i=0}^{m}\mathbf{P}_i(x_s)\mathbf{Y}^{(i)}(x_s) = \mathbf{G}(x_s), \quad s = 0, 1, \ldots, N, \tag{22}$$

or the compact form

$$\sum_{i=0}^{m}\mathbf{P}_i\mathbf{Y}^{(i)} = \mathbf{G}, \tag{23}$$

where

$$\begin{gathered} \mathbf{P}_i = \begin{bmatrix} \mathbf{P}_i(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_i(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_i(x_N) \end{bmatrix}, \\ \mathbf{Y}^{(i)} = \begin{bmatrix} \mathbf{Y}^{(i)}(x_0) \\ \mathbf{Y}^{(i)}(x_1) \\ \vdots \\ \mathbf{Y}^{(i)}(x_N) \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{G}(x_0) \\ \mathbf{G}(x_1) \\ \vdots \\ \mathbf{G}(x_N) \end{bmatrix}. \end{gathered} \tag{24}$$

From the relation (18) and the collocation points (7), we have

$$\mathbf{Y}^{(i)}(x_s) = \overline{\mathbf{X}}(x_s)\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}, \quad s = 0, 1, \ldots, N, \tag{25}$$

or, briefly,

$$\mathbf{Y}^{(i)} = \mathbf{X}\left(\overline{\mathbf{B}}\right)^i\overline{\mathbf{D}}\mathbf{A}, \tag{26}$$

where

$$\mathbf{X} = \begin{bmatrix} \overline{\mathbf{X}}(x_0) \\ \overline{\mathbf{X}}(x_1) \\ \vdots \\ \overline{\mathbf{X}}(x_N) \end{bmatrix},$$
$$\overline{\mathbf{X}}(x_s) = \begin{bmatrix} \mathbf{X}(x_s) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x_s) \end{bmatrix}. \tag{27}$$

By substituting (26) into (23), we obtain the fundamental matrix equation as

$$\left\{ \sum_{i=0}^{m} \mathbf{P}_i \mathbf{X} \left(\overline{\mathbf{B}}\right)^i \overline{\mathbf{D}} \right\} \mathbf{A} = \mathbf{G}. \tag{28}$$

In (28) the full dimensions of the matrices $\mathbf{P}_i$, $\mathbf{X}$, $\overline{\mathbf{B}}$, $\overline{\mathbf{D}}$, $\mathbf{A}$, and $\mathbf{G}$ are $k(N+1) \times k(N+1)$, $k(N+1) \times k(N+1)$, $k(N+1) \times k(N+1)$, $k(N+1) \times k(N+1)$, $k(N+1) \times 1$, and $k(N+1) \times 1$, respectively.

The fundamental matrix equation (28) corresponding to (1) can be written in the form

$$\mathbf{WA} = \mathbf{G} \quad \text{or} \quad [\mathbf{W}; \mathbf{G}]. \tag{29}$$

This is a linear system of $k(N+1)$ algebraic equations in the $k(N+1)$ unknown Lucas coefficients such that

$$\mathbf{W} = \sum_{i=0}^{m} \mathbf{P}_i \mathbf{X} \left(\overline{\mathbf{B}}\right)^i \overline{\mathbf{D}} = \left[w_{p,q}\right],$$
$$p, q = 1, 2, \ldots, k(N+1). \tag{30}$$

By using the conditions (2) and the relations (18), the matrix form for the conditions is obtained as

$$\sum_{j=0}^{m-1} \left[a_j \overline{\mathbf{X}}(a) + b_j \overline{\mathbf{X}}(b)\right] \left(\overline{\mathbf{B}}\right)^j \overline{\mathbf{D}} \mathbf{A} = \mathbf{C}, \tag{31}$$

where

$$a_j = \begin{bmatrix} a_j^1 & 0 & \cdots & 0 \\ 0 & a_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_j^k \end{bmatrix},$$
$$b_j = \begin{bmatrix} b_j^1 & 0 & \cdots & 0 \\ 0 & b_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_j^k \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix},$$
$$a_j^i = \begin{bmatrix} a_{0,j}^i \\ a_{1,j}^i \\ \vdots \\ a_{m-1,j}^i \end{bmatrix}, \qquad b_j^i = \begin{bmatrix} b_{0,j}^i \\ b_{1,j}^i \\ \vdots \\ b_{m-1,j}^i \end{bmatrix}, \qquad c_i = \begin{bmatrix} c_{i,0} \\ c_{i,1} \\ \vdots \\ c_{i,m-1} \end{bmatrix},$$
$$i = 0, 1, \ldots, k. \tag{32}$$

Hence, the fundamental matrix form for conditions is

$$\mathbf{UA} = \mathbf{C} \quad \text{or} \quad [\mathbf{U}; \mathbf{C}] \tag{33}$$

such that

$$\mathbf{U} = \sum_{j=0}^{m-1} \left[a_j \overline{\mathbf{X}}(a) + b_j \overline{\mathbf{X}}(b)\right] \left(\overline{\mathbf{B}}\right)^j \overline{\mathbf{D}}. \tag{34}$$

Consequently, by replacing the row matrices (33) by last rows of the matrix (29), we obtain the new augmented matrix

$$\left[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}\right]. \tag{35}$$

We do not have to change the last rows of the matrix equation given by (29). If the matrix $\widetilde{\mathbf{W}}$ is singular, then rows of the matrix (33) can be replaced with any rows of the matrix (29). If rank $\widetilde{\mathbf{W}}$ = rank$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = N+1$, then we can write

$$\mathbf{A} = \left(\widetilde{\mathbf{W}}\right)^{-1} \widetilde{\mathbf{G}}. \tag{36}$$

Hence, the unknown Lucas coefficients matrix $\mathbf{A}$ is determined. We can find the Lucas polynomial solutions

$$y_{j,N}(x) = \sum_{n=0}^{N} a_{j,n} L_n(x), \quad j = 1, 2, \ldots, k. \tag{37}$$

4. Residual Correction and Error Estimation

In this section, we will give an error estimation for the Lucas polynomial solutions (4) with the residual error function [21–25]. Moreover, we will improve the solution (4) by means of the residual error function. Firstly, we can define the residual function of the method as

$$R_{i,N}(x) = L\left[y_{j,N}(x)\right] - g_i(x),$$
$$\left(i = 1, 2, \ldots, k, \ j = 1, 2, \ldots, k\right). \tag{38}$$

Here, $y_{j,N}(x)$ represent the Lucas polynomial solutions given by (4) of the problem (1) and (2). Hence, $y_{j,N}(x)$ satisfies the problem

$$L\left[y_{j,N}(x)\right] = \sum_{n=0}^{m} \sum_{j=1}^{k} p_{i,j}^n(x) y_{j,N}^{(n)}(x) = g_i(x) + R_{i,N}(x),$$
$$\left(i = 1, 2, \ldots, k, \ j = 1, 2, \ldots, k, \ n = 0, 1, \ldots, m\right),$$
$$\sum_{n=0}^{m-1} \left[a_{i,n}^j y_{j,N}^{(n)}(a) + b_{i,n}^j y_{j,N}^{(n)}(b)\right] = c_{j,i},$$
$$\left(i = 0, 1, \ldots, m-1, \ j = 1, 2, \ldots, k, \ n = 0, 1, \ldots, m-1\right). \tag{39}$$

Also, the error function $e_{j,N}(x)$ can be defined as

$$e_{j,N}(x) = y_j(x) - y_{j,N}(x), \tag{40}$$

where $y_j(x)$ are the exact solutions of the problem (1) and (2). From (1), (2), (38), and (40), we obtain the error differential equation system

$$L\left[e_{j,N}(x)\right] = L\left[y_j(x)\right] - L\left[y_{j,N}(x)\right] = -R_{i,N}(x) \tag{41}$$

with the homogeneous mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j} e_{j,N}^{(n)}(a) + b_{i,n}^{j} e_{j,N}^{(n)}(b)\right] = 0,$$
$$\left(i = 0, 1, \ldots, m-1,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m-1\right) \tag{42}$$

or, openly, the error problem

$$\sum_{n=0}^{m}\sum_{j=1}^{k} p_{i,j}^{n}(x)\, e_{j,N}^{(n)}(x) = -R_{i,N}(x),$$
$$\left(i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m\right),$$
$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j} e_{j,N}^{(n)}(a) + b_{i,n}^{j} e_{j,N}^{(n)}(b)\right] = 0,$$
$$\left(i = 0, 1, \ldots, m-1,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m-1\right). \tag{43}$$

Here, note that the nonhomogeneous mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j} y_{j}^{(n)}(a) + b_{i,n}^{j} y_{j}^{(n)}(b)\right] = c_{j,i},$$
$$\left(i = 0, 1, \ldots, m-1,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m-1\right),$$
$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j} y_{j,N}^{(n)}(a) + b_{i,n}^{j} y_{j,N}^{(n)}(b)\right] = c_{j,i},$$
$$\left(i = 0, 1, \ldots, m-1,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m-1\right) \tag{44}$$

are reduced to homogeneous mixed conditions

$$\sum_{n=0}^{m-1}\left[a_{i,n}^{j} e_{j,N}^{(n)}(a) + b_{i,n}^{j} e_{j,N}^{(n)}(b)\right] = 0,$$
$$\left(i = 0, 1, \ldots, m-1,\ j = 1, 2, \ldots, k,\ n = 0, 1, \ldots, m-1\right). \tag{45}$$

The error problem (43) can be solved by using the procedure in Section 3. Thus, we obtain the approximation $e_{j,N,M}(x)$ to $e_{j,N}(x)$ as follows:

$$e_{j,N,M}(x) = \sum_{n=0}^{M} a_{j,n}^{*} L_n(x), \quad \left(M > N,\ j = 1, 2, \ldots, k\right). \tag{46}$$

Consequently, the corrected Lucas polynomial solution $y_{j,N,M}(x) = y_{j,N}(x) + e_{j,N,M}(x)$ is obtained by means of the polynomials $y_{j,N}(x)$ and $e_{j,N,M}(x)$, $M > N$. Also, we construct the error function $e_{j,N}(x) = y_j(x) - y_{j,N}(x)$, the estimated error function $e_{j,N,M}(x)$, and the corrected error function $E_{j,N,M}(x) = e_{j,N}(x) - e_{j,N,M}(x) = y_j(x) - y_{j,N,M}(x)$.

5. Numerical Examples

In this section, the several numerical examples are given to demonstrate the efficiency and applicability of our method. The computations related to the examples are calculated by using a computer programme which is called *Maple* and the figures are drawn in *Matlab*. In tables and figures, we calculate the values of the Lucas polynomial solution $y_{j,N}(x)$, the corrected Lucas polynomial solution $y_{j,N,M}(x) = y_{j,N}(x) + e_{j,N,M}(x)$, the actual absolute error function $|e_{j,N}(x)| = |y_j(x) - y_{j,N}(x)|$, and the estimated absolute error function $|e_{j,N,M}(x)|$.

Example 1. Let us consider the system of second-order linear differential equations given by

$$y_1^{(2)}(x) + xy_1(x) + xy_2(x) = 2,$$
$$y_2^{(2)}(x) + 2xy_2(x) + 2xy_1(x) = -2, \tag{47}$$
$$0 \le x \le 1$$

with the boundary conditions

$$y_1(0) = 0, \qquad y_1(1) = 0,$$
$$y_2(0) = 0, \qquad y_2(1) = 0 \tag{48}$$

which has the exact solutions $y_1(x) = x^2 - x$ and $y_2(x) = -x^2 + x$ [30]. In this problem $k = 2$, $m = 2$, $p_{1,1}^{2} = 1$, $p_{1,1}^{1} = 0$, $p_{1,1}^{0} = x$, $p_{1,2}^{2} = 0$, $p_{1,2}^{1} = 0$, $p_{1,2}^{0} = x$, $g_1(x) = 2$, $p_{2,1}^{2} = 0$, $p_{2,1}^{1} = 0$, $p_{2,1}^{0} = 2x$, $p_{2,2}^{2} = 1$, $p_{2,2}^{1} = 0$, $p_{2,2}^{0} = 2x$, and $g_2(x) = -2$.

The approximate solutions $y_{1,2}(x)$ and $y_{2,2}(x)$ for $N = 2$ are given by

$$y_{i,2}(x) = \sum_{n=0}^{2} a_{i,n} L_n(x), \quad (i = 1, 2). \tag{49}$$

The set of the collocation points given by (7) for $a = 0$, $b = 1$, and $N = 2$ is calculated as

$$\left\{x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1\right\}. \tag{50}$$

From (28), the fundamental matrix equation of the problem (47) is written as

$$\left\{\mathbf{P}_0\mathbf{X}\overline{\mathbf{D}} + \mathbf{P}_1\mathbf{X}\overline{\mathbf{B}}\,\overline{\mathbf{D}} + \mathbf{P}_2\mathbf{X}\left(\overline{\mathbf{B}}\right)^2\overline{\mathbf{D}}\right\}\mathbf{A} = \mathbf{G}. \tag{51}$$

By applying the procedure in Section 3, we obtain the Lucas polynomial solutions for $N = 2$ as $y_{1,2}(x) = x^2 - x$ and $y_{2,2}(x) = -x^2 + x$, which are the exact solutions.

Table 1: Numerical results of the exact solutions $y_i(x)$ and the approximate solutions $\{y_{i,N}(x), y_{i,N,M}(x)\}$ for $i = 1, 2$, $N = 6$ and $M = 8, 11$ of problem (52).

x_i	Exact solution	Lucas polynomial solutions		
	$y_1(x_i)$	$y_{1,6}(x_i)$	$y_{1,6,8}(x_i)$	$y_{1,6,11}(x_i)$
0.1	0.806485719482	0.806485689143	0.806485719287	0.806485719292
0.2	0.625251708173	0.625251667103	0.625251707990	0.625251708032
0.5	0.145975834384	0.145975783291	0.145975834235	0.145975834133
0.8	−0.252886020789	−0.252886103976	−0.252886020930	−0.252886021210
1.0	−0.482526627107	−0.482526066408	−0.482526628764	−0.482526626284
x_i	$y_2(x_i)$	$y_{2,6}(x_i)$	$y_{2,6,8}(x_i)$	$y_{2,6,11}(x_i)$
0.1	0.098405441705	0.098405456828	0.098405441810	0.098405441788
0.2	0.193895101008	0.193895121487	0.193895101121	0.193895101070
0.5	0.466457257480	0.466457282978	0.466457257577	0.466457257606
0.8	0.724228025488	0.724228067006	0.724228025625	0.724228025747
1.0	0.890857245275	0.890856966173	0.890857246295	0.890857244962

Table 2: Comparison of the actual and estimated absolute errors for $N = 6$ and $M = 8, 11$ of the problem (52).

x_i	Actual absolute errors	Estimated absolute errors	
	$\lvert e_{1,6}(x_i)\rvert = \lvert y_1(x_i) - y_{1,6}(x_i)\rvert$	$\lvert e_{1,6,8}(x_i)\rvert$	$\lvert e_{1,6,11}(x_i)\rvert$
0.1	$3.0339e-8$	$3.0144e-8$	$3.0149e-8$
0.2	$4.1070e-8$	$4.0888e-8$	$4.0930e-8$
0.5	$5.1093e-8$	$5.0951e-8$	$5.0851e-8$
0.8	$8.3187e-8$	$8.3065e-8$	$8.2796e-8$
1.0	$5.6070e-7$	$5.6232e-7$	$5.5982e-7$
x_i	$\lvert e_{2,6}(x_i)\rvert = \lvert y_2(x_i) - y_{2,6}(x_i)\rvert$	$\lvert e_{2,6,8}(x_i)\rvert$	$\lvert e_{2,6,11}(x_i)\rvert$
0.1	$1.5123e-8$	$1.5017e-8$	$1.5040e-8$
0.2	$2.0479e-8$	$2.0367e-8$	$2.0418e-8$
0.5	$2.5498e-8$	$2.5404e-8$	$2.5377e-8$
0.8	$4.1518e-8$	$4.1390e-8$	$4.1270e-8$
1.0	$2.7910e-7$	$2.8011e-7$	$2.7877e-7$

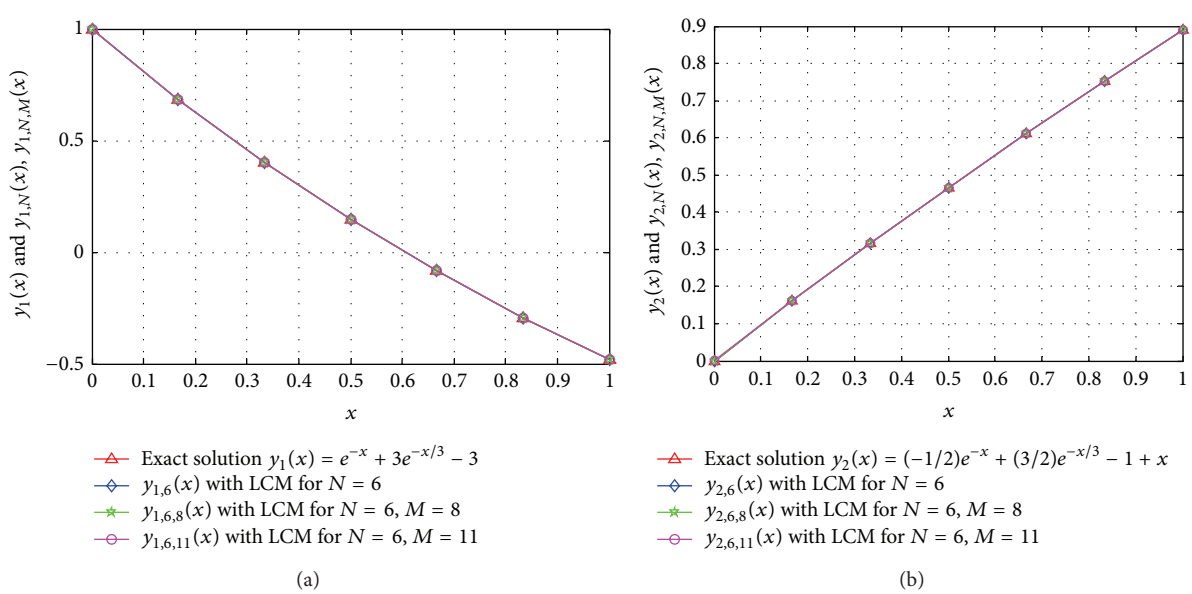

Figure 1: (a) Comparison of the exact solution $y_1(x)$ and the approximate solutions $y_{1,N}(x)$ and $y_{1,N,M}(x)$. (b) Comparison of the exact solution $y_2(x)$ and the approximate solutions $y_{2,N}(x)$ and $y_{2,N,M}(x)$.

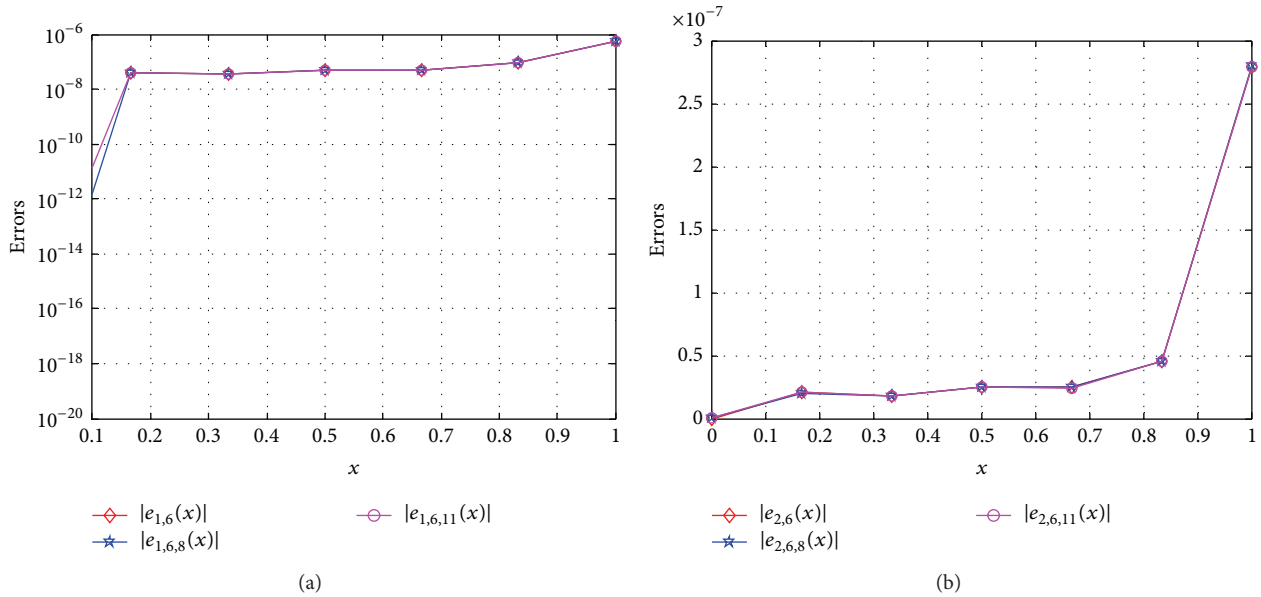

Figure 2: (a) Comparison of the actual and estimated absolute errors for $y_1(x)$. (b) Comparison of the actual and estimated absolute errors for $y_2(x)$.

Table 3: Numerical results of the exact solutions $y_1(x)$, $y_2(x)$ and the Lucas polynomial solutions $y_{1,N}(x)$, $y_{2,N}(x)$ for $N = 6, 8, 11$ of problem (52).

x_i	Exact solution	Lucas polynomial solutions		
	$y_1(x_i)$	$y_{1,6}(x_i)$	$y_{1,8}(x_i)$	$y_{1,11}(x_i)$
0.1	0.806485719482	0.806485689143	0.806485719432	0.806485719479
0.2	0.625251708173	0.625251667103	0.625251708124	0.625251708169
0.5	0.145975834384	0.145975783291	0.145975834317	0.145975834379
0.8	−0.252886020789	−0.252886103976	−0.252886020880	−0.252886020798
1.0	−0.482526627107	−0.482526066408	−0.482526625954	−0.482526627122
x_i	$y_2(x_i)$	$y_{2,6}(x_i)$	$y_{2,8}(x_i)$	$y_{2,11}(x_i)$
0.1	0.098405441705	0.098405456828	0.098405441730	0.098405441706
0.2	0.193895101008	0.193895121487	0.193895101033	0.193895101010
0.5	0.466457257480	0.466457282978	0.466457257512	0.466457257479
0.8	0.724228025488	0.724228067006	0.724228025534	0.724228025484
1.0	0.890857245275	0.890856966173	0.890857244698	0.890857245264

Table 4: Comparison of the absolute error functions $|e_{1,N}(x)|$, $|e_{2,N}(x)|$ for $N = 5, 6, 8, 11$ of problem (52).

	Lucas collocation method			
x_i	$\lvert e_{1,5}(x_i)\rvert$	$\lvert e_{1,6}(x_i)\rvert$	$\lvert e_{1,8}(x_i)\rvert$	$\lvert e_{1,11}(x_i)\rvert$
0.1	$5.9187e-7$	$3.0339e-8$	$4.9550e-11$	$2.3912e-12$
0.2	$1.0110e-6$	$4.1070e-8$	$4.9303e-11$	$2.8583e-12$
0.5	$1.0978e-6$	$5.1093e-8$	$6.4840e-11$	$4.8214e-12$
0.8	$9.0094e-7$	$8.3187e-8$	$9.1255e-11$	$8.8867e-12$
1.0	$1.3659e-5$	$5.6070e-7$	$1.1536e-9$	$1.4366e-11$
x_i	$\lvert e_{2,5}(x_i)\rvert$	$\lvert e_{2,6}(x_i)\rvert$	$\lvert e_{2,8}(x_i)\rvert$	$\lvert e_{2,11}(x_i)\rvert$
0.1	$2.9327e-7$	$1.5123e-8$	$2.4764e-11$	$1.5621e-12$
0.2	$5.0134e-7$	$2.0479e-8$	$2.4641e-11$	$1.5043e-12$
0.5	$5.4596e-7$	$2.5498e-8$	$3.2407e-11$	$2.5725e-12$
0.8	$4.5116e-7$	$4.1518e-8$	$4.5610e-11$	$6.7968e-12$
1.0	$6.7555e-6$	$2.7910e-7$	$5.7653e-10$	$1.3288e-11$

Table 5: Comparison of the absolute error functions $|e_{1,N}(x)|$ and $|e_{2,N}(x)|$ for different methods of problem (52).

x_i	Chebyshev method in [12] $\lvert e_{1,5}(x_i)\rvert$	Stehfest Method in [29]	Present method $\lvert e_{1,5}(x_i)\rvert$	Present method $\lvert e_{1,11}(x_i)\rvert$
0.1	$4.5105e-5$	$6.7614e-5$	$5.9187e-7$	$2.3912e-12$
0.2	$7.9850e-5$	$8.4949e-5$	$1.0110e-6$	$2.8583e-12$
0.5	$9.7191e-5$	$3.1897e-3$	$1.0978e-6$	$4.8214e-12$
0.8	$8.0060e-5$	$5.2028e-3$	$9.0094e-7$	$8.8867e-12$
1.0	$1.0677e-4$	$1.1938e-2$	$1.3659e-5$	$1.4366e-11$
x_i	Chebyshev method in [12] $\lvert e_{2,5}(x_i)\rvert$	Stehfest Method in [29]	Present method $\lvert e_{2,5}(x_i)\rvert$	Present method $\lvert e_{2,11}(x_i)\rvert$
0.1	$2.2477e-5$	$8.4086e-6$	$2.9327e-7$	$1.5621e-12$
0.2	$3.9847e-5$	$1.9575e-5$	$5.0134e-7$	$1.5043e-12$
0.5	$4.8907e-5$	$2.2420e-4$	$5.4596e-7$	$2.5725e-12$
0.8	$4.0642e-5$	$4.6470e-4$	$4.5116e-7$	$6.7968e-12$
1.0	$5.3904e-5$	$4.7100e-4$	$6.7555e-6$	$1.3288e-11$

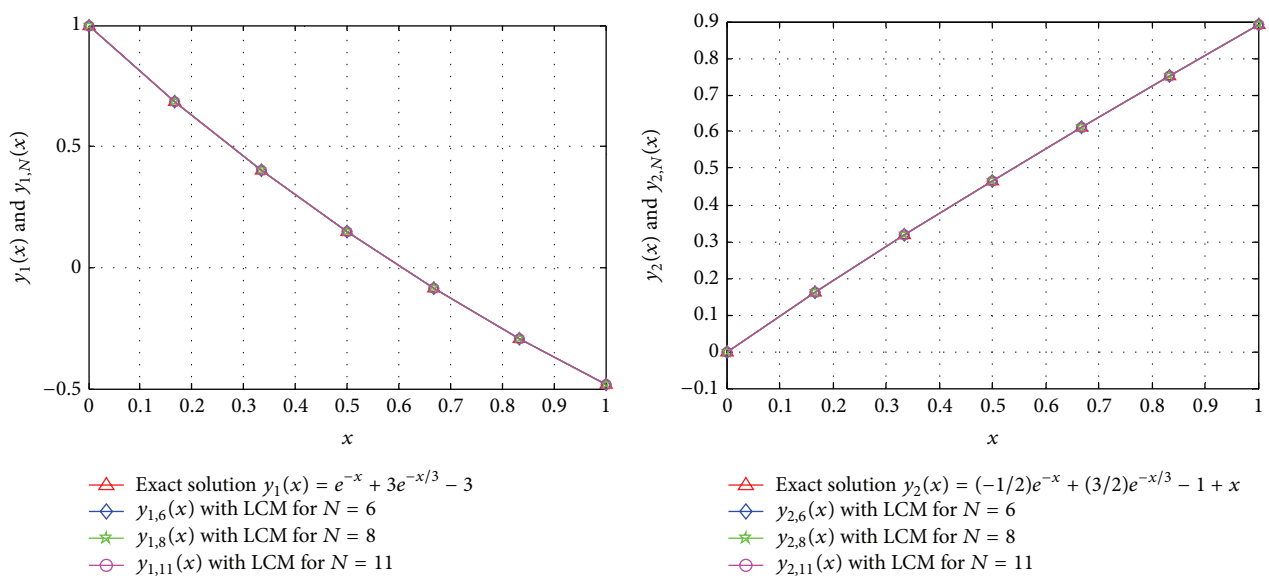

Figure 3: (a) Comparison of the exact solution $y_1(x)$ and the approximate solutions $y_{1,N}(x)$. (b) Comparison of the exact solution $y_2(x)$ and the approximate solutions $y_{2,N}(x)$.

Example 2. Let us consider the linear differential equations system given by

$$y_1^{(1)}(x) + y_2^{(1)}(x) + y_2(x) = x - e^{-x},$$
$$y_1^{(1)}(x) + 4y_2^{(1)}(x) + y_1(x) = 1 + 2e^{-x}, \qquad (52)$$
$$0 \le x \le 1$$

with the initial conditions

$$y_1(0) = 1, \qquad y_2(0) = 0 \qquad (53)$$

which has the exact solutions $y_1(x) = e^{-x} + 3e^{-x/3} - 3$ and $y_2(x) = -e^{-x}/2 + 3e^{-x/3}/2 - 1 + x$ [12, 29]. In this problem $k = 2$, $m = 1$, $p_{1,1}^1 = 1$, $p_{1,1}^0 = 0$, $p_{1,2}^1 = 1$, $p_{1,2}^0 = 1$, $g_1(x) = x - e^{-x}$, $p_{2,1}^1 = 1$, $p_{2,1}^0 = 1$, $p_{2,2}^1 = 4$, $p_{2,2}^0 = 0$, and $g_2(x) = 1 + 2e^{-x}$.

We can write the fundamental matrix equation of the problem (52) from (28) as

$$\left\{\mathbf{P}_0\mathbf{X}\overline{\mathbf{D}} + \mathbf{P}_1\mathbf{X}\overline{\mathbf{B}}\,\overline{\mathbf{D}}\right\}\mathbf{A} = \mathbf{G}. \qquad (54)$$

By using our method, the approximate solutions of the problem (52) for $N = 6$ are obtained as

$$\begin{aligned} y_{1,6}(x) = {} & -1.999999999999999904x \\ & + 0.6666586621368288521x^2 \\ & - 0.185113238503384888x^3 \\ & + (0.429421011617410024e - 1)\,x^4 \\ & - (0.7937368973290355e - 2)\,x^5 \\ & + (0.9237777698612969e - 3)\,x^6 \end{aligned}$$

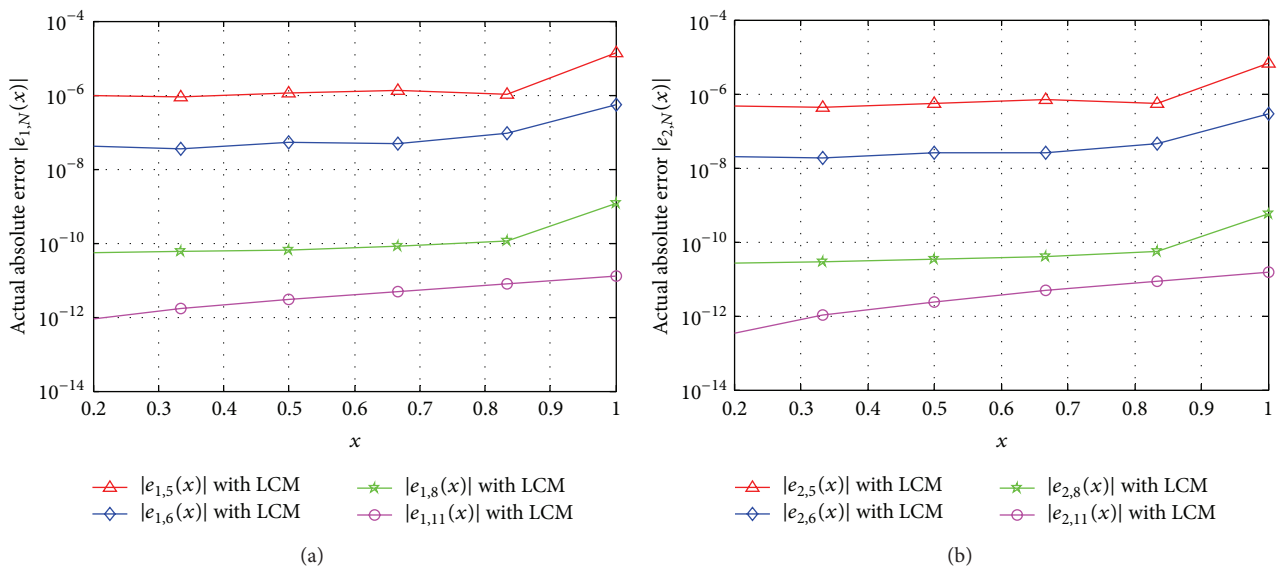

FIGURE 4: (a) Comparison of the absolute error functions $|e_{1,N}(x)|$. (b) Comparison of the absolute error functions $e_{2,N}(x)$.

TABLE 6: Numerical results of the exact solutions and the approximate solutions for $N = 2, 5, 9$ of problem (60).

x_i	Exact solution	Lucas polynomial solutions		
	$y_1(x_i)$	$y_{1,2}(x_i)$	$y_{1,5}(x_i)$	$y_{1,9}(x_i)$
0.1	1.105170918076	1.106706136074	1.105169808728	1.105170918072
0.2	1.221402758160	1.226824544296	1.221400910855	1.221402758157
0.5	1.648721270700	1.667653401853	1.648719504255	1.648721270696
0.8	2.225540928492	2.229192708743	2.225540191066	2.225540928487
1.0	2.718281828459	2.670613607410	2.718246576634	2.718281828322
	$y_2(x_i)$	$y_{2,2}(x_i)$	$y_{2,5}(x_i)$	$y_{2,9}(x_i)$
0.1	0.099833416646	0.098994748986	0.099833619019	0.099833416647
0.2	0.198669330795	0.195978995944	0.198669599126	0.198669330795
0.5	0.479425538604	0.474868724650	0.479425418757	0.479425538603
0.8	0.717356090899	0.735663935104	0.717355091659	0.717356090898
1.0	0.841470984808	0.899474898600	0.841480130265	0.841470984845
	$y_3(x_i)$	$y_{3,2}(x_i)$	$y_{3,5}(x_i)$	$y_{3,9}(x_i)$
0.1	2.100175083354	2.101927846772	2.100174590140	2.100175083352
0.2	2.201469336001	2.207711387088	2.201468393096	2.201469335999
0.5	2.526303832591	2.548196169301	2.526302577195	2.526303832587
0.8	2.922247637840	2.923382193410	2.922246514015	2.922247637835
1.0	3.258584134327	3.192784677202	3.258568370866	3.258584134265

$$\begin{aligned} &+ 1.0000000000000000808, \\ y_{2,6}(x) = {} & 1.0000000000000000105x \\ & - 0.16666267720756320 49x^2 \\ & + (0.74038219115823775e - 1)\, x^3 \\ & - (0.199282827658208166e - 1)\, x^4 \\ & + (0.3866617014883018e - 2)\, x^5 \\ & - (0.4569099838690821e - 3)\, x^6 \\ & + (0.108e - 16). \end{aligned} \tag{55}$$

In order to calculate the corrected Lucas polynomial solutions, let us consider the error problem

$$\begin{aligned} e_{1,6}^{(1)}(x) + e_{2,6}^{(1)}(x) + e_{2,6}(x) &= -R_{1,6}(x), \\ e_{1,6}^{(1)}(x) + 4e_{2,6}^{(1)}(x) + e_{1,6}(x) &= -R_{2,6}(x) \end{aligned} \tag{56}$$

such that $e_{1,6}(0) = 0$, $e_{2,6}(0) = 0$, and the residual functions are

$$\begin{aligned} R_{1,6}(x) &= y_{1,6}^{(1)}(x) + y_{2,6}^{(1)}(x) + y_{2,6}(x) - x + e^{-x}, \\ R_{2,6}(x) &= y_{1,6}^{(1)}(x) + 4y_{2,6}^{(1)}(x) + y_{1,6}(x) - 1 - 2e^{-x}. \end{aligned} \tag{57}$$

Table 7: Comparison of the actual absolute errors $|e_{1,N}(x)|$, $|e_{2,N}(x)|$ for $N = 2, 5, 9$ of problem (60).

x_i	$\lvert e_{1,2}(x_i)\rvert$	$\lvert e_{1,5}(x_i)\rvert$	$\lvert e_{1,9}(x_i)\rvert$
0.1	$1.5352e-3$	$1.1093e-6$	$3.5633e-12$
0.2	$5.4218e-3$	$1.8473e-6$	$3.1738e-12$
0.5	$1.8932e-2$	$1.7664e-6$	$4.0266e-12$
0.8	$3.6518e-3$	$7.3743e-7$	$5.3975e-12$
1.0	$4.7668e-2$	$3.5252e-5$	$1.3694e-10$
x_i	$\lvert e_{2,2}(x_i)\rvert$	$\lvert e_{2,5}(x_i)\rvert$	$\lvert e_{2,9}(x_i)\rvert$
0.1	$8.3867e-4$	$2.0237e-7$	$6.1484e-13$
0.2	$2.6903e-3$	$2.6833e-7$	$2.8246e-13$
0.5	$4.5568e-3$	$1.1985e-7$	$4.6422e-13$
0.8	$1.8308e-2$	$9.9924e-7$	$1.5204e-12$
1.0	$5.8004e-2$	$9.1455e-6$	$3.7266e-11$
x_i	$\lvert e_{3,2}(x_i)\rvert$	$\lvert e_{3,5}(x_i)\rvert$	$\lvert e_{3,9}(x_i)\rvert$
0.1	$1.7528e-3$	$4.9321e-7$	$1.7336e-12$
0.2	$6.2421e-3$	$9.4291e-7$	$1.8815e-12$
0.5	$2.1892e-2$	$1.2554e-6$	$3.0301e-12$
0.8	$1.1346e-3$	$1.1238e-6$	$4.3129e-12$
1.0	$6.5799e-2$	$1.5763e-5$	$6.2478e-11$

Table 8: Comparison of the errors of the methods for problem (60).

x_i	Differential transform method in [2]	Present method $\lvert e_{1,9}(x_i)\rvert$
0.1	$2.0092e-11$	$3.5633e-12$
0.2	$2.6046e-9$	$3.1738e-12$
0.5	$1.6526e-6$	$4.0266e-12$
0.8	$4.6173e-5$	$5.3975e-12$
1.0	$2.2627e-4$	$1.3694e-10$
x_i	Differential transform method in [2]	Present method $\lvert e_{2,9}(x_i)\rvert$
0.1	$1.9839e-11$	$6.1484e-13$
0.2	$2.5383e-9$	$2.8246e-13$
0.5	$1.5447e-6$	$4.6422e-13$
0.8	$4.1242e-5$	$1.5204e-12$
1.0	$1.9568e-4$	$3.7266e-11$
x_i	Differential transform method in [2]	Present method $\lvert e_{3,9}(x_i)\rvert$
0.1	$2.0340e-11$	$1.7336e-12$
0.2	$2.6681e-9$	$1.8815e-12$
0.5	$1.7493e-6$	$3.0301e-12$
0.8	$5.0305e-5$	$4.3129e-12$
1.0	$2.5080e-4$	$6.2478e-11$

By solving the error problem (56) for $M = 8$, the estimated Lucas error functions $e_{1,6,8}(x)$ and $e_{2,6,8}(x)$ to $e_{1,6}(x)$ and $e_{2,6}(x)$ are obtained as

$$\begin{aligned} e_{1,6,8}(x) = &-(0.1210e-17)\,x \\ &-(0.178885917374467780e-3)\,x^7 \\ &+(0.142216618722969108e-4)\,x^8 \\ &+(0.79326892394531528e-5)\,x^2 \\ &-(0.71083092941970440e-4)\,x^3 \\ &+(0.263310871469073366 0e-3)\,x^4 \\ &-(0.486294856240169128e-3)\,x^5 \\ &+(0.450236321527052776 4e-3)\,x^6 \\ &-(0.6384e-18), \\ e_{2,6,8}(x) = &(0.6913e-18)\,x \\ &+(0.892342831286434209e-4)\,x^7 \\ &-(0.710875043057030262e-5)\,x^8 \\ &-(0.39530091035351939e-5)\,x^2 \\ &+(0.354285130160153836e-4)\,x^3 \\ &-(0.131258747033565960 4e-3)\,x^4 \\ &+(0.242451368617999419 3e-3)\,x^5 \\ &-(0.224513547727827588 96e-3)\,x^6 \\ &-(0.18124e-18). \end{aligned} \tag{58}$$

Thus, we can calculate the corrected Lucas polynomial solutions $y_{1,6,8}(x)$ and $y_{2,6,8}(x)$ as

$$\begin{aligned} y_{1,6,8}(x) = &-2x + 0.6666665948x^2 \\ &-0.1851843216x^3 + (0.4320541203e-1)\,x^4 \\ &-(0.8423663829e-2)\,x^5 \\ &+(0.1374014091e-2)\,x^6 + 1 \\ &-(0.178885917374467780e-3)\,x^7 \\ &+(0.142216618722969108e-4)\,x^8, \end{aligned}$$

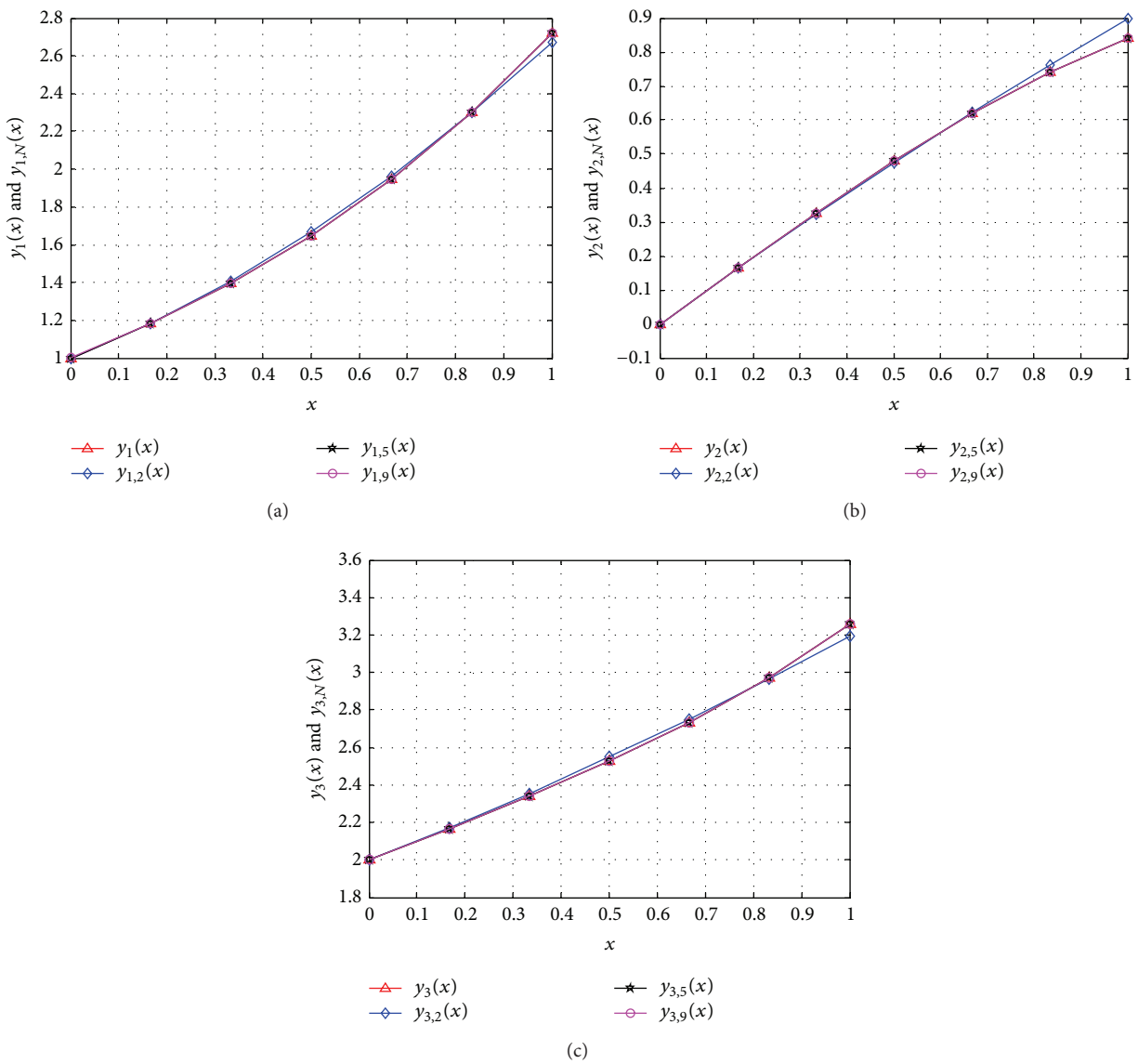

FIGURE 5: (a) Comparison of the exact solution $y_1(x)$ and the approximate solutions $y_{1,N}(x)$. (b) Comparison of the exact solution $y_2(x)$ and the approximate solutions $y_{2,N}(x)$. (c) Comparison of the exact solution $y_3(x)$ and the approximate solutions $y_{3,N}(x)$.

$$\begin{aligned} y_{2,6,8}(x) = {} & x - 0.1666666302x^2 \\ & + (0.7407364763e-1)\,x^3 \\ & - (0.2005954152e-1)\,x^4 \\ & + (0.4109068384e-2)\,x^5 \\ & - (0.6814235316e-3)\,x^6 + (0.1061876e-16) \\ & + (0.892342831286434209e-4)\,x^7 \\ & - (0.710875043057030262e-5)\,x^8. \end{aligned} \tag{59}$$

It is seen from Table 1 and Figures 1(a) and 1(b) that the accuracy of solution increases when the values of N and M increase.

Table 2 and Figures 2(a) and 2(b) display that the actual and estimated errors are very close to zero and almost identical.

Table 3 and Figures 3(a) and 3(b) show that when the value of N increases, the accuracy of solution increases.

Table 4 and Figures 4(a) and 4(b) show that the value of N is increased; the actual absolute errors decrease rapidly.

In addition, this problem was solved by Akyüz-Daşoğlu and Sezer [12] and Davies and Crann [29]. Now, let us compare our method (LCM) with the other methods (Chebyshev method and Stehfest method) given by [12, 29]. Table 5 indicates this comparison.

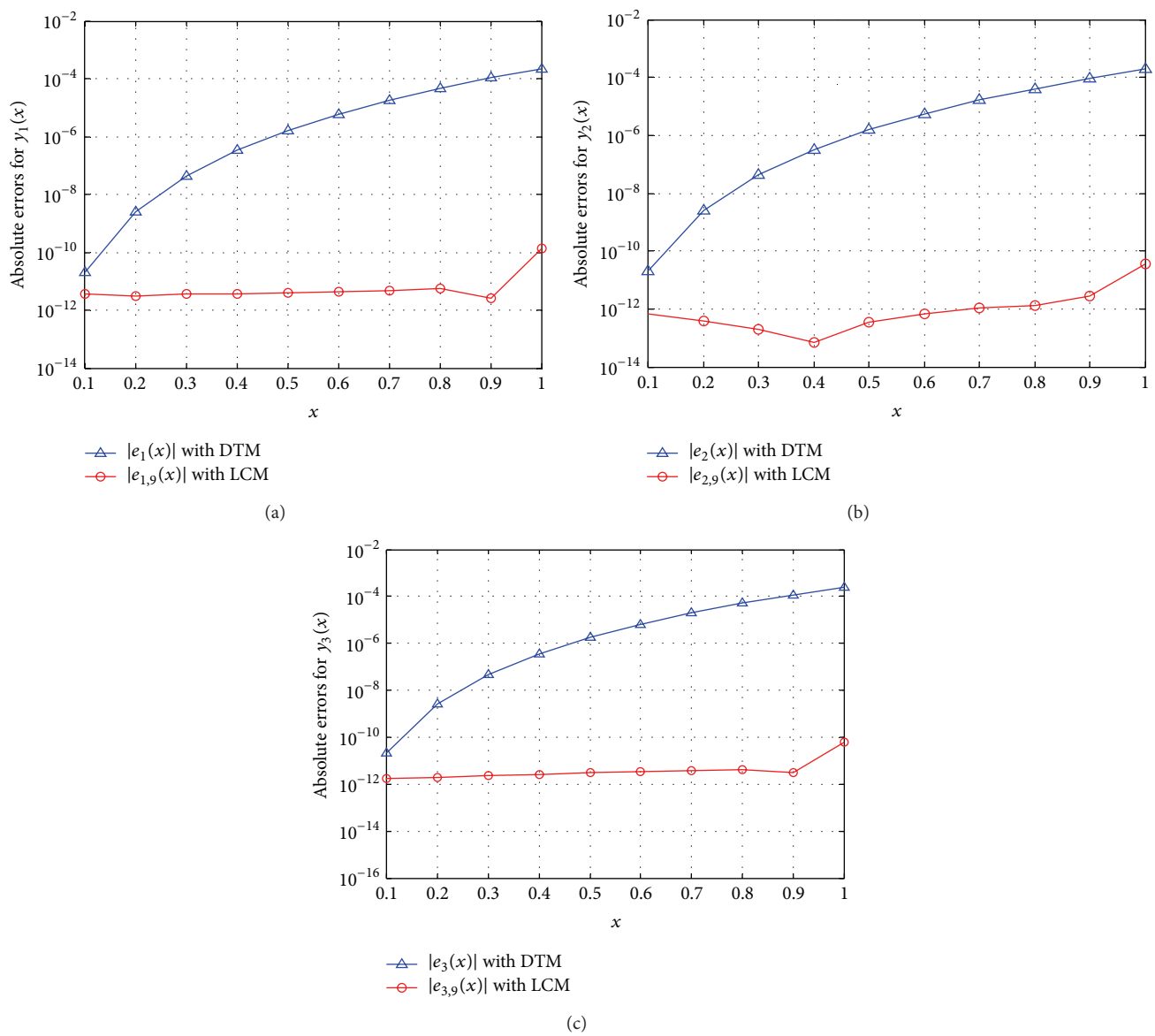

Figure 6: (a) Comparison of the absolute error functions for $y_1(x)$. (b) Comparison of the absolute error functions for $y_2(x)$. (c) Comparison of the absolute error functions for $y_3(x)$.

It is seen from Table 5 that the present method gives better approximations than the other methods given by [12, 29].

Example 3. Let us consider the linear differential equations system given by

$$\begin{aligned} y_1^{(1)}(x) - y_3(x) &= -\cos(x), \\ y_2^{(1)}(x) - y_3(x) &= -e^x, \\ y_3^{(1)}(x) - y_1(x) + y_2(x) &= 0, \\ 0 \le x \le 1 \end{aligned} \tag{60}$$

with the initial conditions

$$y_1(0) = 1, \qquad y_2(0) = 0, \qquad y_3(0) = 2 \tag{61}$$

which has the exact solutions $y_1(x) = e^x$, $y_2(x) = \sin(x)$, and $y_3(x) = e^x + \cos(x)$ [2]. By using the method, the approximate solutions of the problem (60) for $N = 2, 5$ are obtained as

$$\begin{aligned} y_{1,2}(x) &= 1 + x + 0.67061360741023699830x^2, \\ y_{2,2}(x) &= -(0.1e-18) + x - 0.1005251013995184324x^2, \\ y_{3,2}(x) &= 2 + x + 0.19278467720243885767x^2, \\ y_{1,5}(x) &= 0.99999999999999999824 \\ &\quad + 1.0000000000000000125x \\ &\quad + 0.49976858483072423438x^2 \\ &\quad + 0.16822545527123026922x^3 \end{aligned}$$

$$
\begin{aligned}
&+ (0.3772109990725761363e - 1)\, x^4 \\
&+ (0.12531436624829093170e - 1)\, x^5, \\
y_{2,5}(x) = {} & +1.000000000000000009x \\
&+ (0.45327625709752e - 4)\, x^2 \\
&- 0.166988949339599508x^3 \\
&+ (0.782824607296818e - 3)\, x^4 \\
&+ (0.7640927371818304e - 2)\, x^5 + (0.6e - 17), \\
y_{3,5}(x) = {} & 1.9999999999999999978 \\
&+ 1.0000000000000000004x \\
&- (0.938977776578073e - 4)\, x^2 \\
&+ 0.1672264869687396031x^3 \\
&+ (0.820919229141536686e - 1)\, x^4 \\
&+ (0.93438587611964760e - 2)\, x^5.
\end{aligned} \tag{62}
$$

It is seen from Table 6 and Figures 5(a), 5(b), and 5(c) that the accuracy increases as the N increase.

Table 7 shows that while the value of N is increased, the errors decrease rapidly. Now, we compare the present method with the differential transform method given by [2].

It is seen from Table 8 that the present method (LCM) is very effective compared to the differential transform method (DTM) for problem (60).

Figures 6(a), 6(b), and 6(c) display the actual absolute error functions obtained by present method for $N = 9$ and the differential transform method. These figures display that the results gained by the present method are better than those obtained by the differential transform method.

6. Conclusions

It is known that solving the high-order linear differential equations system is usually very difficult analytically. In this case, it is required to approximate solutions. In this paper, a new method based on the Lucas polynomials with the help of the residual error function for solving system of high-order linear differential equations numerically is presented. When the obtained results are investigated in examples, it can be seen that the developed method is very effective compared to the others. Also, it can be seen from the tables and the figures that the accuracy increased when the value of N is increased. The approximate solutions are obtained in a short time with computer programmes such as *Maple*, *Mathematica*, and *Matlab*. We have used the *Maple* and *Matlab* for computations and graphics, respectively. Additionally, the presented method can be applied to the other system of linear integral and integrodifferential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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