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Research Article

Periodic Wave Solutions and Their Limit Forms of the Modified Novikov Equation

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The modified Novikov equation $u_t - u_{txx} + (b+1)u^2u_x = buu_xu_{xx} + u^2u_{xxx}$ is studied by using the bifurcation theory of dynamical system and the method of phase portraits analysis. The existences, dynamic properties, and limit forms of periodic wave solutions for b being a negative even are investigated. All possible exact parametric representations of the different kinds of nonlinear waves also are presented.

1. Introduction

The Novikov equation

$$u_t - u_{xxt} + 4u^2u_x = 3uu_xu_{xx} + u^2u_{xxx} \tag{1}$$

was discovered by Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [1]. The perturbative symmetry approach yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate (1) and find its first few symmetries, and he subsequently found a scalar Lax pair for it (also see [2]) and then proved that the equation is integrable. Hone and Wang [3] have shown that (1) admits peakon solutions like the CH and the DP equations. Jiang and Ni [4] have shown that (1) possesses the blow-up phenomenon. The existence and uniqueness of global weak solutions for (1) were studied in [5]. Bozhkov et al. [6] found the Lie point symmetries of (1) and demonstrate that it is strictly self-adjoint. Li [7] obtained exact cuspon wave solution and compactons and found that the corresponding traveling system of (1) has no one-peakon solution. The Cauchy problem of (1) was investigated in [8, 9].

The modified Novikov equation reads as

$$u_t - u_{txx} + (b+1)u^2u_x = buu_xu_{xx} + u^2u_{xxx}, \tag{2}$$

where b is a real parameter. Clearly, letting $b = 3$, (2) becomes the Novikov equation (1). Lai and Wu [10] considered the local strong and weak solutions of (2). The global solution and blow-up phenomena of (2) were investigated in [11]. The Cauchy problem of (2) was studied in [12, 13].

In this paper, we consider the existences, dynamic properties, and limit forms of periodic wave solutions of (2) for b being a negative even using the bifurcation theory of dynamical system and the method of phase portraits analysis [7, 14], we also will present some new explicit exact solutions of (2).

Using transformation

$$u(x,t) = \phi(\xi) = \phi(x - ct), \tag{3}$$

where $c\,(\neq 0)$ is the wave speed, (2) can be rewritten as

$$-c\phi' + c\phi''' + (b+1)\phi^2\phi' = b\phi\phi'\phi'' + \phi^2\phi''', \tag{4}$$

where "$'$" is the derivative with respect to ξ.

Integrating (4) once, it follows that

$$\begin{aligned}\left(\phi^2 - c\right)\phi'' = {} & -c\phi + \frac{1}{3}(b+1)\phi^3 - \frac{1}{2}(b-2)\phi\left(\phi'\right)^2 \\ & + \frac{1}{2}(b-2)\int\left(\phi'\right)^2 d\phi.\end{aligned} \tag{5}$$

Let $y = \phi'$, and then (5) becomes

$$\left(\phi^2 - c\right)\frac{dy^2}{d\phi} = -2c\phi + \frac{2}{3}(b+1)\phi^3 - (b-2)\phi y^2 + (b-2)\int y^2 d\phi. \tag{6}$$

Differentiating both sides of (6) with respect to ϕ, we have

$$\left(\phi^2 - c\right)\frac{d^2y^2}{d\phi^2} + b\phi\frac{dy^2}{d\phi} = -2c + 2(b+1)\phi^2. \tag{7}$$

It implies that

$$\frac{dy^2}{d\phi} = 2\phi + A\left(\phi^2 - c\right)^{-b/2}, \tag{8}$$

where A is the integral constant.

For simplicity, we only consider the special case $b = -2m$ and $m \in Z^+$ in this paper. For this special case, (2) becomes

$$u_t - u_{txx} + (1-2m)u^2u_x = -2muu_xu_{xx} + u^2u_{xxx}, \tag{9}$$

and (8) can be rewritten as

$$\frac{dy^2}{d\phi} = 2\phi + A\left(\phi^2 - c\right)^m. \tag{10}$$

We see from (10) that

$$y^2 = \phi^2 + A\int\left(\phi^2 - c\right)^m d\phi + 2h, \tag{11}$$

where h is an integral constant. Thus, the function

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{1}{2}\phi^2 - \frac{1}{2}A\int\left(\phi^2 - c\right)^m d\phi = h \tag{12}$$

is a first integral of (9). The dynamics of (9) is equivalent to the system

$$\begin{aligned}\frac{d\phi}{d\xi} &= \frac{\partial H}{\partial y} = y,\\ \frac{dy}{d\xi} &= -\frac{\partial H}{\partial \phi} = \phi + \frac{1}{2}A\left(\phi^2 - c\right)^m,\end{aligned} \tag{13}$$

where $A \neq 0$; otherwise system (13) becomes a linear system.

For a fixed h, the level curve $H(\phi, y) = h$ defined by (12) determines a set of invariant curves of system (13) which contains different branches of curves. As h is varied, it defines different families of orbits of (13) with different dynamical behaviors.

The remainder of this paper is organized as follows: In Section 2, we consider bifurcation sets and phase portraits of (13). Existences and limit forms of periodic wave solutions of (9) are stated in Section 3. Some explicit exact traveling wave solutions of (9) are presented in Section 4. A short conclusion will be given in Section 5.

2. Bifurcation Analysis of (13)

Obviously, the equilibrium point of system (13) is just the intersection point of the straight line $y = \phi$ and the curve defined by $y = -(1/2)A(\phi^2 - c)^m$ (see Figures 1(a)–1(h)). Clearly, system (13) does not have any equilibrium point when $c < (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$. There exists only one equilibrium point $(\phi_0, 0)$ of (13) satisfying $\phi_0 + (1/2)A(\phi_0^2 - c)^m = 0$ when $c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, and $\phi_0 > 0$ when $A < 0$, $\phi_0 < 0$ when $A > 0$. System (13) has two equilibrium points $(\phi_1, 0)$, $(\phi_2, 0)$ satisfying $\phi_i + (1/2)A(\phi_i^2 - c)^m = 0$, $i = 1, 2$, when $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, and $0 < \phi_2 < \phi_1$ when $A < 0$, $\phi_2 < \phi_1 < 0$ when $A > 0$.

Let $M(\phi_e, 0)$ be the coefficient matrix of the linearized system of (13) at equilibrium point $(\phi_e, 0)$, $J(\phi_e, 0) = \det(M(\phi_e, 0))$, we have

$$J(\phi_e, 0) = \frac{mA\phi_e\left(\phi_e^2 - c\right)^m}{c - \phi_e^2} - 1. \tag{14}$$

By the bifurcation theory of dynamical system, we know that $(\phi_e, 0)$ is a saddle point if $J(\phi_e, 0) < 0$, a center point if $J(\phi_e, 0) > 0$, and a cusp if $J(\phi_e, 0) = 0$ and the Poincaré index of $(\phi_e, 0)$ is zero.

By using the properties of equilibrium points and the bifurcation theory of dynamical system, we can show that bifurcation sets and phase portraits of (13) are as drawn in Figure 2.

3. Existences of Traveling Wave Solutions of (9)

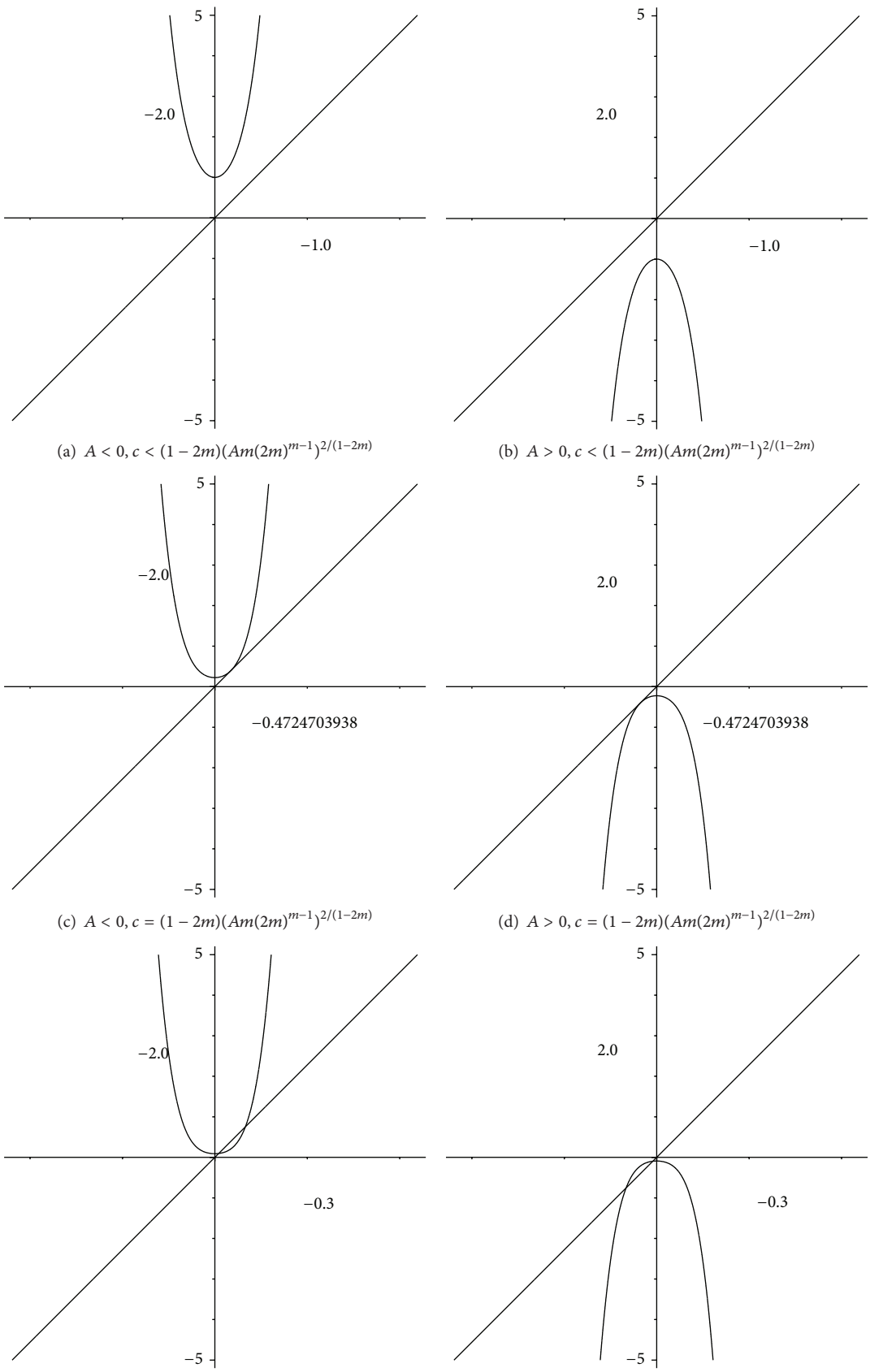
Liu and Guo [15] investigated the periodic blow-up solutions and their limit forms of a generalized Camassa-Holm equation. We consider existences of periodic blow-up wave solutions and other traveling wave solutions of (9) in this section. Denote that $h_i = H(\phi_i, 0)$, $i = 0, 1, 2$, and from Figure 2, we have the following results.

Theorem 1. *(i) When $A < 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, for $h \in (h_1, h_2)$, there exists a family of uncountably infinite many smooth periodic wave solutions of (9). Moreover, the smooth periodic wave solutions converge to a solitary wave solution of peak type as h approaches h_2.*

(ii) When $A > 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, for $h \in (h_2, h_1)$, there exists a family of uncountably infinite many smooth periodic wave solutions of (9). Moreover, the smooth periodic wave solutions converge to a solitary wave solution of valley type as h approaches h_1.

Theorem 2. *(i) When $A < 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, for $h \in (h_1, h_2)$, there exists a family of uncountably infinite many periodic blow-up wave solutions of (9). Moreover, the periodic blow-up wave solutions converge to a blow-up wave solution as h approaches h_2.*

(ii) When $A > 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, for $h \in (h_2, h_1)$, there exists a family of uncountably infinite many periodic blow-up wave solutions of (9). Moreover, the periodic



(a) $A < 0, c < (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(b) $A > 0, c < (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

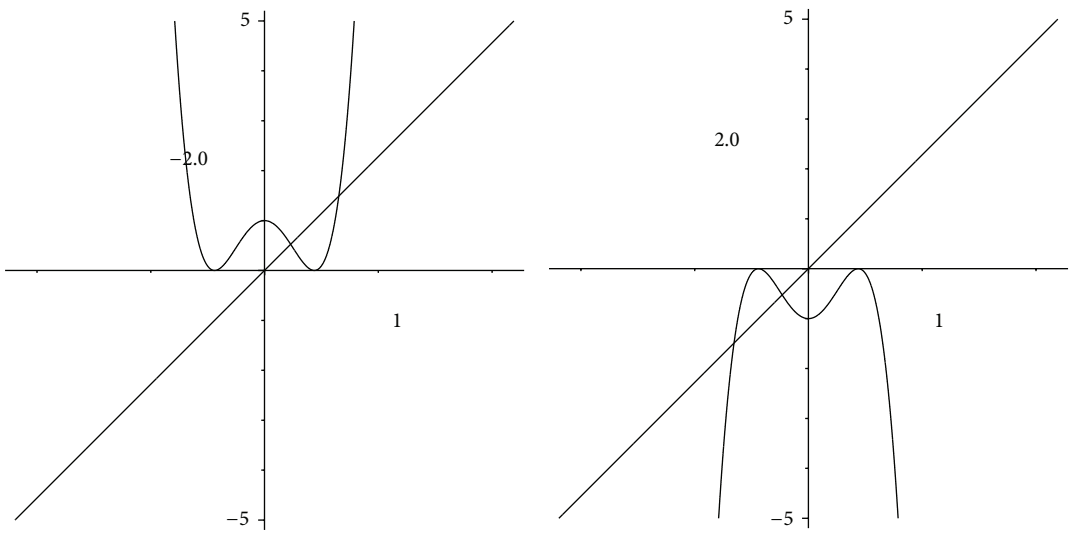
(c) $A < 0, c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(d) $A > 0, c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(e) $A < 0, (1-2m)(Am(2m)^{m-1})^{2/(1-2m)} < c < 0$

(f) $A > 0, (1-2m)(Am(2m)^{m-1})^{2/(1-2m)} < c < 0$

Figure 1: Continued.



(g) $A < 0, c > 0$ (h) $A > 0, c > 0$

Figure 1: The intersection points of $y = \phi$ and $y = -(1/2)A(\phi^2 - c)^m$, $m \in Z^+$.

blow-up wave solutions converge to a blow-up wave solution as h approaches h_1.

Theorem 3. *When $A \neq 0$, $c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, for $h \in (-\infty, h_0)$, there exists a family of uncountably infinite many periodic blow-up wave solutions of (9). Moreover, the periodic blow-up wave solutions converge to a blow-up wave solution as h approaches h_0.*

4. Explicit Exact Traveling Solutions of (9)

4.1. Solitary Wave Solutions. From Figure 2(e), we see that there is a homoclinic orbit connecting with the saddle point $(\phi_2, 0)$ and passing point $(\phi_M, 0)$ when $A < 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$.

When $m = 1$, expression of the homoclinic orbit is

$$y = \pm(\phi - \phi_2)\sqrt{-\frac{1}{3}A(\phi_M - \phi)}, \quad \phi_2 < \phi \le \phi_M, \tag{15}$$

where $\phi_2 = (-1 + \sqrt{1 + A^2c})/A$, $\phi_M = -(1 + 2\sqrt{1 + A^2c})/A$.

Substituting (15) into the $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yields

$$\int_{\phi}^{\phi_M} \frac{ds}{(s - \phi_2)\sqrt{\phi_M - s}} = \sqrt{-\frac{1}{3}A}\,|\xi|. \tag{16}$$

Completing above integral, we can get a solitary wave solution of peak type of (9) as follows:

$$u(x,t) = \phi_M - (\phi_M - \phi_2)\tanh^2(\omega_1(x - ct)), \tag{17}$$

where $\omega_1 = (1/2)\sqrt{-(1/3)A(\phi_M - \phi_2)}$.

When $m = 2$, expression of the homoclinic orbit is

$$y = \pm(\phi - \phi_2)\sqrt{-\frac{1}{5}A(\phi_M - \phi)\left((\phi - b_1)^2 + a_1^2\right)}, \quad \phi_2 < \phi \le \phi_M, \tag{18}$$

where $(\phi-\phi_2)^2(-(1/5)A(\phi_M-\phi)((\phi-b_1)^2+a_1^2)) = (1/5)A\phi^5 - (2/3)Ac\phi^3 + \phi^2 + Ac\phi + 2h_2$, $h_2 = -(1/10)A\phi_2^5 + (1/3)Ac\phi_2^3 - (1/2)\phi_2^2 - (1/2)Ac^2\phi_2$. For example, letting $A = -0.2$, $c = -1.0$, we get that $\phi_2 \doteq 0.1020955659$, $\phi_M \doteq 2.456246444$, $b_1 \doteq -1.330218788$, and $a_1^2 \doteq 8.129812158$.

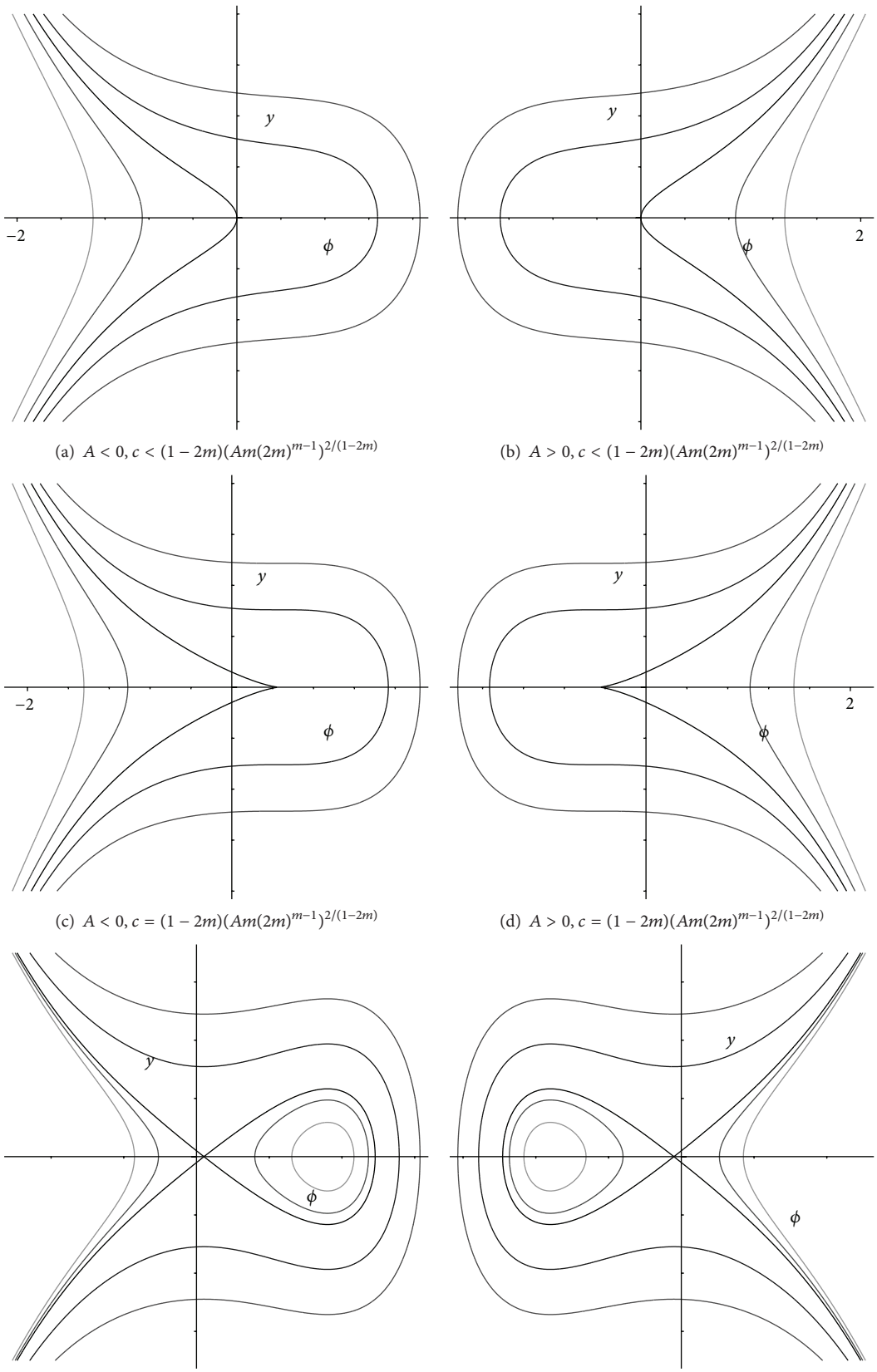
Substituting (18) into the $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yields

$$\int_{\phi}^{\phi_M} \frac{ds}{(s - \phi_2)\sqrt{(\phi_M - s)\left((s - b_1)^2 + a_1^2\right)}} = \sqrt{-\frac{1}{5}A}\,|\xi|. \tag{19}$$

Completing above integral, we can get the implicit representation of the solitary wave solution of peak type of (9) for $u = \phi(x - ct)$ as follows:

$$\pm(x - ct) = \frac{1}{\alpha_1(\phi_M - \phi_2 - B_1)\sqrt{-(1/5)AB_1}}\left\{\mathrm{F}(\varphi_1, k_1) - \frac{1}{1+\alpha_1}\left[\Pi\left(\varphi_1, \frac{\alpha_1^2}{\alpha_1^2 - 1}, k_1\right) - \alpha_1 f_1\right]\right\}, \tag{20}$$

where $\mathrm{F}(\cdot, k)$, $\Pi(\cdot,\cdot,k)$ are the elliptic integrals of the first and third kind, respectively, with the modulus k [16] and $B_1 = \sqrt{(b_1 - \phi_M)^2 + a_1^2}$, $\alpha_1 = (\phi_M - \phi_2 + B_1)/(\phi_M - \phi_2 - B_1)$,



(a) $A < 0, c < (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(b) $A > 0, c < (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(c) $A < 0, c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(d) $A > 0, c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(e) $A < 0, c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

(f) $A > 0, c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$

Figure 2: Bifurcation sets and phase portraits of system (13).

$\varphi_1 = \arccos((B_1 - \phi_M + \phi)/(B_1 + \phi_M - \phi))$, $k_1 = \sqrt{(B_1 - b_1 + \phi_M)/2B_1}$, $\bar{k}_1 = \sqrt{1-k_1^2}$, and $f_1 = \sqrt{(\alpha_1^2-1)/(k_1^2+\bar{k}_1^2\alpha_1^2)}\ln|\sqrt{(k_1^2+\bar{k}_1^2\alpha_1^2)(1-k_1^2\sin^2\varphi_1)} + \sqrt{(\alpha_1^2-1)(1-\cos^2\varphi_1)}/(\sqrt{(k_1^2+\bar{k}_1^2\alpha_1^2)(1-k_1^2\sin^2\varphi_1)} - \sqrt{(\alpha_1^2-1)(1-\cos^2\varphi_1)})|$.

From Figure 2(f), we see that there is a homoclinic orbit connecting with the saddle point $(\phi_1, 0)$ and passing point $(\phi_m, 0)$ when $A > 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$.

When $m = 1$, expression of the homoclinic orbit is

$$y = \pm(\phi_1 - \phi)\sqrt{\frac{1}{3}A(\phi - \phi_m)}, \quad \phi_m \le \phi < \phi_1, \tag{21}$$

where $\phi_1 = (-1+\sqrt{1+A^2c})/A$, $\phi_m = -(1+2\sqrt{1+A^2c})/A$.

Substituting (21) into the $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yields

$$\int_{\phi_m}^{\phi} \frac{ds}{(\phi_1 - s)\sqrt{s-\phi_m}} = \sqrt{\frac{1}{3}A}\,|\xi|. \tag{22}$$

Completing above integral, we can get a solitary wave solution of valley type of (9) as follows:

$$u(x,t) = \phi_m + (\phi_1 - \phi_m)\tanh^2(\omega_2(x-ct)), \tag{23}$$

where $\omega_2 = (1/2)\sqrt{(1/3)A(\phi_1-\phi_m)}$.

When $m = 2$, expression of the homoclinic orbit is

$$y = \pm(\phi_1 - \phi)\sqrt{\frac{1}{5}A(\phi-\phi_m)\left((\phi-b_2)^2 + a_2^2\right)}, \quad \phi_m \le \phi < \phi_1, \tag{24}$$

where $(\phi_1-\phi)^2((1/5)A(\phi-\phi_m)((\phi-b_2)^2+a_2^2)) = (1/5)A\phi^5 - (2/3)Ac\phi^3 + \phi^2 + Ac\phi + 2h_1$, $h_1 = -(1/10)A\phi_1^5 + (1/3)Ac\phi_1^3 - (1/2)\phi_1^2 - (1/2)Ac^2\phi_1$. For example, letting $A = 0.2$, $c = -1.0$, we get that $\phi_1 \doteq -0.1020955659$, $\phi_m \doteq -2.456246444$, $b_2 \doteq 1.330218788$, and $a_2^2 \doteq 8.129812158$.

Substituting (24) into the $d\phi/d\xi = y$ and integrating it along the homoclinic orbit yields

$$\int_{\phi_m}^{\phi} \frac{ds}{(\phi_1 - s)\sqrt{(s-\phi_m)\left((s-b_2)^2+a_2^2\right)}} = \sqrt{\frac{1}{5}A}\,|\xi|. \tag{25}$$

Completing above integral, we can get the implicit representation of the solitary wave solution of valley type of (9) for $u = \phi(x-ct)$ as follows:

$$\begin{aligned} \pm(x-ct) &= \frac{1}{\alpha_2(\phi_1-\phi_m-B_2)\sqrt{(1/5)AB_2}}\left\{\mathrm{F}(\varphi_2,k_2)\right. \\ &\quad \left. - \frac{1}{1+\alpha_2}\left[\Pi\left(\varphi_2, \frac{\alpha_2^2}{\alpha_2^2-1}, k_2\right) - \alpha_2 f_2\right]\right\}, \end{aligned} \tag{26}$$

where $B_2 = \sqrt{(b_2-\phi_m)^2+a_2^2}$, $\alpha_2 = (\phi_1-\phi_m+B_2)/(\phi_1-\phi_m-B_2)$, $\varphi_2 = \arccos((B_2+\phi_m-\phi)/(B_2-\phi_m+\phi))$, $k_2 = \sqrt{(B_2+b_2-\phi_m)/2B_2}$, $\bar{k}_2 = \sqrt{1-k_2^2}$, and $f_2 = \sqrt{(\alpha_2^2-1)/(k_2^2+\bar{k}_2^2\alpha_2^2)}\ln|\sqrt{(k_2^2+\bar{k}_2^2\alpha_2^2)(1-k_2^2\sin^2\varphi_2)} + \sqrt{(\alpha_2^2-1)(1-\cos^2\varphi_2)}/\sqrt{(k_2^2+\bar{k}_2^2\alpha_2^2)(1-k_2^2\sin^2\varphi_2)} - \sqrt{(\alpha_2^2-1)(1-\cos^2\varphi_2)}|$.

4.2. Blow-Up Wave Solutions. From Figure 2(c), we see that there is an open orbit passing the cusp $(\phi_0, 0)$ when $A < 0$, $c = (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$.

When $m = 1$, expression of the open orbit is

$$y = \pm(\phi_0 - \phi)\sqrt{-\frac{1}{3}A(\phi_0-\phi)}, \quad -\infty < \phi \le \phi_0, \tag{27}$$

where $\phi_0 = -1/A$.

Substituting (27) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{-\infty}^{\phi} \frac{ds}{(\phi_0 - s)\sqrt{\phi_0 - s}} = \sqrt{-\frac{1}{3}A}\,|\xi|. \tag{28}$$

Completing above integral, we can get a blow-up wave solution of (9) as follows:

$$u(x,t) = \phi_0 - \frac{1}{(\omega_3(x+(1/A^2)t))^2}, \tag{29}$$

where $\omega_3 = (1/2)\sqrt{-(1/3)A}$.

When $m = 2$, expression of the open orbit is

$$y = \pm(\phi_0 - \phi)\sqrt{-\frac{1}{5}A(\phi_0-\phi)\left((\phi-b_3)^2+a_3^2\right)}, \quad -\infty < \phi \le \phi_0, \tag{30}$$

where $(\phi_0-\phi)^2(-(1/5)A(\phi_0-\phi)((\phi-b_3)^2+a_3^2)) = (1/5)A\phi^5 - (2/3)Ac\phi^3 + \phi^2 + Ac\phi + 2h_0$, $h_0 = -(1/10)A\phi_0^5 + (1/3)Ac\phi_0^3 - (1/2)\phi_0^2 - (1/2)Ac^2\phi_0$. For example, letting $A = -1.2$, $c \doteq -0.6641616057$, we get that $\phi_0 \doteq 0.4705128788$, $b_3 \doteq -0.7057693180$, and $a_3^2 \doteq 3.044055903$.

Substituting (30) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{-\infty}^{\phi} \frac{ds}{(\phi_0 - s)\sqrt{(\phi_0-s)\left((s-b_3)^2+a_3^2\right)}} = \sqrt{-\frac{1}{5}A}\,|\xi|. \tag{31}$$

Completing above integral, we can get the implicit representation of the blow-up wave solution of (9) for $u = \phi(x + (3/4)\sqrt[3]{1/A^2}t)$ as follows:

$$\begin{aligned} \pm\left(x + \frac{3}{4}\sqrt[3]{\frac{1}{A^2}}t\right) &= \frac{1}{B_3\sqrt{-(1/5)AB_3}}\left\{\mathrm{F}(\varphi_3,k_3)\right. \\ &\quad \left. - 2\mathrm{E}(\varphi_3,k_3) + \frac{2\sin\varphi_3\sqrt{1-k_3^2\sin^2\varphi_3}}{1+\cos\varphi_3}\right\}, \end{aligned} \tag{32}$$

where $E(\cdot, k)$ is the elliptic integral of the second kind with the modulus k [16] and $B_3 = \sqrt{(b_3 - \phi_0)^2 + a_3^2}$, $\varphi_3 = \arccos((\phi_0 - B_3 - \phi)/(\phi_0 + B_3 - \phi))$, $k_3 = \sqrt{(B_3 - b_3 + \phi_0)/2B_3}$.

From Figure 2(d), we see that there is an open orbit passing the cusp $(\phi_0, 0)$ when $A > 0$, $c = (1 - 2m)(Am(2m)^{m-1})^{2/(1-2m)}$.

When $m = 1$, expression of the open orbit is

$$y = \pm(\phi - \phi_0)\sqrt{\frac{1}{3}A(\phi - \phi_0)}, \quad \phi_0 \le \phi < +\infty, \tag{33}$$

where $\phi_0 = -1/A$.

Substituting (33) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{\phi}^{+\infty} \frac{ds}{(s - \phi_0)\sqrt{s - \phi_0}} = \sqrt{\frac{1}{3}A}\,|\xi|. \tag{34}$$

Completing above integral, we can get a blow-up wave solution of (9) as follows:

$$u(x,t) = \phi_0 + \frac{1}{(\omega_4(x + (1/A^2)t))^2}, \tag{35}$$

where $\omega_4 = (1/2)\sqrt{(1/3)A}$.

When $m = 2$, expression of the open orbit is

$$y = \pm(\phi - \phi_0)\sqrt{\frac{1}{5}A(\phi - \phi_0)\left((\phi - b_4)^2 + a_4^2\right)}, \quad \phi_0 \le \phi < +\infty, \tag{36}$$

where $(\phi - \phi_0)^2((1/5)A(\phi - \phi_0)((\phi - b_4)^2 + a_4^2)) = (1/5)A\phi^5 - (2/3)Ac\phi^3 + \phi^2 + Ac\phi + 2h_0$, $h_0 = -(1/10)A\phi_0^5 + (1/3)Ac\phi_0^3 - (1/2)\phi_0^2 - (1/2)Ac^2\phi_0$. For example, letting $A = 1.2$, $c \doteq -0.6641616057$, we get that $\phi_0 \doteq -0.4705128788$, $b_4 \doteq -0.7057693180$, and $a_4^2 \doteq 3.044055903$.

Substituting (36) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{\phi}^{+\infty} \frac{ds}{(s - \phi_0)\sqrt{(s - \phi_0)\left((s - b_4)^2 + a_4^2\right)}} = \sqrt{\frac{1}{5}A}\,|\xi|. \tag{37}$$

Completing above integral, we can get the implicit representation of the blow-up wave solution of (9) for $u = \phi(x + (3/4)\sqrt[3]{(1/A^2)}t)$ as follows:

$$\pm\left(x + \frac{3}{4}\sqrt[3]{\frac{1}{A^2}}t\right) = \frac{1}{B_4\sqrt{(1/5)AB_4}}\left\{F(\varphi_4, k_4) - 2E(\varphi_4, k_4) + \frac{2\sin\varphi_4\sqrt{1 - k_4^2\sin^2\varphi_4}}{1 + \cos\varphi_4}\right\}, \tag{38}$$

where $B_4 = \sqrt{(b_4 - \phi_0)^2 + a_4^2}$, $\varphi_4 = \arccos((\phi - \phi_0 - B_4)/(\phi - \phi_0 + B_4))$, and $k_4 = \sqrt{(B_4 + b_4 - \phi_0)/2B_4}$.

From Figure 2(e), we see that there is an open orbit passing the saddle point $(\phi_2, 0)$ when $A < 0$, $c > (1 - 2m)(Am(2m)^{m-1})^{2/(1-2m)}$.

When $m = 1$, expression of the open orbit is

$$y = \pm(\phi_2 - \phi)\sqrt{-\frac{1}{3}A(\phi_M - \phi)}, \quad -\infty < \phi \le \phi_2, \tag{39}$$

where ϕ_2, ϕ_M are given in (15).

Substituting (39) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{-\infty}^{\phi} \frac{ds}{(\phi_2 - s)\sqrt{\phi_M - s}} = \sqrt{-\frac{1}{3}A}\,|\xi|. \tag{40}$$

Completing above integral, we can get a blow-up wave solution of (9) as follows:

$$u(x,t) = \phi_M - (\phi_M - \phi_2)\coth^2(\omega_1(x - ct)), \tag{41}$$

where ω_1 is given in (17).

When $m = 2$, expression of the open orbit is

$$y = \pm(\phi_2 - \phi)\sqrt{-\frac{1}{5}A(\phi_M - \phi)\left((\phi - b_1)^2 + a_1^2\right)}, \quad -\infty < \phi \le \phi_2, \tag{42}$$

where ϕ_2, ϕ_M, b_1, and a_1^2 are given in (18).

Substituting (42) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{-\infty}^{\phi} \frac{ds}{(\phi_2 - s)\sqrt{(\phi_M - s)\left((s - b_1)^2 + a_1^2\right)}} = \sqrt{-\frac{1}{5}A}\,|\xi|. \tag{43}$$

Completing above integral, we can get the implicit representation of the blow-up wave solution of (9) for $u = \phi(x - ct)$ as follows:

$$\pm(x - ct) = \frac{1}{\alpha_1(\phi_2 - \phi_M + B_1)\sqrt{-(1/5)AB_1}}\left\{F(\overline{\varphi}_1, k_1) - \frac{1}{1 + \alpha_1}\left[\Pi\left(\overline{\varphi}_1, \frac{\alpha_1^2}{\alpha_1^2 - 1}, k_1\right) + \alpha_1 f_1\right]\right\}, \tag{44}$$

where $\overline{\varphi}_1 = \pi - \varphi_1$ and B_1, α_1, φ_1, k_1, and f_1 are given in (20).

From Figure 2(f), we see that there is an open orbit passing the saddle point $(\phi_1, 0)$ when $A > 0$, $c > (1 - 2m)(Am(2m)^{m-1})^{2/(1-2m)}$.

When $m = 1$, expression of the open orbit is

$$y = \pm(\phi - \phi_1)\sqrt{\frac{1}{3}A(\phi - \phi_m)}, \quad \phi_1 \le \phi < +\infty, \tag{45}$$

where ϕ_1, ϕ_m are given in (21).

Substituting (45) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{\phi}^{+\infty} \frac{ds}{(s-\phi_1)\sqrt{s-\phi_m}} = \sqrt{\frac{1}{3}A}\,|\xi|. \tag{46}$$

Completing above integral, we can get a blow-up wave solution of (9) as follows:

$$u(x,t) = \phi_m + (\phi_1 - \phi_m)\coth^2(\omega_2(x-ct)), \tag{47}$$

where ω_2 is given in (23).

When $m = 2$, expression of the open orbit is

$$y = \pm(\phi-\phi_1)\sqrt{\frac{1}{5}A(\phi-\phi_m)\left((\phi-b_2)^2 + a_2^2\right)}, \quad \phi_1 \le \phi < +\infty, \tag{48}$$

where ϕ_1, ϕ_m, b_2, and a_2^2 are given in (24).

Substituting (48) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{\phi}^{+\infty} \frac{ds}{(s-\phi_1)\sqrt{(s-\phi_m)\left((s-b_2)^2 + a_2^2\right)}} = \sqrt{\frac{1}{5}A}\,|\xi|. \tag{49}$$

Completing above integral, we can get the implicit representation of the blow-up wave solution of (9) for $u = \phi(x-ct)$ as follows:

$$\begin{aligned} \pm(x-ct) &= \frac{1}{\alpha_2(B_2+\phi_m-\phi_1)\sqrt{(1/5)AB_2}}\left\{\mathrm{F}(\overline{\varphi}_2, k_2)\right. \\ &\quad \left. - \frac{1}{1+\alpha_2}\left[\Pi\left(\overline{\varphi}_2, \frac{\alpha_2^2}{\alpha_2^2-1}, k_2\right) + \alpha_2 f_2\right]\right\}, \end{aligned} \tag{50}$$

where $\overline{\varphi}_2 = \pi - \varphi_2$ and B_2, α_2, φ_2, k_2, and f_2 are given in (26).

4.3. Smooth Periodic Wave Solutions. From Figure 2(e), we see that there is a periodic orbit passing points $(\gamma_1, 0)$ and $(\gamma_2, 0)$ when $A < 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, $h \in (h_1, h_2)$.

When $m = 1$, expression of the periodic orbit is

$$y = \pm\sqrt{-\frac{1}{3}A(\gamma_1-\phi)(\phi-\gamma_2)(\phi-\gamma_3)}, \quad \gamma_2 \le \phi \le \gamma_1, \tag{51}$$

where γ_1, γ_2, and γ_3 ($\gamma_3 < 0 < \gamma_2 < \gamma_1$) are three real roots of $(1/3)A\psi^3 + \psi^2 - Ac\psi + 2h = 0$. For example, taking $A = -1.5$, $c = 1.2$, we have $h_1 \doteq -1.802532163$, $h_2 \doteq 0.3062358669$. Letting $h = -0.2$, we get that $\gamma_1 \doteq 3.083397695$, $\gamma_2 \doteq 0.2018680190$, and $\gamma_3 \doteq -1.285265714$.

Substituting (51) into the $d\phi/d\xi = y$ and integrating it along the periodic orbit yields

$$\int_{\gamma_2}^{\phi} \frac{ds}{\sqrt{(\gamma_1-s)(s-\gamma_2)(s-\gamma_3)}} = \sqrt{-\frac{1}{3}A}\,|\xi|. \tag{52}$$

Completing above integral, we can get a smooth periodic wave solution of (9) as follows:

$$u(x,t) = \frac{\gamma_2(\gamma_3-\gamma_1) + \gamma_3(\gamma_1-\gamma_2)\,\mathrm{sn}^2(\omega_5(x-ct), k_5)}{(\gamma_3-\gamma_1) + (\gamma_1-\gamma_2)\,\mathrm{sn}^2(\omega_5(x-ct), k_5)}, \tag{53}$$

where $\mathrm{sn}(\cdot, k)$ is the Jacobian elliptic function with the modulus k [16], and $\omega_5 = (1/2)\sqrt{-(1/3)A(\gamma_1-\gamma_3)}$, $k_5 = \sqrt{(\gamma_1-\gamma_2)/(\gamma_1-\gamma_3)}$.

From Figure 2(f), we see that there is a periodic orbit passing points $(\gamma_2, 0)$ and $(\gamma_3, 0)$ when $A > 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, $h \in (h_2, h_1)$.

When $m = 1$, expression of the periodic orbit is

$$y = \pm\sqrt{\frac{1}{3}A(\gamma_1-\phi)(\gamma_2-\phi)(\phi-\gamma_3)}, \quad \gamma_3 \le \phi \le \gamma_2, \tag{54}$$

where γ_1, γ_2, and γ_3 ($\gamma_3 < \gamma_2 < 0 < \gamma_1$) are three real roots of $(1/3)A\psi^3 + \psi^2 - Ac\psi + 2h = 0$. For example, taking $A = 1.5$, $c = 1.2$, we have $h_1 \doteq 0.3062358669$, $h_2 \doteq -1.802532163$. Letting $h = -1.3$, we get that $\gamma_1 \doteq 1.751240424$, $\gamma_2 \doteq -1.134926411$, and $\gamma_3 \doteq -2.616314014$.

Substituting (54) into the $d\phi/d\xi = y$ and integrating it along the periodic orbit yields

$$\int_{\phi}^{\gamma_2} \frac{ds}{\sqrt{(\gamma_1-s)(\gamma_2-s)(s-\gamma_3)}} = \sqrt{\frac{1}{3}A}\,|\xi|. \tag{55}$$

Completing above integral, we can get a smooth periodic wave solution of (9) as follows:

$$u(x,t) = \frac{\gamma_2(\gamma_3-\gamma_1) + \gamma_1(\gamma_2-\gamma_3)\,\mathrm{sn}^2(\omega_6(x-ct), k_6)}{(\gamma_3-\gamma_1) + (\gamma_2-\gamma_3)\,\mathrm{sn}^2(\omega_6(x-ct), k_6)}, \tag{56}$$

where $\omega_6 = (1/2)\sqrt{(1/3)A(\gamma_1-\gamma_3)}$, $k_6 = \sqrt{(\gamma_2-\gamma_3)/(\gamma_1-\gamma_3)}$.

4.4. Periodic Blow-Up Wave Solutions. From Figure 2(e), we see that there is an open orbit passing point $(\gamma_3, 0)$ when $A < 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, $h \in (h_1, h_2)$.

When $m = 1$, expression of the open orbit is

$$y = \pm\sqrt{-\frac{1}{3}A(\gamma_1-\phi)(\gamma_2-\phi)(\gamma_3-\phi)}, \quad -\infty < \phi \le \gamma_3, \tag{57}$$

where γ_1, γ_2, and γ_3 are given in (51).

Substituting (57) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{-\infty}^{\phi} \frac{ds}{\sqrt{(\gamma_1-s)(\gamma_2-s)(\gamma_3-s)}} = \sqrt{-\frac{1}{3}A}\,|\xi|. \tag{58}$$

Completing above integral, we can get a periodic blow-up wave solution of (9) as follows:

$$u(x,t) = \gamma_1 - (\gamma_1 - \gamma_3)\,\mathrm{ns}^2(\omega_5(x-ct), k_5), \tag{59}$$

where $\mathrm{ns}(\cdot, k)$ is the Jacobian elliptic function with the modulus k [16], and ω_5, k_5 are given in (53).

From Figure 2(f), we see that there is an open orbit passing point $(\gamma_1, 0)$ when $A > 0$, $c > (1-2m)(Am(2m)^{m-1})^{2/(1-2m)}$, $h \in (h_2, h_1)$.

When $m = 1$, expression of the periodic orbit is

$$y = \pm\sqrt{\frac{1}{3}A(\phi-\gamma_1)(\phi-\gamma_2)(\phi-\gamma_3)}, \qquad \gamma_1 \le \phi < +\infty, \tag{60}$$

where γ_1, γ_2, and γ_3 are given in (54).

Substituting (60) into the $d\phi/d\xi = y$ and integrating it along the open orbit yields

$$\int_{\phi}^{+\infty} \frac{ds}{\sqrt{(s-\gamma_1)(s-\gamma_2)(s-\gamma_3)}} = \sqrt{\frac{1}{3}A}\,|\xi|. \tag{61}$$

Completing above integral, we can get a periodic blow-up wave solution of (9) as follows:

$$u(x,t) = \gamma_3 + (\gamma_1 - \gamma_3)\,\mathrm{ns}^2(\omega_6(x-ct), k_6), \tag{62}$$

where ω_6, k_6 are given in (56).

5. Conclusion

In this paper, the modified Novikov equation is studied by using the bifurcation theory of dynamical system and the method of phase portraits analysis. For a special case, the existences of different kinds of nonlinear waves and limit forms of two periodic wave solutions of the equation are stated in Theorems 1–3. All possible exact parametric representations of the nonlinear waves also are presented in (17), (20), (23), and (29) and so forth. The previous results of the equation are enriched.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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