Research Article

Online Scheduling with Delivery Time on a Bounded Parallel Batch Machine with Limited Restart

Hailing Liu,^{1,2} Long Wan,³ Zhigang Yan,⁴ and Jinjiang Yuan¹

¹School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan 450001, China

²College of Science, Henan Institute of Engineering, Zhengzhou, Henan 451191, China

³School of Information Technology, Jiangxi University of Finance and Economics, Nanchang 330013, China

⁴Accounting School, Jiangxi University of Finance and Economics, Nanchang 330013, China

Correspondence should be addressed to Long Wan; cocu3328@163.com

Received 23 August 2014; Accepted 28 September 2014

Academic Editor: Yunqiang Yin

Copyright © 2015 Hailing Liu et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We consider the online (over time) scheduling of equal length jobs on a bounded parallel batch machine with batch capacity b to minimize the time by which all jobs have been delivered with limited restart. Here, “restart” means that a running batch may be interrupted, losing all the work done on it, and jobs in the interrupted batch are then released and become independently unscheduled jobs, called restarted jobs. “Limited restart” means that a running batch which contains some restarted jobs cannot be restarted again. When $b = 2$, we propose a best possible online algorithm $H(b = 2)$ with a competitive ratio of $1 + \alpha$, where α is the positive solution of $2\alpha(1 + \alpha) = 1$. When $b \geq 3$, we present a best possible online algorithm $H(b \geq 3)$ with a competitive ratio of $1 + \beta$, where β is the positive solution of $\beta(1 + \beta)^2 = 1$.

1. Introduction

In this paper, we consider the online scheduling of equal length jobs on a bounded parallel batch machine with batch capacity b to minimize the time by which all jobs have been delivered with limited restart. Here, a bounded parallel batch machine is a machine which can process up to b jobs simultaneously as a batch, where b is the batch capacity. The processing time of a batch is the longest processing time of jobs in the batch. Jobs in a batch have the same starting time and the same completion time. According to the characteristic of batch capacity b , there are the bounded model and the unbounded model. The bounded model means that b is finite; that is, $b < \infty$. The unbounded model means that b can be as large as possible; that is, $b = \infty$. In this paper, we study the bounded model, that is, bounded parallel batch machine.

In this paper, online means that each job becomes available at its arrival time which is unknown beforehand and the characteristics of the job are unknown until it arrives.

Each job J_j has an arrival time r_j , a processing time p_j , and a delivery time q_j . In our research, we assume that all jobs have equal processing times. By scaling, we may assume that each job has a processing time 1. Each job needs to be processed on the bounded parallel batch machine and then delivered to the destination by a delivery vehicle. Here we assume that there are sufficiently many delivery vehicles, which means that a job can be transported as soon as it is completed on the bounded parallel batch machine. Let C_j be the completion time of J_j on the bounded parallel batch machine and let L_j be the time by which J_j has been delivered. Then $L_j = C_j + q_j$. Let $L_{\max} = \max_j \{L_j : L_j = C_j + q_j\}$. Our objective is to minimize $L_{\max}$, that is, the time by which all jobs have been delivered.

The quality of an online algorithm is measured by the competitive ratio. An online algorithm is called ρ -competitive if, for any input instance, it produces a schedule with an objective value not worse than ρ times the value of an optimal offline schedule.

Restart (see Hoogeveen et al. [1]) means that a running task may be interrupted, losing all the work done on it. The jobs in the interrupted task, which are called restarted jobs, are then released and become independently unscheduled jobs which can be scheduled anew later. Using restarts means that we have a chance to change our original mind to get a better schedule according to the newly arrived jobs and so can efficiently improve the quality of some online algorithms. For example, for the problem of minimizing the time by which all jobs have been delivered on a single machine without restarts, the best possible online algorithm has a competitive ratio of $(\sqrt{5} + 1)/2$. While using restarts, a best possible online algorithm with a competitive ratio of $3/2$ was given in Akker et al. [2]. Other researches with restarts can see Epstein and Stee [3], Stee and Poutré [4], and Yuan et al. [5].

Limited restart is first introduced by Fu et al. [6]. Limited restart means that once a batch contains some restarted jobs, we cannot restart the batch again. Limited restart implies that a job can be restarted at most once. Limited restart is of practical value. In fact, too many restarts of a job may cause the waste of cost and improve the possibility of a spoiled product.

Parallel batching scheduling is motivated by burn in operations in semiconductor manufacture (see [7, 8]). For online scheduling on a single unbounded parallel batch machine to minimize makespan, that is, the time by which all jobs are completed on the parallel batch machine, Deng et al. [9] and Zhang et al. [10] independently gave the same online algorithm with the competitive ratio of $(\sqrt{5} + 1)/2$ and proved it is best possible. For online scheduling on a single bounded parallel batch machine to minimize makespan, Poon and Yu [11] proved that any FBLPT-based algorithm is 2-competitive. Particularly for batch capacity $b = 2$, they presented an online algorithm with the competitive ratio of $7/4$. For online scheduling on m unbounded parallel batch machines to minimize makespan, Liu et al. [12] and Tian et al. [13] independently presented two different but best possible online algorithms with a competitive ratio of $1 + \alpha_m$, where α_m is the positive solution of the equation $\alpha_m^2 + m\alpha_m - 1 = 0$. For online scheduling on a single parallel batch machine to minimize $L_{\max}$, that is, the time by which all jobs have been delivered, Tian et al. [14] gave a 2-competitive online algorithm for the unbounded model and a 3-competitive online algorithm for the bounded model. Moreover, when each job has the same processing time, they provided two $(\sqrt{5} + 1)/2$ -competitive optimal online algorithms for the two models, respectively. Yuan et al. [15] gave a best possible online algorithm with a competitive ratio of $(\sqrt{5} + 1)/2$ for the two models of $q_j \leq p_j$ and $[p_i > p_j \text{ implying } q_i \geq q_j]$. Fang et al. [16] studied the restricted model that all jobs have their processing times in $[p, ((1 + \sqrt{5})/2)p]$ and provided a class of optimal online algorithms with a competitive ratio of $(1 + \sqrt{5})/2$. Tian et al. [17] presented a $(2\sqrt{2} - 1)$ -competitive online algorithm for the unbounded model. Liu and Lu [18] studied online scheduling on m unbounded parallel batch machines to minimize $L_{\max}$. For the restricted model of $[p_i > p_j \text{ implying } q_i \geq q_j]$, they gave a best possible online algorithm with competitive ratio of $1 + \alpha_m$, where α_m is the

positive solution of $\alpha_m^2 + m\alpha_m = 1$. For the general case, they gave an online algorithm with competitive ratio of $1 + 2/\lfloor m \rfloor$.

For the problem of minimizing makespan on an unbounded parallel batch machine with restarts, Fu et al. [19] showed that any online algorithm cannot have a competitive ratio of less than $(5 - \sqrt{5})/2$ and Yuan et al. [5] gave a best possible online algorithm matching the lower bound. For the corresponding problem using limited restart, Fu et al. [6] gave an online algorithm with a competitive ratio of $3/2$ and proved that it is best possible. For the problem of minimizing makespan on two unbounded parallel batch machines with limited restart, Fu et al. [20] provided a best possible online algorithm which is $(\sqrt{3} + 1)/2$ -competitive under the second-restart assumption. Recently, Liu and Yuan [21] presented best possible online algorithms for minimizing makespan of equal length jobs on a bounded parallel batch machine with limited restart or restart.

Let J be a job set. Then $|J|$ is used to denote the number of jobs in J . Let $q(J)$ denote the largest delivery time of jobs in J and let $r(J)$ denote the latest arrival time of jobs whose delivery times are $q(J)$ in J . For a batch B in some schedule, $S(B)$ is starting time of B on the parallel batch machine. Batch B is called *full* if $|B| = b$ and *nonfull* if $|B| < b$. In an online algorithm H which generates a schedule consisting of n processing batches, let B_i denote the i th processing batch such that $S(B_1) < S(B_2) < \dots < S(B_n)$. For simplicity, we write $S(B_i)$ as S_i . Let $U(t)$ be the set of the available unscheduled jobs at time t in H .

If, in an online algorithm H , B_k is being processed at the current time t , then the following subprocedure which is called *Restart* (k, t) may be used.

Restart (k, t). At time t , interrupt B_k and select $\min\{b, |B_k \cup U(t)|\}$ jobs in $B_k \cup U(t)$ such that each job selected has a delivery time not less than the largest delivery time of jobs not selected in $B_k \cup U(t)$ as the batch B_{k+1} starting at time t . Then reset k to be $k + 1$.

This paper studies the online scheduling of equal length jobs on a bounded parallel batch machine with batch capacity b to minimize $L_{\max}$ with limited restart. Here $L_{\max}$ means the time by which all jobs have been delivered. This problem can be denoted by $1 \mid \text{online}, r_j, p_j = 1, \text{P-batch}, b < \infty, \text{L-restart} \mid L_{\max}$. For simplicity, when $b = 2$, the problem is shortly written as P-batch ($b = 2$). When $b \geq 3$, the problem is shortly written as P-batch ($b \geq 3$).

Let α be the positive solution of $2\alpha(1 + \alpha) = 1$ and let β be the positive solution of $\beta(1 + \beta)^2 = 1$.

This paper is organized as follows. In Section 2, we give a best possible online algorithm with a competitive ratio of $1 + \alpha$ for problem P-batch ($b = 2$). In Section 3, we give a best possible online algorithm with a competitive ratio of $1 + \beta$ for problem P-batch ($b \geq 3$).

2. Problem P-Batch ($b = 2$)

For a job instance I , let $L_{\text{on}} = L_{\text{on}}(I)$ and $L_{\text{opt}} = L_{\text{opt}}(I)$ denote, respectively, the values of $L_{\max}$ given by an online algorithm and an offline optimal algorithm.

Theorem 1. Any online algorithm for problem P -batch ($b = 2$) cannot have a competitive ratio of less than $1 + \alpha$.

Proof. Let ϵ be a sufficiently small positive number. For any online algorithm H , we consider the following instance. At time 0, the first job J_1 with $q_1 = 0$ arrives and no jobs arrive before the starting time of J_1 .

If $S_1 \geq \alpha$, no jobs arrive later. We have $L_{\text{on}} = S_1 + 1$ and $L_{\text{opt}} = 1$. Hence, $L_{\text{on}}/L_{\text{opt}} = (S_1 + 1)/1 \geq 1 + \alpha$.

If $S_1 < \alpha$, then, at time $S_1 + \alpha$, the second job J_2 with $q_2 = 0$ arrives and no jobs arrive before the starting time of J_2 . We distinguish the following two cases.

Case 1 ($S_2 \geq S_1 + 1$). Then no jobs arrive later. Then $L_{\text{on}} \geq S_1 + 2$ and $L_{\text{opt}} = S_1 + \alpha + 1$. Thus $L_{\text{on}}/L_{\text{opt}} \geq (S_1 + 2)/(S_1 + \alpha + 1) \geq (\alpha + 2)/(2\alpha + 1) = 1 + \alpha$.

Case 2 ($S_2 < S_1 + 1$). Then B_2 interrupts B_1 .

If B_2 is a batch containing only one job J_2 , no jobs arrive later. We have $L_{\text{on}} \geq S_2 + 2 \geq S_1 + \alpha + 2$ and $L_{\text{opt}} = S_1 + \alpha + 1$. Hence, $L_{\text{on}}/L_{\text{opt}} \geq (S_1 + \alpha + 2)/(S_1 + \alpha + 1) \geq 1 + 1/(2\alpha + 1) > 1 + \alpha$.

If B_2 is a batch consisting of $\{J_1, J_2\}$ and $S_2 \leq 2\alpha$, then job J_3 with $q_3 = 1$ arrives at time $S_2 + \epsilon$ and no jobs arrive later. As B_2 cannot be restarted, we have $L_{\text{on}} \geq S_2 + 2 + q_3 = S_2 + 2 + 1$. In the offline setting, we can schedule J_3 starting at time $S_2 + \epsilon$ and $\{J_1, J_2\}$ as a single batch starting at time $S_2 + 1 + \epsilon$. So $L_{\text{opt}} \leq S_2 + 2 + \epsilon$. Then when $\epsilon \rightarrow 0$, we have $L_{\text{on}}/L_{\text{opt}} \geq (S_2 + 3)/(S_2 + 2 + \epsilon) \rightarrow (S_2 + 3)/(S_2 + 2) \geq 1 + 1/(2\alpha + 2) = 1 + \alpha$.

If B_2 is a batch consisting of $\{J_1, J_2\}$ and $S_2 > 2\alpha$, then job J_3 with $q_3 = 0$ arrives at time $S_2 + \epsilon$ and no jobs arrive later. As B_2 cannot be restarted, we have $L_{\text{on}} \geq S_2 + 2$. In the offline setting, we can schedule J_1 starting at time 0 and $\{J_2, J_3\}$ as a single batch starting at time $\max\{1, S_2 + \epsilon\}$. So $L_{\text{opt}} \leq \max\{2, S_2 + 1 + \epsilon\}$. Then when $\epsilon \rightarrow 0$, we have $L_{\text{on}}/L_{\text{opt}} \geq (S_2 + 2)/\max\{2, S_2 + 1 + \epsilon\} \rightarrow (S_2 + 2)/\max\{2, S_2 + 1\} = \min\{(S_2 + 2)/2, 1 + 1/(S_2 + 1)\} \geq \min\{1 + \alpha, 1 + 1/(\alpha + 2)\} = 1 + \alpha$.

The theorem follows. $\square$

Algorithm $H(b = 2)$. Initially set $k = 0$. At the current time instant t , do the following.

S(1): if $k = 0$, $U(t) \neq \emptyset$, and $t \geq \alpha$, then select $\min\{2, |U(t)|\}$ jobs in $U(t)$ such that each job selected has a delivery time not less than the largest delivery time of jobs not selected in $U(t)$ as the batch B_{k+1} starting at time t . Then reset k to be $k + 1$.

S(2): if $k \geq 1$, the machine is idle, and $U(t) \neq \emptyset$, then select $\min\{2, |U(t)|\}$ jobs in $U(t)$ such that each job selected has a delivery time not less than the largest delivery time of jobs not selected in $U(t)$ as the batch B_{k+1} starting at time t . Then reset k to be $k + 1$.

S(3): if the machine is occupied by B_k , $k \in \{1, 2\}$, $U(t) \neq \emptyset$, and each batch starting earlier than S_k is not interrupted, do the following.

S(3.1): if B_k is nonfull and $t = S_k + \alpha$, then do Restart (k, t).

S(3.2): if $k = 1$, B_k is full, $t = S_k + \alpha$, and some job in $U(t)$ has a delivery time larger than the delivery time of some job in B_k , then do Restart (k, t).

S(4): otherwise, do nothing but wait.

From algorithm $H(b = 2)$, we have the following observation.

Observation 1. (1) For each batch B_i in algorithm $H(b = 2)$, we have $S_i \geq \alpha$.

(2) All jobs arrive at or after $S_1 - \alpha$. And if $S_1 > \alpha$, then all jobs arrive at or after S_1 .

(3) If B_j interrupts B_{j-1} , then $S_j = S_{j-1} + \alpha$.

(4) If $k \geq 3$, then $S_k \geq 2\alpha + 1$.

Theorem 2. For problem P -batch ($b = 2$), $H(b = 2)$ is a best possible online algorithm with a competitive ratio of $1 + \alpha$.

Proof. Suppose to the contrary that I is an instance so that $L_{\text{on}}/L_{\text{opt}} > 1 + \alpha$. Let σ be the schedule generated by running algorithm $H(b = 2)$, for instance, I . For simplicity, in the following we always write $q(B_i)$ as q_i and write $r(B_i)$ as r_i without causing confusion. By the definitions of $q(B_i)$ and $r(B_i)$, we know that q_i is the largest delivery time of jobs in B_i and r_i is the latest arrival time of jobs whose delivery times are q_i in B_i . Let B_n be the earliest starting batch in σ which assumes L_{on} and is not interrupted. Then $L_{\text{on}} = S_n + 1 + q_n$. Let π be an offline optimal schedule, for instance, I . As $S_1 \leq \max\{r_1, \alpha\}$, then the assumption that $L_{\text{on}}/L_{\text{opt}} > 1 + \alpha$ implies $n \geq 2$.

Let $C_{\text{on}} = S_n + 1$; that is, C_{on} is the completion time of B_n on the parallel batch machine. Let t be the minimum time instant such that, in σ , there are no idle times between t and C_{on} and each batch processed in the time interval $[t, C_{\text{on}}]$ is not interrupted. Suppose that there are h batches $B_{n-h+1}, \dots, B_n$ processed in the time interval $[t, C_{\text{on}}]$ in σ . Then $C_{\text{on}} = t + h$. Set $n' = n - h + 1$. So $S_{n'} = t$.

Claim 1 (B_n does not interrupt B_{n-1} and $h \geq 2$). Otherwise, B_n interrupts B_{n-1} . Then $L_{\text{on}} = S_n + 1 + q_n = S_{n-1} + \alpha + 1 + q_n$. By algorithm $H(b = 2)$, we have $n = 2, 3$. Let J' be the job with the largest delivery time among jobs arriving in the interval $(S_{n-1}, S_n]$, let q' be the delivery time of J' , and let r' be the arrival time of J' . Then $q_n = \max\{q_{n-1}, q'\}$. If $q' \geq q_{n-1}$, then $q_n = q'$, $L_{\text{opt}} \geq r' + 1 + q' > S_{n-1} + 1 + q'$, $L_{\text{on}} = S_{n-1} + \alpha + 1 + q'$, and, accordingly, $L_{\text{on}}/L_{\text{opt}} < 1 + \alpha$, a contradiction. If there exists only one batch in the offline optimal schedule π , then $L_{\text{opt}} \geq r' + 1 + q_n > S_{n-1} + 1 + q_n$, and accordingly, $L_{\text{on}}/L_{\text{opt}} \leq (S_{n-1} + \alpha + 1 + q_n)/(S_{n-1} + 1 + q_n) < 1 + \alpha$, a contradiction. So in the following we assume that $q' < q_{n-1}$, that is, $q_n = q_{n-1}$, and assume that there exist at least two batches in the offline optimal schedule π . Then $L_{\text{on}} = S_{n-1} + \alpha + 1 + q_{n-1}$ and $L_{\text{opt}} \geq 2$.

If $n = 2$, then $L_{\text{on}} = S_1 + \alpha + 1 + q_1$. By Observation 1, we have $L_{\text{opt}} \geq S_1 - \alpha + 1 + q_1$. Noticing that $L_{\text{opt}} \geq 2$, we have $L_{\text{on}}/L_{\text{opt}} \leq 1 + \alpha$, a contradiction.

If $n = 3$, then $L_{\text{on}} = S_2 + \alpha + 1 + q_2$. By algorithm $H(b = 2)$, we know that B_2 does not interrupt B_1 ; that is, $S_2 \geq S_1 + 1$. If $S_2 > S_1 + 1$, then all jobs in B_2 arrive at or after S_2 , and so $L_{\text{opt}} \geq S_2 + 1 + q_2$. Thus, $L_{\text{on}}/L_{\text{opt}} < 1 + \alpha$, a contradiction. If

$S_2 = S_1 + 1$, then $L_{\text{on}} = S_1 + \alpha + 2 + q_2$. If, in the offline optimal schedule π , J_2 is scheduled after the first starting batch, then $L_{\text{opt}} \geq S_1 - \alpha + 2 + q_2$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + 2 + q_2)/(S_1 - \alpha + 2 + q_2) \leq 1 + \alpha$, a contradiction. If, in the offline optimal schedule π , J_2 is scheduled in the first starting batch and $r_2 \geq S_1 + \alpha$, considering there exist at least two batches in the offline optimal schedule π , then $L_{\text{opt}} \geq \max\{S_1 + \alpha + 1 + q_2, S_1 + \alpha + 2\}$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + 2 + q_2)/\max\{S_1 + \alpha + 1 + q_2, S_1 + \alpha + 2\} \leq 1 + 1/(2\alpha + 2) = 1 + \alpha$, a contradiction. If, in the offline optimal schedule π , J_2 is scheduled in the first starting batch and $r_2 < S_1 + \alpha$, then B_1 is full and the smallest delivery time of jobs in B_1 is not less than q_2 . Thus $L_{\text{opt}} \geq S_1 - \alpha + 2 + q_2$, and so $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + 2 + q_2)/(S_1 - \alpha + 2 + q_2) \leq 1 + \alpha$, a contradiction.

The above analysis means that B_n does not interrupt B_{n-1} . Note that $n \geq 2$. If $h = 1$, then $S_n > S_{n-1} + 1$. From S(2) in algorithm $H(b = 2)$, we know that $S_n = r_n$. Thus $L_{\text{on}} = S_n + 1 + q_n = r_n + 1 + q_n \leq L_{\text{opt}}$, a contradiction. Thus, $h \geq 2$. Claim 1 follows.

Claim 2 (some batch in the $h-1$ batches $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ is nonfull). Otherwise, all of the $h-1$ batches $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ are full.

If the delivery time of some job in $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ is less than q_n , let B_k be the latest starting batch in $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ such that some job in B_k has a delivery time less than q_n . Then each job in batches starting in the time interval (S_k, S_n) and J_n have delivery times not less than q_n and then arrive after S_k . Then $L_{\text{opt}} \geq S_k + n - k + q_n$, $L_{\text{on}} = S_k + n - k + 1 + q_n$, and $L_{\text{on}} - L_{\text{opt}} \leq 1$.

If $k \geq 3$ or $[k = 2 \text{ and } n \geq 4]$, then $S_k + n - k \geq 2 + 2\alpha$, and so $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(2\alpha + 2) = \alpha$, a contradiction.

If $k = 2$ and $n = 3$, then we have $r_3 > S_2$ and $L_{\text{opt}} \geq r_3 + 1 + q_3 > S_2 + 1 + q_3$. As B_2 is full, there exist at least two batches in the offline optimal schedule π . Suppose that B_2 interrupts B_1 , which means that $S_2 = S_1 + \alpha$ and $L_{\text{on}} = S_1 + \alpha + 2 + q_3$. If J_3 is scheduled in the first starting batch in the offline optimal schedule π , then $L_{\text{opt}} \geq r_3 + 2 \geq S_1 + \alpha + 2 \geq 2 + 2\alpha$. Thus, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(2\alpha + 2) = \alpha$, a contradiction. If J_3 is scheduled after the first starting batch in the offline optimal schedule π , then $L_{\text{opt}} \geq S_1 - \alpha + 2 + q_3$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + 2 + q_3)/(S_1 - \alpha + 2 + q_3) \leq 1 + \alpha$, a contradiction. So we suppose that B_2 does not interrupt B_1 ; that is, $S_2 \geq S_1 + 1$. Note that now $L_{\text{on}} = S_2 + 2 + q_3$. If $S_2 > S_1 + 1$, then each job in $\{B_2, B_3\}$ arrives at or later than S_2 . Considering that B_2 is full, then $L_{\text{opt}} \geq S_2 + 2 \geq 3 + \alpha$. Thus, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(\alpha + 3) < \alpha$, a contradiction. If $S_2 = S_1 + 1$, then $L_{\text{on}} = S_2 + 2 + q_3 = S_1 + 3 + q_3$. If B_1 is full, then $L_{\text{opt}} \geq 3$. Thus, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/3 < \alpha$, a contradiction. If B_1 is nonfull, considering that B_2 does not interrupt B_1 , then each job in $\{B_2, B_3\}$ arrives later than $S_1 + \alpha$ which means that $L_{\text{opt}} \geq S_1 + \alpha + 2 \geq 2\alpha + 2$. So $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(2\alpha + 2) = \alpha$, a contradiction.

If $k = 1$, noticing that now B_1 is full and B_2 does not interrupt B_1 , then each job in batches starting in the time interval (S_1, S_n) and J_n have delivery times not less than q_n and arrive after $S_1 + \alpha$ by S(3) in algorithm $H(b = 2)$. So $L_{\text{opt}} \geq S_1 + \alpha + n - 1 + q_n$ and $L_{\text{on}} = S_1 + n + q_n$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq$

$(S_1 + n + q_n)/(S_1 + \alpha + n - 1 + q_n) \leq (2 + \alpha)/(2\alpha + 1) = 1 + \alpha$, a contradiction.

So in the following we assume the delivery time of each job in $\{B_{n'}, B_{n'+1}, \dots, B_{n-1}\}$ is not less than q_n . Then $L_{\text{opt}} \geq S_1 - \alpha + h + q_n$.

If $B_{n'}$ interrupts $B_{n'-1}$, then $n' = 2, 3$. If $n' = 2$, then $L_{\text{on}} = S_2 + h + q_n = S_1 + \alpha + h + q_n$. Considering $h \geq 2$ and $S_1 \geq \alpha$, we have $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + h + q_n)/(S_1 - \alpha + h + q_n) \leq 1 + \alpha$, a contradiction. If $n' = 3$, then B_2 is nonfull, $S_2 \geq S_1 + 1$, and $L_{\text{on}} = S_3 + h + q_n = S_2 + \alpha + h + q_n$. Note that now there is only one job in B_2 . If $q_2 < q_n$, considering jobs in the set of $\{B_{n'}, B_{n'+1}, \dots, B_{n-1}, J_n\}$ all have delivery times not less than q_n , then these jobs arrive after S_2 , and, accordingly, $L_{\text{opt}} \geq S_2 + h + q_n$. So $L_{\text{on}}/L_{\text{opt}} \leq (S_2 + \alpha + h + q_n)/(S_2 + h + q_n) < 1 + \alpha$, a contradiction. If $q_2 \geq q_n$ and $r_2 < S_1 + \alpha$, then $S_2 = S_1 + 1$, $L_{\text{on}} = S_1 + \alpha + h + 1 + q_n$, B_1 is full, and the delivery time of any job in B_1 is not less than q_n . And so $L_{\text{opt}} \geq S_1 - \alpha + h + 1 + q_n$. So $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + h + 1 + q_n)/(S_1 - \alpha + h + 1 + q_n) < 1 + \alpha$, a contradiction. If $q_2 \geq q_n$ and $S_1 + \alpha \leq r_2 \leq S_1 + 1$, considering that there is only one job in B_2 , then $S_2 = S_1 + 1$ and $L_{\text{on}} = S_1 + \alpha + h + 1 + q_n$, and each job in $\{B_{n'}, B_{n'+1}, \dots, B_n\}$ arrives at or after $S_1 + \alpha$. So $L_{\text{opt}} \geq S_1 + \alpha + h + q_n$. Thus $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \alpha + h + 1 + q_n)/(S_1 + \alpha + h + q_n) \leq 1 + 1/(2 + 2\alpha) = 1 + \alpha$, a contradiction. If $q_2 \geq q_n$ and $r_2 > S_1 + 1$, then $S_2 > S_1 + 1$ and each job in $\{B_{n'}, B_{n'+1}, \dots, B_n\}$ arrives at or after S_2 , and so $L_{\text{opt}} \geq S_2 + h + q_n$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_2 + \alpha + h + q_n)/(S_2 + h + q_n) \leq 1 + \alpha$, a contradiction.

If $n' = 1$ or $[n' \geq 2 \text{ and } B_{n'} \text{ does not interrupt } B_{n'-1}]$, then there exists some time instant t' such that $t' < S_{n'}$ and the time interval $[t', S_{n'})$ is an idle time in σ . Then by S(1), S(2) in algorithm $H(b = 2)$, the arrival time of any job in $\{B_{n'}, B_{n'+1}, \dots, B_n\}$ is not less than $S_{n'} - \alpha$. Thus $L_{\text{opt}} \geq S_{n'} - \alpha + h + q_n$, and so $L_{\text{on}}/L_{\text{opt}} \leq (S_{n'} + h + q_n)/(S_{n'} - \alpha + h + q_n) < 1 + \alpha$, a contradiction. Claim 2 follows.

Let $e \leq n - 1$ be maximum so that B_e is a nonfull batch. From Claim 2, we have $e \geq n'$.

Claim 3 ($e \leq n - 2$). Otherwise, suppose that $e = n - 1$. Since B_{n-1} is nonfull, we have $r_n > S_{n-1}$, $L_{\text{on}} = S_e + 2 + q_n$, and $L_{\text{opt}} \geq r_n + 1 + q_n > S_e + 1 + q_n$. If some batch starting before S_e is interrupted, noticing that B_e is a nonfull batch, then the interrupted batch must start before S_{e-1} , and then $S_e \geq S_3 \geq 2\alpha + 1$. Hence, $L_{\text{on}}/L_{\text{opt}} \leq (S_e + 2 + q_n)/(S_e + 1 + q_n) \leq 1 + 1/(2\alpha + 2) = 1 + \alpha$, a contradiction. So in the following we assume that each batch starting before S_e is not interrupted. If $n \leq 3$, then $e \leq 2$. As B_e is a nonfull batch and not interrupted, by S(3.1) in algorithm $H(b = 2)$, all jobs in B_n arrive after $S_e + \alpha$, which means that $L_{\text{opt}} \geq S_e + \alpha + 1 + q_n$. Hence, $L_{\text{on}}/L_{\text{opt}} \leq (S_e + 2 + q_n)/(S_e + \alpha + 1 + q_n) \leq (2 + \alpha)/(2\alpha + 1) = 1 + \alpha$, a contradiction. If $n \geq 4$, then $S_e \geq S_3 \geq 1 + 2\alpha$, and accordingly, $L_{\text{on}}/L_{\text{opt}} \leq (S_e + 2 + q_n)/(S_e + 1 + q_n) \leq 1 + 1/(2\alpha + 2) = 1 + \alpha$, a contradiction. Claim 3 follows.

By the definition of B_e and Claim 3, $n - e \geq 2$, $B_{e+1}, \dots, B_{n-1}$ are all full and each job in batches starting after S_e arrives after S_e . And $L_{\text{on}} = S_e + n - e + 1 + q_n$.

If each job in $B_{e+1}, \dots, B_{n-1}$ has a delivery time not less than q_n , then $L_{\text{opt}} \geq S_e + n - e + q_n$. If $e \geq 2$, then $S_e \geq 2\alpha$. Then $L_{\text{on}}/L_{\text{opt}} \leq (S_e + n - e + 1 + q_n)/(S_e + n - e + q_n) \leq$

$1 + 1/(2\alpha + 2) = 1 + \alpha$, a contradiction. If $e = 1$, considering B_e is nonfull and not interrupted, then, by S(3.1) in algorithm $H(b = 2)$, each job in batches starting after S_1 arrives after $S_1 + \alpha$. Thus $L_{\text{opt}} \geq S_1 + \alpha + n - 1 + q_n$. Hence, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + n + q_n)/(S_1 + \alpha + n - 1 + q_n) \leq (3 + \alpha)/(2\alpha + 2) < 1 + \alpha$, a contradiction.

So in the following we assume that some job in $B_{e+1}, \dots, B_{n-1}$ has a delivery time less than q_n . Let $e + 1 \leq k' \leq n - 1$ be maximum so that some job in $B_{k'}$ has a delivery time less than q_n . Then $k' \geq 2$ and batches starting in the time interval $(S_{k'}, S_n)$ are all full. Also each job in batches starting in the time interval $(S_{k'}, S_n)$ and J_n have delivery times not less than q_n and arrive after $S_{k'}$. Thus $L_{\text{opt}} \geq S_{k'} + n - k' + q_n$. Noting that $L_{\text{on}} = S_{k'} + n - k' + 1 + q_n$, we have $L_{\text{on}} - L_{\text{opt}} \leq 1$. If $k' \geq 3$, then $S_{k'} \geq 1 + 2\alpha$. Hence, $L_{\text{on}}/L_{\text{opt}} \leq (S_{k'} + n - k' + 1 + q_n)/(S_{k'} + n - k' + q_n) \leq 1 + 1/(2\alpha + 2) = 1 + \alpha$, a contradiction. If $k' = 2$, then $e = 1$, $n \geq 3$, and $S_{k'} \geq 2\alpha$. As now B_1 is nonfull and not interrupted, then by S(3.1) in algorithm $H(b = 2)$, each job in batches starting after S_1 arrives after $S_1 + \alpha$. Thus $L_{\text{opt}} \geq S_1 + \alpha + 2 \geq 2\alpha + 2$. So $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(2\alpha + 2) = \alpha$, a contradiction.

From the above discussion we conclude that algorithm $H(b = 2)$ has a competitive ratio of at most $1 + \alpha$. By Theorem 1, we know that algorithm $H(b = 2)$ is an optimal online algorithm with a competitive ratio of $1 + \alpha$. This completes the proof. $\square$

3. Problem P-Batch ($b \geq 3$)

Note that β is the positive solution of $\beta(1 + \beta)^2 = 1$.

Theorem 3. For problem P-batch ($b \geq 3$), there exists no online algorithm with a competitive ratio of less than $1 + \beta$.

Proof. Theorem 2.4 in Liu and Yuan [21] shows that, for problem 1 | online, $r_j, p_j = 1$, P-batch, $b < \infty$, L-restart | C_{max} with $b \geq 3$, no online algorithm has a competitive ratio of less than $1 + \beta$. Here C_{max} is makespan, namely, the maximum completion time of all jobs on the parallel batch machine. Problem 1 | online, $r_j, p_j = 1$, P-batch, $b < \infty$, L-restart | C_{max} is the special case of 1 | online, $r_j, p_j = 1$, P-batch, $b < \infty$, L-restart | L_{max} . So for problem P-batch ($b \geq 3$), no online algorithm has a competitive ratio of less than $1 + \beta$. The theorem follows. $\square$

Algorithm $H(b \geq 3)$. Initially set $k = 0$. At the current time instant t , do the following.

S(1): if $k = 0$, $U(t) \neq \emptyset$, and $t \geq \beta$, then select $\min\{b, |U(t)|\}$ jobs in $U(t)$ such that each job selected has a delivery time not less than the largest delivery time of jobs not selected in $U(t)$ as the batch B_{k+1} starting at time t . Then reset k to be $k + 1$.

S(2): if $k \geq 1$, the machine is idle, and $U(t) \neq \emptyset$, select $\min\{b, |U(t)|\}$ jobs in $U(t)$ such that each job selected has a delivery time not less than the largest delivery time of jobs not selected in $U(t)$ as the batch B_{k+1} starting at time t . Then reset k to be $k + 1$.

S(3): if $S_1 < 1/\beta - 1$, $U(t) \neq \emptyset$, the machine is occupied by B_k , $k \in \{1, 2\}$, and each batch starting earlier than S_k is not interrupted, do the following.

S(3.1): if $k = 1$, B_k is nonfull, $t = (1 + \beta)S_k + \beta$, and $|B_k| + |U(t)| \leq b$, then let q' be the largest delivery time of jobs in $U(t)$. If $q' \geq q_1$, or $q' \geq \beta$, or $[q' < \beta$, and $q_1 < 1 - \beta]$, then do Restart(k, t).

S(3.2): if $k = 1$, B_k is full, $t = S_k + \beta$, the largest delivery time of jobs in $U(t)$ is not less than $\beta^2 + \beta$, and the delivery time of some job in B_1 is less than the delivery time of some job in $U(t)$, then do Restart(k, t).

S(3.3): if $k = 2$, B_k is nonfull, $t = S_k + \beta$, $|B_k| + |U(t)| \leq b$, and $\{\text{either } B_1 \text{ is full or } [B_1 \text{ is nonfull and the number of jobs arriving in the time interval } [0, (1 + \beta)S_1 + \beta] \text{ is not more than } b]\}$, then do Restart(k, t).

S(4): otherwise, do nothing but wait.

From algorithm $H(b \geq 3)$ we have the following observation.

Observation 2. (1) The earliest arrival time of all jobs is not less than $S_1 - \beta$. If $S_1 > \beta$, then all jobs arrive at or after S_1 .

(2) If $S_1 \geq 1/\beta - 1$ or $\{S_1 < 1/\beta - 1, B_1 \text{ is nonfull, and the number of jobs arriving in the time interval } [0, (1 + \beta)S_1 + \beta] \text{ is more than } b\}$, then there exists no batch which is interrupted.

(3) If $k \geq 3$, then $S_k \geq 2\beta + 1$.

(4) If $S_1 < 1/\beta - 1$, then $(1 + \beta)S_1 + \beta < S_1 + 1$.

Theorem 4. For problem P-batch ($b \geq 3$), $H(b \geq 3)$ is a best possible online algorithm with a competitive ratio of $1 + \beta$.

Proof. Suppose to the contrary that I is an instance so that $L_{\text{on}}/L_{\text{opt}} > 1 + \beta$. Let σ be the schedule generated by running algorithm $H(b \geq 3)$ for the instance I and let π be an offline optimal schedule for the instance I . As the proof in Theorem 2, q_i is used to denote the largest delivery time of jobs in B_i and r_i is the latest arrival time of jobs whose delivery times are q_i in B_i . Let B_n be the earliest starting batch in σ which is not interrupted and assumes L_{on} . Then $L_{\text{on}} = S_n + 1 + q_n$. As $S_1 \leq \max\{r_1, \beta\}$, then the assumption that $L_{\text{on}}/L_{\text{opt}} > 1 + \beta$ implies $n \geq 2$.

Let $C_{\text{on}} = S_n + 1$; that is, C_{on} is the completion time of B_n on the parallel batch machine. Let t be the minimum time instant such that, in σ , there are no idle times between t and C_{on} and each batch processed in the time interval $[t, C_{\text{on}}]$ is not interrupted. Suppose that there are h batches $B_{n-h+1}, \dots, B_n$ processed in the time interval $[t, C_{\text{on}}]$ in σ . Then $C_{\text{on}} = t + h$. Set $n' = n - h + 1$. So $S_{n'} = t$.

Claim 1 (B_n does not interrupt B_{n-1} and $h \geq 2$). Otherwise, B_n is a restricted batch. By algorithm $H(b \geq 3)$, we have $n = 2, 3$ and $S_1 < 1/\beta - 1$; that is, $(1 + \beta)S_1 + \beta < S_1 + 1$. Let q'' be the largest delivery time of jobs arriving in the interval $(S_{n-1}, S_n]$, let J'' be the job with delivery time q'' , and let r'' be the arrival time of J'' . Then $q_n = \max\{q_{n-1}, q''\}$. If $q'' \geq q_{n-1}$, then $q_n = q''$, $L_{\text{opt}} \geq S_{n-1} + 1 + q''$, $L_{\text{on}} = S_n + 1 + q'' \leq (1 + \beta)S_{n-1} + \beta + 1 + q''$, and, accordingly, $L_{\text{on}}/L_{\text{opt}} \leq 1 + \beta$, a contradiction. So in the following we assume that $q'' < q_{n-1}$; that is, $q_n = q_{n-1}$. Then $L_{\text{on}} = S_n + 1 + q_{n-1}$. If, in the offline optimal schedule π , there exists only one batch, or J_{n-1} and J''

are scheduled in one batch, or the batch containing J_{n-1} starts later than the batch containing J'' , then $L_{\text{opt}} \geq S_{n-1} + 1 + q_{n-1}$. Considering $L_{\text{on}} = S_n + 1 + q_{n-1} \leq (1 + \beta)S_{n-1} + \beta + 1 + q_{n-1}$, then $L_{\text{on}}/L_{\text{opt}} \leq 1 + \beta$, a contradiction. So in the following we assume that, in the offline optimal schedule π , there exist at least 2 batches, J_{n-1} and J'' are scheduled in two different batches, and the batch containing J_{n-1} starts earlier than the batch containing J'' . Then we easily get $L_{\text{opt}} \geq \max\{r_{n-1} + 1 + q_{n-1}, r_{n-1} + 2 + q''\} \geq S_1 - \beta + 2 + q''$.

If $n = 2$ and B_1 is full, then $q'' < q_1$, $S_2 = S_1 + \beta$, and $L_{\text{on}} = S_1 + \beta + 1 + q_1$. By Observation 2, we have $L_{\text{opt}} \geq S_1 - \beta + 1 + q_1$. Considering $L_{\text{opt}} \geq 2$, then $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq (2\beta)/2 = \beta$, a contradiction.

If $n = 2$ and B_1 is nonfull, then $L_{\text{on}} = (1 + \beta)S_1 + \beta + 1 + q_1$. If $q_1 < 1 - \beta$, then $L_{\text{on}} < (1 + \beta)S_1 + 2$. Noticing that $L_{\text{opt}} \geq S_1 - \beta + 2$, then $L_{\text{on}}/L_{\text{opt}} \leq ((1 + \beta)S_1 + 2)/(S_1 - \beta + 2) < 1 + \beta$, a contradiction. Hence we have $q_1 \geq 1 - \beta$. As B_2 interrupts B_1 , considering $q'' < q_1$ and $q_1 \geq 1 - \beta$, then, by S(3.1) in algorithm $H(b \geq 3)$, the only possibility is that $q'' \geq \beta$. Then $L_{\text{opt}} \geq S_1 - \beta + 2 + q'' \geq S_1 + 2$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq ((1 + \beta)S_1 + \beta + 1 + q_1)/\max\{S_1 - \beta + 1 + q_1, S_1 + 2\} \leq 1 + (\beta(S_1 + 2))/(S_1 + 2) = 1 + \beta$, a contradiction.

If $n = 3$, then $S_3 = S_2 + \beta$ and $L_{\text{on}} = S_2 + \beta + 1 + q_2$. Also noticing that now B_2 does not interrupt B_1 , we have $S_2 \geq S_1 + 1$. Note that $q'' < q_{n-1}$ and in the offline optimal schedule π , the batch containing $J_{n-1} = J_2$ starts earlier than the batch containing J'' . If $r_2 \leq S_1$, then $S_2 = S_1 + 1$, $L_{\text{on}} = S_1 + \beta + 2 + q_2$, B_1 is full and each job in B_1 has a delivery time not less than q_2 , and, accordingly, $L_{\text{opt}} \geq S_1 - \beta + 2 + q_2$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \beta + 2 + q_2)/(S_1 - \beta + 2 + q_2) \leq 1 + \beta$, a contradiction. If $r_2 \geq S_2$, then $L_{\text{opt}} \geq S_2 + 1 + q_2$. Thus $L_{\text{on}}/L_{\text{opt}} \leq (S_2 + \beta + 1 + q_2)/(S_2 + 1 + q_2) < 1 + \beta$, a contradiction. If $(1 + \beta)S_1 + \beta \leq r_2 < S_2$, then $S_2 = S_1 + 1$, $L_{\text{on}} = S_1 + \beta + 2 + q_2$, and $L_{\text{opt}} \geq (1 + \beta)S_1 + \beta + 1 + q_2$. Thus $L_{\text{on}}/L_{\text{opt}} < (S_1 + \beta + 2 + q_2)/((1 + \beta)S_1 + \beta + 1 + q_2) \leq (2\beta + 2 + q_2)/((1 + \beta)\beta + \beta + 1 + q_2) < 1 + \beta$, a contradiction. So in the following we assume that $S_1 < r_2 < (1 + \beta)S_1 + \beta$. Then $S_2 = S_1 + 1$, $L_{\text{on}} = S_1 + \beta + 2 + q_2$, and $L_{\text{opt}} \geq r_2 + 2 > S_1 + 2$. If $q_2 \leq \beta + \beta^2$, then $L_{\text{on}}/L_{\text{opt}} < (S_1 + \beta + 2 + \beta + \beta^2)/(S_1 + 2) \leq 1 + \beta$, a contradiction. If $q_2 > \beta + \beta^2$, noticing that B_2 does not interrupt B_1 and B_3 interrupts B_2 , by S(3) in algorithm $H(b \geq 3)$, then B_1 is full. If each job in the full batch B_1 has a delivery time not less than q_2 , then $L_{\text{opt}} \geq S_1 - \beta + 2 + q_2$, and then $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \beta + 2 + q_2)/(S_1 - \beta + 2 + q_2) \leq 1 + \beta$, a contradiction. So some job in the full batch B_1 has a delivery time less than q_2 . As B_1 is full, $q_2 > \beta + \beta^2$, some job in B_1 has a delivery time less than q_2 , and B_2 does not interrupt B_1 , then $r_2 > S_1 + \beta$, and so $L_{\text{opt}} \geq \max\{S_1 + \beta + 1 + q_2, S_1 + \beta + 2\}$. Thus, $L_{\text{on}}/L_{\text{opt}} < (S_1 + \beta + 2 + q_2)/\max\{S_1 + \beta + 1 + q_2, S_1 + \beta + 2\} \leq 1 + 1/(2\beta + 2) < 1 + \beta$, a contradiction.

The above analysis means that B_n does not interrupt B_{n-1} . Note that $n \geq 2$. If $h = 1$, then $S_n > S_{n-1} + 1$. From S(2) in algorithm $H(b \geq 3)$, we know that $S_n = r_n$. Thus $L_{\text{on}} = S_n + 1 + q_n = r_n + 1 + q_n \leq L_{\text{opt}}$, a contradiction. Hence, $h \geq 2$. Claim 1 follows.

Claim 2 (some batch in the $h-1$ batches $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ is nonfull). Otherwise, all of the $h-1$ batches $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ are full.

If the smallest delivery time of jobs in $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ is not less than q_n and $B_{n'}$ interrupts B_{n-1} , then $n' = 2, 3$. If $n' = 2$ and B_1 is full, then $L_{\text{on}} = S_{n'} + h + q_n = S_1 + \beta + h + q_n$ and $L_{\text{opt}} \geq S_1 - \beta + h + q_n$. Thus, considering $h \geq 2$ and $S_1 \geq \beta$, we have $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + \beta + h + q_n)/(S_1 - \beta + h + q_n) \leq 1 + \beta$, a contradiction. If $[n' = 2 \text{ and } B_1 \text{ is nonfull}]$ or $n' = 3$, then $B_{n'-1}$ is nonfull and each job in batches starting after $S_{n'}$ arrives after $S_{n'}$. Then $L_{\text{opt}} \geq S_{n'} + h - 1 + q_n$ and $S_{n'} \geq \min\{(1 + \beta)S_1 + \beta, S_1 + 1 + \beta\} \geq \min\{(1 + \beta)^2 - 1, 1 + 2\beta\} \geq (1 + \beta)^2 - 1$. Thus $L_{\text{on}}/L_{\text{opt}} \leq (S_{n'} + h + q_n)/(S_{n'} + h - 1 + q_n) \leq 1 + 1/(1 + \beta)^2 = 1 + \beta$, a contradiction.

If the smallest delivery time of jobs in $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ is not less than q_n and $\{n' = 1 \text{ or } [n' \geq 2 \text{ and } B_{n'} \text{ does not interrupt } B_{n'-1}]\}$, then the arrival time of any job in $B_{n'}, B_{n'+1}, \dots, B_n$ is not less than $S_{n'} - \beta$. Thus $L_{\text{opt}} \geq S_{n'} - \beta + h + q_n$, and so $L_{\text{on}}/L_{\text{opt}} \leq (S_{n'} + h + q_n)/(S_{n'} - \beta + h + q_n) < 1 + \beta$, a contradiction.

So in the following we suppose that the delivery time of some job in $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ is less than q_n . Let B_k be the latest starting batch in $B_{n'}, B_{n'+1}, \dots, B_{n-1}$ such that some job in B_k has a delivery time less than q_n . Then each job in batches starting in the time interval (S_k, S_n) and J_n have delivery times not less than q_n and, accordingly, arrive after S_k . Then $L_{\text{opt}} \geq S_k + n - k + q_n$, $L_{\text{on}} = S_k + n - k + 1 + q_n$, and $L_{\text{on}} - L_{\text{opt}} \leq 1$.

If $k \geq 3$ or $[k = 2 \text{ and } n \geq 4]$, then $L_{\text{opt}} \geq S_k + n - k \geq 2 + 2\beta$, and, accordingly, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(2 + 2\beta) < \beta$, a contradiction.

If $k = 2$, $n = 3$, and $\{B_1 \text{ is nonfull or } [B_1 \text{ is full and } B_2 \text{ does not interrupt } B_1]\}$, then $S_2 \geq \min\{(1 + \beta)S_1 + \beta, S_1 + 1\} \geq (1 + \beta)^2 - 1$. Thus, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(1 + \beta)^2 = \beta$, a contradiction. If $k = 2$, $n = 3$, B_1 is full, and B_2 interrupts B_1 , then some job with a delivery time not less than $\beta^2 + \beta$ arrives after S_1 and $L_{\text{opt}} \geq S_1 + 1 + \beta^2 + \beta \geq (1 + \beta)^2$. Thus, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(1 + \beta)^2 = \beta$, a contradiction.

If $k = 1$ and $n \geq 3$, then $L_{\text{opt}} \geq S_k + n - k \geq 2 + \beta$, and, accordingly, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(\beta + 2) < \beta$, a contradiction. If $k = 1$ and $n = 2$, then there are at least two batches in the offline optimal schedule π . Note that now $r_2 > S_1$. If $J_n = J_2$ is scheduled in the first batch in the offline optimal schedule π , then $L_{\text{opt}} \geq \max\{S_1 + 1 + q_2, S_1 + 2\} \geq 2 + \beta$. Thus, $(L_{\text{on}} - L_{\text{opt}})/L_{\text{opt}} \leq 1/(\beta + 2) < \beta$, a contradiction. If $J_n = J_2$ is scheduled in the batch starting after the first batch in the offline optimal schedule π , then $L_{\text{opt}} \geq S_1 - \beta + 2 + q_2$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + 2 + q_2)/(S_1 - \beta + 2 + q_2) \leq 1 + \beta$, a contradiction. Claim 2 follows.

Let $e \leq n - 1$ be maximum so that B_e is a nonfull batch. From Claim 2, we have $k \geq n'$.

Claim 3 ($e \leq n - 2$). Otherwise, suppose that $e = n - 1$. As B_e is a nonfull batch, then $r_n > S_e$, $L_{\text{on}} = S_e + 2 + q_n$, and $L_{\text{opt}} \geq r_n + 1 + q_n > S_e + 1 + q_n$.

If $e \geq 2$, noticing that B_e is a nonfull batch, then $S_e \geq \min\{(1 + \beta)S_1 + \beta, S_1 + 1\} \geq (1 + \beta)^2 - 1$. Then $L_{\text{on}}/L_{\text{opt}} \leq (S_e + 2 + q_n)/(S_e + 1 + q_n) \leq 1 + 1/(1 + \beta)^2 = 1 + \beta$, a contradiction.

If $e = 1$, then $n = 2$, $r_2 > S_1$, $L_{\text{on}} = S_1 + 2 + q_2$, and $L_{\text{opt}} > S_1 + 1 + q_2$. If $S_1 \geq 1/\beta - 1$, then $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + 2 + q_2)/(S_1 + 1 + q_2) \leq 1 + 1/(1/\beta) = 1 + \beta$, a contradiction.

So we suppose that $S_1 < 1/\beta - 1$. If $r_2 \geq (1 + \beta)S_1 + \beta$, then $L_{\text{opt}} \geq r_2 + 1 + q_2 \geq (1 + \beta)^2$. Thus $L_{\text{on}}/L_{\text{opt}} \leq 1 + 1/(1 + \beta)^2 = 1 + \beta$, a contradiction. So in the following we assume that $r_2 < (1 + \beta)S_1 + \beta$. If there is only one batch in the offline optimal schedule π , then $|B_1| + |B_2| \leq b$ and $L_{\text{opt}} \geq S_1 + 1 + q_1$. Also considering that B_2 does not interrupt B_1 , we have that $q_1 \geq 1 - \beta$ and $q_2 < \beta$. Then $L_{\text{opt}} \geq S_1 + 1 + q_1 \geq S_1 + 2 - \beta$. And $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + 2 + q_2)/(S_1 + 1 + q_1) \leq (S_1 + 2 + \beta)/(S_1 + 2 - \beta) \leq 1 + \beta$, a contradiction. So in the following we assume that there are at least two batches in the offline optimal schedule π . If $J_n = J_2$ is scheduled in the first starting batch in the offline optimal schedule π , then $L_{\text{opt}} \geq \max\{S_1 + 1 + q_2, S_1 + 2\} \geq 2 + \beta$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq 1 + 1/(\beta + 2) < 1 + \beta$. If $J_n = J_2$ is scheduled in the batch starting after the first starting batch in the offline optimal schedule π , then $L_{\text{opt}} \geq S_1 - \beta + 2 + q_2$. Thus, $L_{\text{on}}/L_{\text{opt}} \leq (S_1 + 2 + q_2)/(S_1 - \beta + 2 + q_2) \leq 1 + \beta$, a contradiction. Claim 3 follows.

By the definition of B_e and Claim 3, $n - e \geq 2$, $B_{e+1}, \dots, B_{n-1}$ are all full and each job in batches starting after S_e arrives after S_e . And $L_{\text{on}} = S_e + n - e + 1 + q_n$.

If each job in $B_{e+1}, \dots, B_{n-1}$ has a delivery time not less than q_n , then $L_{\text{opt}} \geq S_e + n - e + q_n \geq 2 + \beta$. Then $L_{\text{on}}/L_{\text{opt}} \leq (S_e + n - e + 1 + q_n)/(S_e + n - e + q_n) \leq 1 + 1/(\beta + 2) < 1 + \beta$, a contradiction. So in the following we assume that some job in $B_{e+1}, \dots, B_{n-1}$ has a delivery time less than q_n . Let $e + 1 \leq k' \leq n - 1$ be maximum so that some job in $B_{k'}$ has a delivery time less than q_n . Then $k' \geq 2$ and batches starting in the time interval $(S_{k'}, S_n)$ are all full. And each job in batches starting in the time interval $(S_{k'}, S_n)$ and J_n have delivery times not less than q_n and then arrive after $S_{k'}$. Thus $L_{\text{opt}} \geq S_{k'} + n - k' + q_n$. Note that $L_{\text{on}} = S_{k'} + n - k' + 1 + q_n$. Considering $k' \geq 2$, then $S_{k'} \geq S_e + 1 \geq 1 + \beta$. Hence, $L_{\text{on}}/L_{\text{opt}} \leq (S_{k'} + n - k' + 1 + q_n)/(S_{k'} + n - k' + q_n) \leq 1 + 1/(\beta + 2) < 1 + \beta$, a contradiction.

From the above discussion we conclude that algorithm $H(b \geq 3)$ has a competitive ratio of at most $1 + \beta$. By Theorem 3 we know that algorithm $H(b \geq 3)$ is an optimal online algorithm with a competitive ratio of $1 + \beta$. This completes the proof. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Natural Sciences Foundation (Grant no. 20142BAB211017) of Jiangxi Province and the School Subject (Grant no. 06162015) of Jiangxi University of Finance and Economics.

References

- [1] H. Hoogeveen, C. N. Potts, and G. J. Woeginger, "On-line scheduling on a single machine: maximizing the number of early jobs," *Operations Research Letters*, vol. 27, no. 5, pp. 193–197, 2000.
- [2] M. van den Akker, H. Hoogeveen, and N. Vakhania, "Restarts can help in the on-line minimization of the maximum delivery time on a single machine," *Journal of Scheduling*, vol. 3, no. 6, pp. 333–341, 2000.
- [3] L. Epstein and R. van Stee, "Lower bounds for on-line single-machine scheduling," *Theoretical Computer Science*, vol. 299, no. 1, pp. 439–450, 2003.
- [4] R. van Stee and H. La Poutré, "Minimizing the total completion time on-line on a single machine, using restarts," *Journal of Algorithms*, vol. 57, no. 2, pp. 95–129, 2005.
- [5] J. J. Yuan, R. Y. Fu, C. T. Ng, and T. C. E. Cheng, "A best online algorithm for unbounded parallel-batch scheduling with restarts to minimize makespan," *Journal of Scheduling*, vol. 14, no. 4, pp. 361–369, 2011.
- [6] R. Fu, J. Tian, J. Yuan, and C. He, "On-line scheduling on a batch machine to minimize makespan with limited restarts," *Operations Research Letters*, vol. 36, no. 2, pp. 255–258, 2008.
- [7] R. Uzsoy, C. Y. Lee, and L. A. Martin-Vega, "A review of production planning and scheduling models in the semiconductor industry, part I: system characteristics, performance evaluation and production planning," *IIE Transaction on Scheduling and Logistics*, vol. 26, no. 5, pp. 47–61, 1994.
- [8] R. Uzsoy, C.-Y. Lee, and L. A. Martin-Vega, "Review of production planning and scheduling models in the semiconductor industry part II: shop-floor control," *IIE Transaction on Scheduling and Logistics*, vol. 26, no. 5, pp. 44–55, 1994.
- [9] X. Deng, C. K. Poon, and Y. Zhang, "Approximation algorithms in batch processing," *Journal of Combinatorial Optimization*, vol. 7, no. 3, pp. 247–257, 2003.
- [10] G. Zhang, X. Cai, and C. K. Wong, "On-line algorithms for minimizing makespan on batch processing machines," *Naval Research Logistics*, vol. 48, no. 3, pp. 241–258, 2001.
- [11] C. K. Poon and W. Yu, "On-line scheduling algorithms for a batch machine with finite capacity," *Journal of Combinatorial Optimization*, vol. 9, no. 2, pp. 167–186, 2005.
- [12] P. Liu, X. Lu, and Y. Fang, "A best possible deterministic on-line algorithm for minimizing makespan on parallel batch machines," *Journal of Scheduling*, vol. 15, no. 1, pp. 77–81, 2012.
- [13] J. Tian, T. C. Cheng, C. T. Ng, and J. Yuan, "Online scheduling on unbounded parallel-batch machines to minimize the makespan," *Information Processing Letters*, vol. 109, no. 21–22, pp. 1211–1215, 2009.
- [14] J. Tian, R. Y. Fu, and J. J. Yuan, "On-line scheduling with delivery time on a single batch machine," *Theoretical Computer Science*, vol. 374, no. 1–3, pp. 49–57, 2007.
- [15] J. J. Yuan, S. Li, J. Tian, and R. Y. Fu, "A best on-line algorithm for the single machine parallel-batch scheduling with restricted delivery times," *Journal of Combinatorial Optimization*, vol. 17, no. 2, pp. 206–213, 2009.
- [16] Y. Fang, P. H. Liu, and X. W. Lu, "Optimal on-line algorithms for one batch machine with grouped processing times," *Journal of Combinatorial Optimization*, vol. 22, no. 4, pp. 509–516, 2011.
- [17] J. Tian, T. C. Cheng, C. T. Ng, and J. Yuan, "An improved on-line algorithm for single parallel-batch machine scheduling with delivery times," *Discrete Applied Mathematics*, vol. 160, no. 7–8, pp. 1191–1210, 2012.
- [18] P. Liu and X. Lu, "Online unbounded batch scheduling on parallel machines with delivery times," *Journal of Combinatorial Optimization*, 2014.
- [19] R. Y. Fu, T. Ji, J. J. Yuan, and Y. X. Lin, "Online scheduling in a parallel batch processing system to minimize makespan using

- restarts,” *Theoretical Computer Science*, vol. 374, no. 1–3, pp. 196–202, 2007.
- [20] R. Fu, T. C. Cheng, C. T. Ng, and J. Yuan, “Online scheduling on two parallel-batching machines with limited restarts to minimize the makespan,” *Information Processing Letters*, vol. 110, no. 11, pp. 444–450, 2010.
- [21] H. Liu and J. Yuan, “Online scheduling of equal length jobs on a bounded parallel batch machine with restart or limited restart,” *Theoretical Computer Science*, vol. 543, pp. 24–36, 2014.