Research Article

Phase Synchronization Is the Amplified Result by the Hilbert Transform

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We investigate the interplay between phase synchronization and amplitude synchronization in nonlinear dynamical systems. It is numerically found that phase synchronization intends to be established earlier than amplitude synchronization. Nevertheless, amplitude synchronization (or the state with large correlation between the amplitudes) is crucial for the maintenance of a high correlation between two time series. A breakdown of high correlation in amplitudes will lead to a desynchronization of two time series. It is shown that these unique features are caused essentially by the Hilbert transform. This leads to a deep concern and criticism on the current usage of phase synchronization.

1. Introduction

Synchronization of chaotic systems has been an important area in nonlinear dynamics [1]. “Synchronization” is defined as a complete coincidence of two variables (or time series) that are belonging to different systems [2] while the appearance of some functional relations between two variables is termed as “generalized synchronization” [3, 4]. Instead of focusing on the synchronization of time series, Rosenblum et al. introduced the concept of *phase synchronization* to describe how the coupled chaotic oscillators could present a nearly perfect locking of phase, whereas the amplitude remained chaotic [5]. In [5], the phase of a time series was defined based on the Hilbert transform [6]. This definition is also very attractive in characterization of chaos [7]. In a more explicit form, the Hilbert transform of a time series $x(t)$ follows:

$$\hat{x}(t) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau, \quad (1)$$

where P.V. means the Cauchy principal value for the integral. Thus, a new complex quantity $\psi(t)$ can be introduced; that is,

$$\psi(t) = x(t) + i\hat{x}(t) = A_H(t) e^{i\theta_H(t)}, \quad (2)$$

where $A_H(t)$ is the phase and $\theta_H(t)$ is the amplitude [5, 7] and they form a conjugate pair. One can also define the phase angle to be the projection of phase point on the x - y plane with the phase angle $\theta_T(t) = \tan^{-1}(y/x)$ and the amplitude $A_T(t) = \sqrt{x^2 + y^2}$. Alternatively by using the Poincare section, one can also choose a phase:

$$\theta_P(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2n\pi, \quad (3)$$

where $t_n < t < t_{n+1}$ and t_n denotes the n th crossing, but there is no conjugate amplitude for $\theta_P(t)$. In this comment, we reexamine the virtue of using the phase variable as the indicator of synchronization of the time series. We believe that phase synchronization may not be a good tracer to the actual synchronization of the time series. Then our work will

explain the onset mechanism of phase synchronization and it leads to a deep concern on the current usage of phase synchronization.

2. Give an Example of Coupled Rossler Model

Let us recalculate the same coupled Rossler model as in [5] to explore the synchronization. The model follows:

$$\begin{aligned}\dot{x}_{1,2} &= -\omega_{1,2}y_{1,2} - z_{1,2} + C(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2}x_{1,2} + 0.15y_{1,2}, \\ \dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10),\end{aligned}\quad (4)$$

where C is the strength of coupling, $\omega_{1,2} = 1 \pm \Delta\omega$, and $\Delta\omega = 0.015$ which indicates that there is a frequency mismatch between two oscillators. Because of finite frequency mismatch, there is no exact synchronization in time series. Synchronization can be found for C not only in the range $[0, 1]$, as reported in [5], region I, but also in $[2.9, 3.45]$, region II. Thus, in such a case, “synchronization” only implies a high correlation between two variables, such as time series. By using the Hilbert transform, the variables x_i have the phase θ_i and the amplitude A_i ($i = 1, 2$). It has been reported that θ_1 and θ_2 move together and get nearly synchronized, while A_1 and A_2 remain irregular and unrelated in a range of small C [5]. Since then, cited and extended works have been expanded dramatically [8–19]. However, it should be emphasized that as *phase* is introduced, the influence of its counterpart, that is, *amplitude*, and their interplay should not be ignored. Unfortunately, although the concept of phase synchronization has been extensively addressed, the correlation between phase synchronization and amplitude synchronization remains to be clarified.

3. Results and Discussion

Let us use a common measure, the mean square error, to quantify the degree of synchronization. For two time series, x_1 and x_2 , the mean square error is defined as

$$\sigma_x = \frac{1}{T} \int_0^T [x_1(t) - x_2(t)]^2 dt, \quad (5)$$

where the integration time T should be sufficiently long. For the phase and the amplitude deduced by the Hilbert transform, we denote their mean square errors to be H_θ and H_A . For phase angle defined by projection (here x_i and y_i of the Rossler model), the corresponding mean square errors are T_θ and T_A , while in the case of the Poincare section (here a typical section in the x_i - y_i plane) we use P_θ to denote it. Obviously, to be good indicators of the synchronization, variations of these quantities should faithfully reflect the true status of the coupled chaotic oscillators. As shown in Figure 1, where the values of the mean square errors in region I are plotted, the mean square errors of the phase variables are insensitive to the true state of synchronization in time series. In contrast, the mean square errors of the conjugate amplitudes reflect more faithfully to the status of oscillators

FIGURE 1: The mean square error (mse) diagram of the coupled Rossler model. σ_x is the mse of the time series. H_θ , T_θ and P_θ are the mse for the phase variables while H_A and T_A are for the amplitudes.

in this case. Similar feature can also be found in region II as well as for different coupled oscillators.

4. Conclusion

The result shows that phase synchronization will intend to be established earlier than amplitude synchronization under the Hilbert transform. This unique feature is novel, but it is caused by the Hilbert transform. The transformation on the phase part is nonlinear. This nonlinear transform is the generic mechanism for the novel features that have been reported on the phase synchronization [5, 6, 8–19]. Therefore, we believe the role of “phase” and phase synchronization may not be a good indicator to the true synchronization in the time series. The relevance of phase variables to the synchronization of the original variables in nonlinear dynamical systems seems to be a mathematical consequence of the transform one used. Thus, the works on phase synchronization [6, 13–19] may be worthwhile to be reconsidered.

Conflict of Interests

There is no conflict of interests related to this paper.

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