Research Article

Exact Static Analysis of In-Plane Curved Timoshenko Beams with Strong Nonlinear Boundary Conditions

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Analytical solutions have been developed for nonlinear boundary problems. In this paper, the shifting function method is applied to develop the static deflection of in-plane curved Timoshenko beams with nonlinear boundary conditions. Three coupled governing differential equations are derived via the Hamilton's principle. The mathematical modeling of the curved beam system can be decomposed into a complete sixth-order ordinary differential characteristic equation and the associated boundary conditions. It is shown that the proposed method is valid and performs well for problems with strong nonlinearity.

1. Introduction

Curved beam structures are widely used in engineering fields, such as mechanical, civil, and aerospace engineering. Reviews of research on such structures have been conducted by Henrych [1], Markus and Nanasi [2], Chidamparam and Leissa [3], and Auciello and De Rosa [4]. The in-plane and out-of-plane problems of plane curved beams have been studied. In most general problems, they are coupled. However, if the cross section of the curved beam is doubly symmetric and the plane is a principal plane of the cross section, then the in-plane and out-of-plane problems are uncoupled.

Many investigators have studied linear problems about the static and free vibration behaviors of curved beams. Washizu [5] obtained the equations for the coupled motions of a curved and pretwisted bar. Rao [6] used Hamilton's principle to derive the governing equations of motion in consideration of the effects of rotatory inertia and shearing deformation. Fettahtlioglu and Mayers [7] studied the static deflection of a ring. Bickford and Maganty [8] used a formulation similar to that of Rao and developed equations of motion for out-of-plane vibrations of symmetrical cross-section thick rings, accounting for curvature variation through the thickness. Their frequency predictions were validated with the experimental data of Kuhl [9]. Based on the moment-displacement relationships presented by Rao, Silva

and Urgueira [10] derived the dynamic stiffness matrices for the out-of-plane vibration of curved beams using dynamic equilibrium equations. Lee and Chao [11] developed the exact out-of-plane vibration solutions of curved nonuniform beams. In the book by Cook and Young [12], the exact static analysis of extensional circular curved Timoshenko beams with some special conditions was revealed. Based on the generalized Green function, Lin [13] developed the exact solution of extensible curved Timoshenko beams. Lee and Wu [14] studied the exact in-plane vibration solutions of extensible curved nonuniform Timoshenko beams.

For the beams with time dependent boundary conditions, Lee and Lin [15] generalized the solution method of Mindlin and Goodman [16] and developed the shifting function method to study the problems with general time dependent elastic boundary conditions. Recently, Lee et al. [17, 18] extend the shifting function method to study the exact large deflection of a Bernoulli-Euler beam and Timoshenko beam with nonlinear boundary conditions. From the existing literatures, it shows that exact solutions for curved beam problems with nonlinear boundary conditions are not available.

In the previous studies [17, 18], the governing differential equations are fourth-order differential equations. In the present study, one extends the previous studies and the shifting function method [15] to study the exact large

FIGURE 1: Geometry and coordinate system of uniform curved beam with nonlinear boundary conditions, subjected to loads $P_v(s)$, $P_u(s)$, and $M_z(s)$ in r , θ , and z directions, respectively.

static deflection of in-plane curved Timoshenko beams with nonlinear boundaries. The beam with doubly symmetric cross section is considered. The three coupled governing differential equations for the in-plane curved uniform beams of constant radius are derived via Hamilton's principle. These three coupled governing differential equations are decoupled and reduced into a sixth-order differential equation with nonlinear boundary conditions. Consequently, the shifting function method is extended and applied to develop the exact solution of the system. It can be shown that the proposed method is valid for problems with strong nonlinearity.

2. Mathematical Modeling of the Curved Beam System

Consider the static response of an extensional curved uniform Timoshenko beam resting on an elastic foundation with nonlinear boundary conditions, subjected to loads $P_v(s)$, $P_u(s)$, and $M_z(s)$, as shown in Figure 1. If the thickness of the curved beam is small in comparison with the radius of the curved beam without considering the warping effects, the displacement fields of the curved beam in cylindrical coordinates are

$$\begin{aligned} u_r(r, s, z) &= v(s), \\ u_\theta(r, s, z) &= u(s) - \bar{r}\psi_z(s), \\ u_z(r, s, z) &= 0, \end{aligned} \quad (1)$$

where u_r , u_θ , and u_z denote the displacement of the curved beam in the r , θ , and z directions, respectively, and s is the arc length along the neutral axis. R is the constant radius of the curved beam and $s = R\theta$, v and u are the neutral axis displacements of the curved beam in the r and θ directions, respectively, and ψ_z is the angle of rotation due to bending in the z direction. $\bar{r}$ is measured inward from the neutral axis in the r direction.

Substituting (1) into the strain-displacement relations in cylindrical coordinates, only two nonzero strains, namely, $\epsilon_{\theta\theta}$ and $\gamma_{r\theta}$, are obtained:

$$\begin{aligned} \epsilon_{\theta\theta} &= \frac{R}{R - \bar{r}} \left[\left(\frac{\partial u}{\partial s} - \frac{v}{R} \right) - \bar{r} \frac{\partial \psi_z}{\partial s} \right], \\ \gamma_{r\theta} &= -\frac{R}{R - \bar{r}} \left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right). \end{aligned} \quad (2)$$

When $\bar{r}$ is small in comparison with R , the two strains reduce to

$$\begin{aligned} \epsilon_{\theta\theta} &= \left(\frac{\partial u}{\partial s} - \frac{v}{R} \right) - \bar{r} \frac{\partial \psi_z}{\partial s}, \\ \gamma_{r\theta} &= -\left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right). \end{aligned} \quad (3)$$

The strain energy of the curved beam is

$$\begin{aligned} V^* &= \frac{1}{2} \int_0^L \int_A E \left[\left(\frac{\partial u}{\partial s} - \frac{v}{R} \right)^2 - 2\bar{r} \frac{\partial \psi_z}{\partial s} \left(\frac{\partial u}{\partial s} - \frac{v}{R} \right) \right. \\ &\quad \left. + r^2 \left(\frac{\partial \psi_z}{\partial s} \right)^2 \right] dA ds + \frac{1}{2} \int_0^L \int_A k_z G \left(\frac{\partial v}{\partial s} + \frac{u}{R} \right. \\ &\quad \left. - \psi_z \right)^2 dA ds, \end{aligned} \quad (4)$$

where L is the length of the neutral axis. E , G , and A denote Young's modulus, shear modulus, and cross-section area of the curved beam, respectively, and k_z denotes the shear correction factors of the curved beam section about the z -axes. Since the cross section of the curved beam considered is doubly symmetric, the integral of the second term in the square brackets vanishes. Equation (4) becomes

$$\begin{aligned} V^* &= \frac{1}{2} \int_0^L \left[EA \left(\frac{\partial u}{\partial s} - \frac{v}{R} \right)^2 + EI_z \left(\frac{\partial \psi_z}{\partial s} \right)^2 \right] ds \\ &\quad + \frac{1}{2} \int_0^L k_z GA \left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right)^2 ds, \end{aligned} \quad (5)$$

where I_z denotes the second area moments of inertia of the curved beam section about the z -axes.

Considering the linear and nonlinear spring and moment on the boundary of the curved beam end and the force and moment loads on the curved beam, the potential energy and work are, respectively,

$$\begin{aligned}
 U^* = & \int [K_{VL}v(0) + K_{NVL}v^3(0)] dv(0) \\
 & + \int [K_{VR}v(L) + K_{NVR}v^3(L)] dv(L) \\
 & + \int [K_{UL}u(0) + K_{NUL}u^3(0)] du(0) \\
 & + \int [K_{UR}u(L) + K_{NUR}u^3(L)] du(L) \\
 & + \int [K_{\theta L}\psi_z(0) + K_{N\theta L}\psi_z^3(0)] d\psi_z(0) \\
 & + \int [K_{\theta R}\psi_z(L) + K_{N\theta R}\psi_z^3(L)] d\psi_z(L), \\
 W^* = & \int_0^L [P_v(s)v(s) + P_u(s)u(s) + M_z(s)\psi_z(s)] ds \\
 & + f_1v(0) + f_2u(0) + f_3\psi_z(0) + f_4v(L) \\
 & + f_5u(L) + f_6\psi_z(L),
 \end{aligned} \quad (6)$$

where K_{VL} , K_{UL} , $K_{\theta L}$, K_{VR} , K_{UR} , and $K_{\theta R}$ are the linear spring and rotational stiffness constants in the r , θ , and z directions at the left and right ends of the curved beam, respectively. K_{NVL} , K_{NUL} , $K_{N\theta L}$, K_{NVR} , K_{NUR} , and $K_{N\theta R}$ are the nonlinear spring and rotational stiffness constants in the r , θ , and z directions at the left and right ends of the curved beam, respectively. $P_v(s)$, $P_u(s)$, and $M_z(s)$ are the force and moment loads in the r , θ , and z directions, respectively, and f_1 , f_2 , f_3 , f_4 , f_5 , and f_6 are the force and moment loads in the r , θ , and z directions at the left and right ends of the curved beam, respectively.

The general form of Hamilton's principle is

$$\delta \int_{t_1}^{t_2} (-V^* - U^* + W^*) dt = 0. \quad (8)$$

Based on Hamilton's principle, the governing differential equations and the associated boundary conditions for the curved uniform Timoshenko beam can be derived. The governing differential equations for the in-plane are three coupled differential equations:

$$\begin{aligned}
 -k_z GA \left(\frac{\partial^2 v}{\partial s^2} + \frac{1}{R} \frac{\partial u}{\partial s} - \frac{\partial \psi_z}{\partial s} \right) - \frac{EA}{R} \left(\frac{\partial u}{\partial s} - \frac{v}{R} \right) &= P_v, \\
 -EA \left(\frac{\partial^2 u}{\partial s^2} - \frac{1}{R} \frac{\partial v}{\partial s} \right) + \frac{k_z GA}{R} \left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right) &= P_u, \\
 -EI_z \left(\frac{\partial^2 \psi_z}{\partial s^2} \right) - k_z GA \left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right) &= M_z.
 \end{aligned} \quad (9)$$

The associated boundary conditions are as follows:

(i) At $s = 0$

$$\begin{aligned}
 -K_{VL}v - K_{NVL}v^3 + k_z GA \left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right) &= -f_1, \\
 -K_{UL}u - K_{NUL}u^3 + EA \left(\frac{\partial u}{\partial s} - \frac{v}{R} \right) &= -f_2, \\
 -K_{\theta L}\psi_z - K_{N\theta L}\psi_z^3 + EI_z \left(\frac{\partial \psi_z}{\partial s} \right) &= -f_3.
 \end{aligned} \quad (10)$$

(ii) At $s = L$

$$\begin{aligned}
 K_{VR}v + K_{NVR}v^3 + k_z GA \left(\frac{\partial v}{\partial s} + \frac{u}{R} - \psi_z \right) &= f_4, \\
 K_{UR}u + K_{NUR}u^3 + EA \left(\frac{\partial u}{\partial s} - \frac{v}{R} \right) &= f_5, \\
 K_{\theta R}\psi_z + K_{N\theta R}\psi_z^3 + EI_z \left(\frac{\partial \psi_z}{\partial s} \right) &= f_6.
 \end{aligned} \quad (11)$$

In terms of the following nondimensional quantities,

$$\begin{aligned}
 V(\xi) &= \frac{v}{L}, \\
 U(\xi) &= \frac{u}{L}, \\
 \Psi(\xi) &= \psi_z, \\
 \bar{P}_v(\xi) &= \frac{P_v L^3}{EI_z}, \\
 \bar{P}_u(\xi) &= \frac{P_u L^3}{EI_z}, \\
 \bar{M}_z(\xi) &= \frac{M_z L^2}{EI_z}, \\
 f_1^* &= \frac{f_1 L^2}{EI_z}, \\
 f_2^* &= \frac{f_2 L^2}{EI_z}, \\
 f_3^* &= \frac{f_3 L}{EI_z}, \\
 f_4^* &= \frac{f_4 L^2}{EI_z}, \\
 f_5^* &= \frac{f_5 L^2}{EI_z}, \\
 f_6^* &= \frac{f_6 L}{EI_z}, \\
 \beta_{VL} &= \frac{K_{VL} L^3}{EI_z},
 \end{aligned}$$

$$\beta_{NVL} = \frac{K_{NVL}L^5}{EI_z},$$

$$\beta_{UL} = \frac{K_{UL}L^3}{EI_z},$$

$$\beta_{NUL} = \frac{K_{NUL}L^5}{EI_z},$$

$$\beta_{\theta L} = \frac{K_{\theta L}L}{EI_z},$$

$$\beta_{N\theta L} = \frac{K_{N\theta L}L}{EI_z},$$

$$\beta_{VR} = \frac{K_{VR}L^3}{EI_z},$$

$$\beta_{NVR} = \frac{K_{NVR}L^5}{EI_z},$$

$$\beta_{UR} = \frac{K_{UR}L^3}{EI_z},$$

$$\beta_{NUR} = \frac{K_{NUR}L^5}{EI_z},$$

$$\beta_{\theta R} = \frac{K_{\theta R}L}{EI_z},$$

$$\beta_{N\theta R} = \frac{K_{N\theta R}L}{EI_z},$$

$$\theta_0 = \frac{L}{R},$$

$$\mu = \frac{EI_z}{k_z GAL^2},$$

$$\varsigma = \frac{AL^2}{I_z},$$

$$\xi = \frac{s}{L},$$

(12)

the nondimensional coupled governing characteristic differential equations are

$$-\frac{1}{\mu} \left(\frac{\partial^2 V}{\partial \xi^2} + \theta_0 \frac{\partial U}{\partial \xi} - \frac{\partial \Psi}{\partial \xi} \right) - \varsigma \theta_0 \left(\frac{\partial U}{\partial \xi} - \theta_0 V \right) = \bar{P}_v, \quad (13)$$

$$-\varsigma \left(\frac{\partial^2 U}{\partial \xi^2} - \theta_0 \frac{\partial V}{\partial \xi} \right) + \frac{\theta_0}{\mu} \left(\frac{\partial V}{\partial \xi} + \theta_0 U - \Psi \right) = \bar{P}_u, \quad (14)$$

$$-\frac{\partial^2 \Psi}{\partial \xi^2} - \frac{1}{\mu} \left(\frac{\partial V}{\partial \xi} + \theta_0 U - \Psi \right) = \bar{M}_z. \quad (15)$$

The associated boundary conditions are as follows:

(i) At $\xi = 0$

$$-\beta_{VL}V - \beta_{NVL}V^3 + \frac{1}{\mu} \left(\frac{\partial V}{\partial \xi} + \theta_0 U - \Psi \right) = -f_1^*, \quad (16)$$

$$-\beta_{UL}U - \beta_{NUL}U^3 + \varsigma \left(\frac{\partial U}{\partial \xi} - \theta_0 V \right) = -f_2^*, \quad (17)$$

$$-\beta_{\theta L}\Psi - \beta_{N\theta L}\Psi^3 + \frac{\partial \Psi}{\partial \xi} = -f_3^*. \quad (18)$$

(ii) At $\xi = 1$

$$\beta_{VR}V + \beta_{NVR}V^3 + \frac{1}{\mu} \left(\frac{\partial V}{\partial \xi} + \theta_0 U - \Psi \right) = f_4^*, \quad (19)$$

$$\beta_{UR}U + \beta_{NUR}U^3 + \varsigma \left(\frac{\partial U}{\partial \xi} - \theta_0 V \right) = f_5^*, \quad (20)$$

$$\beta_{\theta R}\Psi + \beta_{N\theta R}\Psi^3 + \frac{\partial \Psi}{\partial \xi} = f_6^*. \quad (21)$$

The coupled differential equations (13)–(15) can be decoupled to get two equations, Ψ and V :

$$\begin{aligned} \Psi = & \frac{1}{\theta_0 \delta_1} \left\{ \delta_2 U^{(4)} + \delta_3 U^{(2)} + \delta_1 \theta_0^2 U \right. \\ & + \left[\frac{1}{\varsigma} - \mu \theta_0^2 \left(\frac{1}{\varsigma} + \mu \right) \right] \bar{P}_u \\ & \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} + \theta_0 \bar{M}_z \right) \right\}, \end{aligned} \quad (22)$$

$$\begin{aligned} V = & -\frac{1}{\varsigma \theta_0^3 \delta_1} U^{(5)} - \frac{1}{\varsigma \theta_0 \delta_1} U^{(3)} + \frac{1}{\theta_0} U^{(1)} + \frac{1}{\varsigma \theta_0^2} \bar{P}_v \\ & - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} \right. \\ & \left. + \theta_0 \bar{M}_z^{(1)} \right), \end{aligned} \quad (23)$$

where the notation $^{(n)}$ denotes the n th-order differentiation with respect to ξ . Consider

$$\delta_1 = 1 + \theta_0^2 \left(\frac{1}{\varsigma} + \mu \right),$$

$$\delta_2 = \frac{1}{\varsigma} + \mu, \quad (24)$$

$$\delta_3 = 1 + 2\theta_0^2 \left(\frac{1}{\varsigma} + \mu \right).$$

Substituting (22)–(24) into (15) and (16)–(21) yields the complete sixth-order ordinary differential characteristic

equation and the associated boundary conditions in the θ direction, respectively. Consider

$$\begin{aligned} & U^{(6)} + 2\theta_0^2 U^{(4)} + \theta_0^4 U^{(2)} \\ &= \theta_0 \bar{P}_v^{(1)} - \delta_2 \theta_0 \bar{P}_v^{(3)} - \theta_0^2 \bar{P}_u + \mu \theta_0^2 \bar{P}_u^{(2)} \\ &\quad - \frac{1}{\varsigma} \bar{P}_u^{(4)} - \theta_0^3 \bar{M}_z - \theta_0 \bar{M}_z^{(2)}. \end{aligned} \quad (25)$$

(i) At $\xi = 0$

$$\begin{aligned} & \frac{\beta_{VL}}{\varsigma \theta_0^3 \delta_1} U^{(5)} - \frac{1}{\theta_0 \delta_1} U^{(4)} + \frac{\beta_{VL}}{\varsigma \theta_0 \delta_1} U^{(3)} - \frac{\theta_0}{\delta_1} U^{(2)} - \frac{\beta_{VL}}{\theta_0} \\ & \cdot U^{(1)} - \beta_{NVL} V^3 = \beta_{VL} \left[\frac{1}{\varsigma \theta_0^2} \bar{P}_v - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} \right. \\ & \quad \left. + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] \\ & \quad + \frac{1}{\delta_1} \left(\delta_2 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} - \delta_2 \theta_0 \bar{P}_u - \delta_2 \theta_0^2 \bar{M}_z \right) \\ & \quad + \bar{M}_z - f_1^*, \\ & \frac{1}{\delta_1 \theta_0^2} U^{(5)} + \frac{1}{\delta_1} U^{(3)} - \beta_{UL} U - \beta_{NUL} U^3 \\ &= -\frac{1}{\delta_1 \theta_0} \left(\delta_2 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} - \delta_2 \theta_0 \bar{P}_u^{(1)} + \bar{M}_z^{(1)} \right) \\ & \quad + \frac{1}{\theta_0} \bar{P}_v - f_2^*, \end{aligned} \quad (26)$$

$$\begin{aligned} & \frac{\delta_2}{\theta_0 \delta_1} U^{(5)} - \frac{\delta_2 \beta_{\theta L}}{\theta_0 \delta_1} U^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} U^{(3)} - \frac{\delta_3 \beta_{\theta L}}{\theta_0 \delta_1} U^{(2)} \\ & \quad + \theta_0 U^{(1)} - \theta_0 \beta_{\theta L} U - \beta_{N\theta L} \Psi^3 \\ &= \frac{\beta_{\theta L}}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u \right. \\ & \quad \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} + \theta_0 \bar{M}_z \right) \right] \\ & \quad - \frac{1}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u^{(1)} \right. \\ & \quad \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] - f_3^*. \end{aligned}$$

(ii) At $\xi = 1$

$$\begin{aligned} & -\frac{\beta_{VR}}{\varsigma \theta_0^3 \delta_1} U^{(5)} - \frac{1}{\delta_1 \theta_0} U^{(4)} - \frac{\beta_{VR}}{\varsigma \theta_0 \delta_1} U^{(3)} - \frac{\theta_0}{\delta_1} U^{(2)} + \frac{\beta_{VR}}{\theta_0} \\ & \cdot U^{(1)} + \beta_{NVR} V^3 = -\beta_{VR} \left[\frac{1}{\varsigma \theta_0^2} \bar{P}_v - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} \right. \\ & \quad \left. + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] \end{aligned}$$

$$\begin{aligned} & + \frac{1}{\delta_1} \left(\delta_2 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} - \delta_2 \theta_0 \bar{P}_u - \delta_2 \theta_0^2 \bar{M}_z \right) \\ & \quad + \bar{M}_z + f_4^*, \\ & \frac{1}{\delta_1 \theta_0^2} U^{(5)} + \frac{1}{\delta_1} U^{(3)} + \beta_{UR} U + \beta_{NUR} U^3 \\ &= -\frac{1}{\delta_1 \theta_0} \left(\delta_2 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} - \delta_2 \theta_0 \bar{P}_u^{(1)} + \bar{M}_z^{(1)} \right) \\ & \quad + \frac{1}{\theta_0} \bar{P}_v + f_5^*, \\ & \frac{\delta_2}{\theta_0 \delta_1} U^{(5)} + \frac{\delta_2 \beta_{\theta R}}{\theta_0 \delta_1} U^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} U^{(3)} + \frac{\delta_3 \beta_{\theta R}}{\theta_0 \delta_1} U^{(2)} \\ & \quad + \theta_0 U^{(1)} + \theta_0 \beta_{\theta R} U + \beta_{N\theta R} \Psi^3 \\ &= -\frac{\beta_{\theta R}}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u \right. \\ & \quad \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} + \theta_0 \bar{M}_z \right) \right] \\ & \quad - \frac{1}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u^{(1)} \right. \\ & \quad \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] + f_6^*. \end{aligned} \quad (27)$$

3. The Shifting Function Method

In this paper, the solution for the sixth-order differential equation (25) with nonlinear boundary conditions (26)-(27) is derived. The shifting function method developed by Lee and Lin [15], given below, is used:

$$U(\xi) = \bar{a}(\xi) + \sum_{i=1}^6 \bar{f}_i \bar{g}_i(\xi), \quad (28)$$

where

$$\begin{aligned} \bar{f}_1 &= \beta_{NVL} V^3(0), \\ \bar{f}_2 &= \beta_{NUL} U^3(0), \\ \bar{f}_3 &= \beta_{N\theta L} \Psi^3(0), \\ \bar{f}_4 &= -\beta_{NVR} V^3(1), \\ \bar{f}_5 &= -\beta_{NUR} U^3(1), \\ \bar{f}_6 &= -\beta_{N\theta R} \Psi^3(1), \end{aligned} \quad (29)$$

and $\bar{g}_i(\xi)$, $i = 1, 2, 3, 4, 5, 6$, are the shifting functions to be specified, and $\bar{a}(\xi)$ is the transformed function. Substituting

(28)–(29) into (25)–(27) yields the differential equation for $\bar{a}(\xi)$ and the associated boundary conditions:

$$\begin{aligned} & \bar{a}^{(6)} + 2\theta_0^2 \bar{a}^{(4)} + \theta_0^4 \bar{a}^{(2)} \\ &= - \sum_{i=1}^6 \bar{f}_i \left(\bar{g}_i^{(6)} + 2\theta_0^2 \bar{g}_i^{(4)} + \theta_0^4 \bar{g}_i^{(2)} \right) + \theta_0 \bar{P}_v^{(1)} \\ & \quad - \delta_2 \theta_0 \bar{P}_v^{(3)} - \theta_0^2 \bar{P}_u + \mu \theta_0^2 \bar{P}_u^{(2)} - \frac{1}{\varsigma} \bar{P}_u^{(4)} \\ & \quad - \theta_0^3 \bar{M}_z - \theta_0 \bar{M}_z^{(2)}. \end{aligned} \quad (30)$$

(i) At $\xi = 0$

$$\begin{aligned} & \frac{\beta_{VL}}{\varsigma \theta_0^3 \delta_1} \bar{a}^{(5)} - \frac{1}{\theta_0 \delta_1} \bar{a}^{(4)} + \frac{\beta_{VL}}{\varsigma \theta_0 \delta_1} \bar{a}^{(3)} - \frac{\theta_0}{\delta_1} \bar{a}^{(2)} - \frac{\beta_{VL}}{\theta_0} \bar{a}^{(1)} \\ &= \bar{f}_1 - \sum_{i=1}^6 \bar{f}_i \left(\frac{\beta_{VL}}{\varsigma \theta_0^3 \delta_1} \bar{g}_i^{(5)} - \frac{1}{\theta_0 \delta_1} \bar{g}_i^{(4)} + \frac{\beta_{VL}}{\varsigma \theta_0 \delta_1} \bar{g}_i^{(3)} \right. \\ & \quad \left. - \frac{\theta_0}{\delta_1} \bar{g}_i^{(2)} - \frac{\beta_{VL}}{\theta_0} \bar{g}_i^{(1)} \right) + \beta_{VL} \left[\frac{1}{\varsigma \theta_0^2} \bar{P}_v - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} \right. \\ & \quad \left. + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] \\ & \quad + \frac{1}{\delta_1} \left(\delta_2 \bar{P}_v^{(1)} + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(2)} - \delta_2 \theta_0 \bar{P}_u - \delta_2 \theta_0^2 \bar{M}_z \right) \\ & \quad + \bar{M}_z - f_1^*, \\ & \frac{1}{\delta_1 \theta_0^2} \bar{a}^{(5)} + \frac{1}{\delta_1} \bar{a}^{(3)} - \beta_{UL} \bar{a} = \bar{f}_2 - \sum_{i=1}^6 \bar{f}_i \left(\frac{1}{\delta_1 \theta_0^2} \bar{g}_i^{(5)} \right. \\ & \quad \left. + \frac{1}{\delta_1} \bar{g}_i^{(3)} - \beta_{UL} \bar{g}_i \right) - \frac{1}{\delta_1 \theta_0} \left(\delta_2 \bar{P}_v^{(2)} + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(3)} \right. \\ & \quad \left. - \delta_2 \theta_0 \bar{P}_u^{(1)} + \bar{M}_z^{(1)} \right) + \frac{1}{\theta_0} \bar{P}_v - f_2^*, \\ & \frac{\delta_2}{\theta_0 \delta_1} \bar{a}^{(5)} - \frac{\delta_2 \beta_{\theta L}}{\theta_0 \delta_1} \bar{a}^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{a}^{(3)} - \frac{\delta_3 \beta_{\theta L}}{\theta_0 \delta_1} \bar{a}^{(2)} + \theta_0 \bar{a}^{(1)} \\ &= - \theta_0 \beta_{\theta L} \bar{a} = \bar{f}_3 - \sum_{i=1}^6 \bar{f}_i \left(\frac{\delta_2}{\theta_0 \delta_1} \bar{g}_i^{(5)} - \frac{\delta_2 \beta_{\theta L}}{\theta_0 \delta_1} \bar{g}_i^{(4)} \right. \\ & \quad \left. + \frac{\delta_3}{\theta_0 \delta_1} \bar{g}_i^{(3)} - \frac{\delta_3 \beta_{\theta L}}{\theta_0 \delta_1} \bar{g}_i^{(2)} + \theta_0 \bar{g}_i^{(1)} - \theta_0 \beta_{\theta L} \bar{g}_i \right) \\ & \quad + \frac{\beta_{\theta L}}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} \right. \right. \\ & \quad \left. \left. + \theta_0 \bar{M}_z \right) \right] - \frac{1}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u^{(1)} \right. \\ & \quad \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] - f_3^*. \end{aligned} \quad (31)$$

(ii) At $\xi = 1$

$$\begin{aligned} & - \frac{\beta_{VR}}{\varsigma \theta_0^3 \delta_1} \bar{a}^{(5)} - \frac{1}{\delta_1 \theta_0} \bar{a}^{(4)} - \frac{\beta_{VR}}{\varsigma \theta_0 \delta_1} \bar{a}^{(3)} - \frac{\theta_0}{\delta_1} \bar{a}^{(2)} + \frac{\beta_{VR}}{\theta_0} \bar{a}^{(1)} \\ &= \bar{f}_4 - \sum_{i=1}^6 \bar{f}_i \left(- \frac{\beta_{VR}}{\varsigma \theta_0^3 \delta_1} \bar{g}_i^{(5)} - \frac{1}{\delta_1 \theta_0} \bar{g}_i^{(4)} \right. \\ & \quad \left. - \frac{\beta_{VR}}{\varsigma \theta_0 \delta_1} \bar{g}_i^{(3)} - \frac{\theta_0}{\delta_1} \bar{g}_i^{(2)} + \frac{\beta_{VR}}{\theta_0} \bar{g}_i^{(1)} \right) - \beta_{VR} \left[\frac{1}{\varsigma \theta_0^2} \bar{P}_v \right. \\ & \quad \left. - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} \right. \right. \\ & \quad \left. \left. + \theta_0 \bar{M}_z^{(1)} \right) \right] + \frac{1}{\delta_1} \left(\delta_2 \bar{P}_v^{(1)} + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(2)} - \delta_2 \theta_0 \bar{P}_u \right. \\ & \quad \left. - \delta_2 \theta_0^2 \bar{M}_z \right) + \bar{M}_z + f_4^*, \\ & \frac{1}{\delta_1 \theta_0^2} \bar{a}^{(5)} + \frac{1}{\delta_1} \bar{a}^{(3)} + \beta_{UR} \bar{a} = \bar{f}_5 - \sum_{i=1}^6 \bar{f}_i \left(\frac{1}{\delta_1 \theta_0^2} \bar{g}_i^{(5)} \right. \\ & \quad \left. + \frac{1}{\delta_1} \bar{g}_i^{(3)} + \beta_{UR} \bar{g}_i \right) - \frac{1}{\delta_1 \theta_0} \left(\delta_2 \bar{P}_v^{(2)} + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(3)} \right. \\ & \quad \left. - \delta_2 \theta_0 \bar{P}_u^{(1)} + \bar{M}_z^{(1)} \right) + \frac{1}{\theta_0} \bar{P}_v + f_5^*, \\ & \frac{\delta_2}{\theta_0 \delta_1} \bar{a}^{(5)} + \frac{\delta_2 \beta_{\theta R}}{\theta_0 \delta_1} \bar{a}^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{a}^{(3)} + \frac{\delta_3 \beta_{\theta R}}{\theta_0 \delta_1} \bar{a}^{(2)} + \theta_0 \bar{a}^{(1)} \\ &= \theta_0 \beta_{\theta R} \bar{a} = \bar{f}_6 - \sum_{i=1}^6 \bar{f}_i \left(\frac{\delta_2}{\theta_0 \delta_1} \bar{g}_i^{(5)} + \frac{\delta_2 \beta_{\theta R}}{\theta_0 \delta_1} \bar{g}_i^{(4)} \right. \\ & \quad \left. + \frac{\delta_3}{\theta_0 \delta_1} \bar{g}_i^{(3)} + \frac{\delta_3 \beta_{\theta R}}{\theta_0 \delta_1} \bar{g}_i^{(2)} + \theta_0 \bar{g}_i^{(1)} + \theta_0 \beta_{\theta R} \bar{g}_i \right) \\ & \quad - \frac{\beta_{\theta R}}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} \right. \right. \\ & \quad \left. \left. + \theta_0 \bar{M}_z \right) \right] - \frac{1}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u^{(1)} \right. \\ & \quad \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] + f_6^*. \end{aligned} \quad (32)$$

If the shifting functions $\bar{g}_i(\xi)$, $i = 1, 2, 3, 4, 5, 6$, in (28) are chosen to satisfy the differential equation

$$\bar{g}_i^{(6)} + 2\theta_0^2 \bar{g}_i^{(4)} + \theta_0^4 \bar{g}_i^{(2)} = 0 \quad (33)$$

and the following boundary conditions,

(i) at $\xi = 0$

$$\begin{aligned} & \frac{\beta_{VL}}{\varsigma \theta_0^3 \delta_1} \bar{g}_i^{(5)} - \frac{1}{\theta_0 \delta_1} \bar{g}_i^{(4)} + \frac{\beta_{VL}}{\varsigma \theta_0 \delta_1} \bar{g}_i^{(3)} - \frac{\theta_0}{\delta_1} \bar{g}_i^{(2)} \\ & \quad - \frac{\beta_{VL}}{\theta_0} \bar{g}_i^{(1)} = \delta_{ij}, \quad j = 1, \end{aligned}$$

$$\begin{aligned} \frac{1}{\delta_1 \theta_0^2} \bar{g}_i^{(5)} + \frac{1}{\delta_1} \bar{g}_i^{(3)} - \beta_{UL} \bar{g}_i &= \delta_{ij}, \quad j = 2, \\ \frac{\delta_2}{\theta_0 \delta_1} \bar{g}_i^{(5)} - \frac{\delta_2 \beta_{\theta L}}{\theta_0 \delta_1} \bar{g}_i^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{g}_i^{(3)} - \frac{\delta_3 \beta_{\theta L}}{\theta_0 \delta_1} \bar{g}_i^{(2)} \\ + \theta_0 \bar{g}_i^{(1)} - \theta_0 \beta_{\theta L} \bar{g}_i &= \delta_{ij}, \quad j = 3, \end{aligned} \quad (34)$$

(ii) at $\xi = 1$

$$\begin{aligned} -\frac{\beta_{VR}}{\varsigma \theta_0^3 \delta_1} \bar{g}_i^{(5)} - \frac{1}{\delta_1 \theta_0} \bar{g}_i^{(4)} - \frac{\beta_{VR}}{\varsigma \theta_0 \delta_1} \bar{g}_i^{(3)} - \frac{\theta_0}{\delta_1} \bar{g}_i^{(2)} \\ + \frac{\beta_{VR}}{\theta_0} \bar{g}_i^{(1)} &= \delta_{ij}, \quad j = 4, \\ \frac{1}{\delta_1 \theta_0^2} \bar{g}_i^{(5)} + \frac{1}{\delta_1} \bar{g}_i^{(3)} + \beta_{UR} \bar{g}_i &= \delta_{ij}, \quad j = 5, \\ \frac{\delta_2}{\theta_0 \delta_1} \bar{g}_i^{(5)} + \frac{\delta_2 \beta_{\theta R}}{\theta_0 \delta_1} \bar{g}_i^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{g}_i^{(3)} + \frac{\delta_3 \beta_{\theta R}}{\theta_0 \delta_1} \bar{g}_i^{(2)} \\ + \theta_0 \bar{g}_i^{(1)} + \theta_0 \beta_{\theta R} \bar{g}_i &= \delta_{ij}, \quad j = 6, \end{aligned} \quad (35)$$

where δ_{ij} is a Kronecker symbol, then the differential equation (30) and the associated boundary conditions (31)-(32) can be reduced to

$$\begin{aligned} \bar{a}^{(6)} + 2\theta_0^2 \bar{a}^{(4)} + \theta_0^4 \bar{a}^{(2)} \\ = \theta_0 \bar{P}_v^{(1)} - \delta_2 \theta_0 \bar{P}_v^{(3)} - \theta_0^2 \bar{P}_u + \mu \theta_0^2 \bar{P}_u^{(2)} \\ - \frac{1}{\varsigma} \bar{P}_u^{(4)} - \theta_0^3 \bar{M}_z - \theta_0 \bar{M}_z^{(2)}. \end{aligned} \quad (36)$$

(i) At $\xi = 0$

$$\begin{aligned} \frac{\beta_{VL}}{\varsigma \theta_0^3 \delta_1} \bar{a}^{(5)} - \frac{1}{\theta_0 \delta_1} \bar{a}^{(4)} + \frac{\beta_{VL}}{\varsigma \theta_0 \delta_1} \bar{a}^{(3)} - \frac{\theta_0}{\delta_1} \bar{a}^{(2)} - \frac{\beta_{VL}}{\theta_0} \bar{a}^{(1)} \\ = \beta_{VL} \left[\frac{1}{\varsigma \theta_0^2} \bar{P}_v - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} \right. \\ \left. - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] + \frac{1}{\delta_1} \left(\delta_2 \bar{P}_v^{(1)} \right. \\ \left. + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(2)} - \delta_2 \theta_0 \bar{P}_u - \delta_2 \theta_0^2 \bar{M}_z \right) + \bar{M}_z - f_1^*, \\ \frac{1}{\delta_1 \theta_0^2} \bar{a}^{(5)} + \frac{1}{\delta_1} \bar{a}^{(3)} - \beta_{UL} \bar{a} = -\frac{1}{\delta_1 \theta_0} \left(\delta_2 \bar{P}_v^{(2)} \right. \\ \left. + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(3)} - \delta_2 \theta_0 \bar{P}_u^{(1)} + \bar{M}_z^{(1)} \right) + \frac{1}{\theta_0} \bar{P}_v - f_2^*, \\ \frac{\delta_2}{\theta_0 \delta_1} \bar{a}^{(5)} - \frac{\delta_2 \beta_{\theta L}}{\theta_0 \delta_1} \bar{a}^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{a}^{(3)} - \frac{\delta_3 \beta_{\theta L}}{\theta_0 \delta_1} \bar{a}^{(2)} + \theta_0 \bar{a}^{(1)} \end{aligned}$$

$$\begin{aligned} -\theta_0 \beta_{\theta L} \bar{a} &= \frac{\beta_{\theta L}}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u \right. \\ &+ \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} + \theta_0 \bar{M}_z \right) \Big] \\ &- \frac{1}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u^{(1)} \right. \\ &+ \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \Big] - f_3^*. \end{aligned} \quad (37)$$

(ii) At $\xi = 1$

$$\begin{aligned} -\frac{\beta_{VR}}{\varsigma \theta_0^3 \delta_1} \bar{a}^{(5)} - \frac{1}{\delta_1 \theta_0} \bar{a}^{(4)} - \frac{\beta_{VR}}{\varsigma \theta_0 \delta_1} \bar{a}^{(3)} - \frac{\theta_0}{\delta_1} \bar{a}^{(2)} + \frac{\beta_{VR}}{\theta_0} \bar{a}^{(1)} \\ = -\beta_{VR} \left[\frac{1}{\varsigma \theta_0^2} \bar{P}_v - \frac{\delta_2}{\varsigma \theta_0^2 \delta_1} \bar{P}_v^{(2)} + \frac{\delta_2}{\varsigma \theta_0 \delta_1} \bar{P}_u^{(1)} \right. \\ \left. - \frac{1}{\varsigma \theta_0^3 \delta_1} \left(\frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] + \frac{1}{\delta_1} \left(\delta_2 \bar{P}_v^{(1)} \right. \\ \left. + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(2)} - \delta_2 \theta_0 \bar{P}_u - \delta_2 \theta_0^2 \bar{M}_z \right) + \bar{M}_z + f_4^*, \\ \frac{1}{\delta_1 \theta_0^2} \bar{a}^{(5)} + \frac{1}{\delta_1} \bar{a}^{(3)} + \beta_{UR} \bar{a} = -\frac{1}{\delta_1 \theta_0} \left(\delta_2 \bar{P}_v^{(2)} \right. \\ \left. + \frac{1}{\varsigma \theta_0} \bar{P}_u^{(3)} - \delta_2 \theta_0 \bar{P}_u^{(1)} + \bar{M}_z^{(1)} \right) + \frac{1}{\theta_0} \bar{P}_v + f_5^*, \\ \frac{\delta_2}{\theta_0 \delta_1} \bar{a}^{(5)} + \frac{\delta_2 \beta_{\theta R}}{\theta_0 \delta_1} \bar{a}^{(4)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{a}^{(3)} + \frac{\delta_3 \beta_{\theta R}}{\theta_0 \delta_1} \bar{a}^{(2)} + \theta_0 \bar{a}^{(1)} \\ + \theta_0 \beta_{\theta R} \bar{a} = -\frac{\beta_{\theta R}}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u \right. \\ \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(1)} + \frac{1}{\varsigma} \bar{P}_u^{(2)} + \theta_0 \bar{M}_z \right) \right] \\ - \frac{1}{\theta_0 \delta_1} \left[\left(\frac{1}{\varsigma} - \mu \delta_2 \theta_0^2 \right) \bar{P}_u^{(1)} \right. \\ \left. + \delta_2 \left(\delta_2 \theta_0 \bar{P}_v^{(2)} + \frac{1}{\varsigma} \bar{P}_u^{(3)} + \theta_0 \bar{M}_z^{(1)} \right) \right] + f_6^*. \end{aligned} \quad (38)$$

Once the transformed function $\bar{a}(\xi)$ and the shifting functions $\bar{g}_1(\xi)$, $\bar{g}_2(\xi)$, $\bar{g}_3(\xi)$, $\bar{g}_4(\xi)$, $\bar{g}_5(\xi)$, and $\bar{g}_6(\xi)$ are determined, they are substituted into (28), yielding

$$\begin{aligned} U(\xi) &= \bar{a}(\xi) + \beta_{NVL} V^3(0) \bar{g}_1(\xi) + \beta_{NUL} U^3(0) \bar{g}_2(\xi) \\ &+ \beta_{N\theta L} \Psi^3(0) \bar{g}_3(\xi) - \beta_{NVR} V^3(1) \bar{g}_4(\xi) \\ &- \beta_{NUR} U^3(1) \bar{g}_5(\xi) - \beta_{N\theta R} \Psi^3(1) \bar{g}_6(\xi). \end{aligned} \quad (39)$$

Substituting (39) into (22) and (23) yields the solutions $\Psi(\xi)$ and $V(\xi)$. It can be observed that final solutions include the superposition of the linear and the nonlinear parts. The shifting function method can deal with the nonlinear parts of the boundary conditions very well.

4. Verification and Examples

The previous analysis is illustrated using the following example.

Example. Consider the deflection of a beam subjected to uniform distributed load P . The curved beam is clamped at the left end and supported at the right end with linear and nonlinear springs in the r direction. The corresponding coefficients are

$$\begin{aligned}\bar{P}_u &= \bar{M}_z = \beta_{NVL} = \beta_{NUL} = \beta_{N\theta L} = \beta_{UR} \\ &= \beta_{NUR} = \beta_{\theta R} = \beta_{N\theta R} = f_1^* = f_2^* = f_3^* \\ &= f_5^* = f_6^* = 0,\end{aligned}\quad (40)$$

$$\beta_{UL}, \beta_{VL}, \beta_{\theta L} \rightarrow \infty,$$

$$\bar{P}_v = P,$$

where P is constant.

Equations (22) and (23) become

$$\begin{aligned}\Psi &= \frac{1}{\theta_0 \delta_1} (\delta_2 U^{(4)} + \delta_3 U^{(2)} + \delta_1 \theta_0^2 U), \\ V &= -\frac{1}{\zeta \theta_0^3 \delta_1} U^{(5)} - \frac{1}{\zeta \theta_0 \delta_1} U^{(3)} + \frac{1}{\theta_0} U^{(1)} + \frac{P}{\zeta \theta_0^2}.\end{aligned}\quad (41)$$

The complete sixth-order ordinary differential characteristic equation in (25) and the associated boundary conditions in (26)-(27) become

$$U^{(6)} + 2\theta_0^2 U^{(4)} + \theta_0^4 U^{(2)} = 0. \quad (42)$$

(i) At $\xi = 0$

$$\begin{aligned}U^{(5)} + \theta_0^2 U^{(3)} - \zeta \theta_0^2 \delta_1 U^{(1)} &= \theta_0 \delta_1 P, \\ U &= 0,\end{aligned}\quad (43)$$

$$U^{(4)} + \frac{\delta_3}{\delta_2} U^{(2)} + \frac{\theta_0^2 \delta_1}{\delta_2} U = 0.$$

(ii) At $\xi = 1$

$$\begin{aligned}-\frac{\beta_{VR}}{\zeta \theta_0^3 \delta_1} U^{(5)} - \frac{1}{\delta_1 \theta_0} U^{(4)} - \frac{\beta_{VR}}{\zeta \theta_0 \delta_1} U^{(3)} - \frac{\theta_0}{\delta_1} U^{(2)} \\ + \frac{\beta_{VR}}{\theta_0} U^{(1)} + \beta_{NVR} V^3 &= -\frac{\beta_{VR} P}{\zeta \theta_0^2} + f_4^*, \\ \frac{1}{\delta_1 \theta_0^2} U^{(5)} + \frac{1}{\delta_1} U^{(3)} &= \frac{P}{\theta_0}, \\ \frac{\delta_2}{\theta_0 \delta_1} U^{(5)} + \frac{\delta_3}{\theta_0 \delta_1} U^{(3)} + \theta_0 U^{(1)} &= 0.\end{aligned}\quad (44)$$

Let

$$U(\xi) = \bar{a}(\xi) + \bar{f}_4 \bar{g}_4(\xi), \quad (45)$$

where $\bar{f}_4 = -\beta_{NVR} V^3(1)$.

Here $\bar{g}_4(\xi)$ is the shifting function to be specified. $\bar{a}(\xi)$ is the transformed function which satisfies the differential equation (42) and the associated boundary conditions (43)-(44). Consider

$$\bar{a}^{(6)} + 2\theta_0^2 \bar{a}^{(4)} + \theta_0^4 \bar{a}^{(2)} = 0. \quad (46)$$

The boundary conditions are as follows:

(i) At $\xi = 0$

$$\begin{aligned}\bar{a}^{(5)} + \theta_0^2 \bar{a}^{(3)} - \zeta \theta_0^2 \delta_1 \bar{a}^{(1)} &= \theta_0 \delta_1 P, \\ \bar{a} &= 0,\end{aligned}\quad (47)$$

$$\bar{a}^{(4)} + \frac{\delta_3}{\delta_2} \bar{a}^{(2)} + \frac{\theta_0^2 \delta_1}{\delta_2} \bar{a} = 0.$$

(ii) At $\xi = 1$

$$\begin{aligned}-\frac{\beta_{VR}}{\zeta \theta_0^3 \delta_1} \bar{a}^{(5)} - \frac{1}{\delta_1 \theta_0} \bar{a}^{(4)} - \frac{\beta_{VR}}{\zeta \theta_0 \delta_1} \bar{a}^{(3)} - \frac{\theta_0}{\delta_1} \bar{a}^{(2)} + \frac{\beta_{VR}}{\theta_0} \bar{a}^{(1)} \\ = -\frac{\beta_{VR} P}{\zeta \theta_0^2} + f_4^*, \\ \frac{1}{\delta_1 \theta_0^2} \bar{a}^{(5)} + \frac{1}{\delta_1} \bar{a}^{(3)} = \frac{P}{\theta_0}, \\ \frac{\delta_2}{\theta_0 \delta_1} \bar{a}^{(5)} + \frac{\delta_3}{\theta_0 \delta_1} \bar{a}^{(3)} + \theta_0 \bar{a}^{(1)} = 0.\end{aligned}\quad (48)$$

It can be found that the function $\bar{a}(\xi)$ is

$$\begin{aligned}\bar{a}(\xi) &= \bar{K}_1 + \bar{K}_2 \xi + \bar{K}_3 \sin(\theta_0 \xi) + \bar{K}_4 \cos(\theta_0 \xi) \\ &+ \bar{K}_5 \xi \sin(\theta_0 \xi) + \bar{K}_6 \xi \cos(\theta_0 \xi),\end{aligned}\quad (49)$$

where $\bar{K}_1, \bar{K}_2, \bar{K}_3, \bar{K}_4, \bar{K}_5$, and $\bar{K}_6$ are given in Appendix.

The shifting function $\bar{g}_4(\xi)$ satisfies the following differential equation and boundary conditions:

$$\bar{g}_4^{(6)} + 2\theta_0^2 \bar{g}_4^{(4)} + \theta_0^4 \bar{g}_4^{(2)} = 0. \quad (50)$$

(i) At $\xi = 0$

$$\begin{aligned}\bar{g}_4^{(5)} + \theta_0^2 \bar{g}_4^{(3)} - \zeta \theta_0^2 \delta_1 \bar{g}_4^{(1)} &= 0, \\ \bar{g}_4 &= 0,\end{aligned}\quad (51)$$

$$\bar{g}_4^{(4)} + \frac{\delta_3}{\delta_2} \bar{g}_4^{(2)} + \frac{\theta_0^2 \delta_1}{\delta_2} \bar{g}_4 = 0.$$

(ii) At $\xi = 1$

$$\begin{aligned}
 & -\frac{\beta_{VR}}{\zeta\theta_0^3\delta_1}\bar{g}_4^{(5)} - \frac{1}{\delta_1\theta_0}\bar{g}_4^{(4)} - \frac{\beta_{VR}}{\zeta\theta_0\delta_1}\bar{g}_4^{(3)} - \frac{\theta_0}{\delta_1}\bar{g}_4^{(2)} \\
 & + \frac{\beta_{VR}}{\theta_0}\bar{g}_4^{(1)} = 1, \\
 & \frac{1}{\delta_1\theta_0^2}\bar{g}_4^{(5)} + \frac{1}{\delta_1}\bar{g}_4^{(3)} = 0, \\
 & \frac{\delta_2}{\theta_0\delta_1}\bar{g}_4^{(5)} + \frac{\delta_3}{\theta_0\delta_1}\bar{g}_4^{(3)} + \theta_0\bar{g}_4^{(1)} = 0.
 \end{aligned} \tag{52}$$

It can be found that the function $\bar{g}_4(\xi)$ is

$$\begin{aligned}
 \bar{g}_4(\xi) = & \bar{J}_1 + \bar{J}_2\xi + \bar{J}_3\sin(\theta_0\xi) + \bar{J}_4\cos(\theta_0\xi) \\
 & + \bar{J}_5\xi\sin(\theta_0\xi) + \bar{J}_6\xi\cos(\theta_0\xi),
 \end{aligned} \tag{53}$$

where $\bar{J}_1, \bar{J}_2, \bar{J}_3, \bar{J}_4, \bar{J}_5$, and $\bar{J}_6$ are given in Appendix.

Substituting (45), (49), and (53) into (41) yields the exact solutions of $U(\xi)$, $V(\xi)$, and $\Psi(\xi)$, respectively.

Considering uniform straight Timoshenko beams as $\theta_0 = 0$ and $f_4^* = 0$ with (41), (45), (49), and (53), one has

$$\begin{aligned}
 \Psi(\xi) = & a_1\xi + a_2\xi^2 + \frac{1}{6}P\xi^3, \\
 V(\xi) = & \frac{(P\xi^3 - 12P\xi\mu + 12\xi a_1 + 8\xi^2 a_2 - 48\mu a_2)\xi}{24},
 \end{aligned} \tag{54}$$

where

$$\begin{aligned}
 a_1 = & -\frac{-12P + 24\beta_{NVR}V^3(1) - P\beta_{VR}}{8(3 + \beta_{VR} + 3\mu\beta_{VR})}, \\
 a_2 = & -\frac{24P - 24\beta_{NVR}V^3(1) + 5P\beta_{VR} + 12\mu P\beta_{VR}}{16(3 + \beta_{VR} + 3\mu\beta_{VR})}.
 \end{aligned} \tag{55}$$

With both linear and nonlinear spring stiffness constants being zeros (i.e., $\beta_{NVR} = \beta_{VR} = 0$), (54) reduce to

$$\begin{aligned}
 \Psi(\xi) = & \frac{\xi^3 - 3\xi^2 + 3\xi}{6}P, \\
 V(\xi) = & \frac{\xi^4 - 4\xi^3 + 6(1 - 2\mu)\xi^2 + 24\mu\xi}{24}P.
 \end{aligned} \tag{56}$$

For a Bernoulli-Euler beam without shear deformation (i.e., $\mu = 0$), (54) and (55) reduce to

$$\begin{aligned}
 \Psi(\xi) = & a_3\xi + a_4\xi^2 + \frac{1}{6}P\xi^3, \\
 V(\xi) = & \frac{(P\xi^3 + 12\xi a_3 + 8\xi^2 a_4)\xi}{24},
 \end{aligned} \tag{57}$$

TABLE 1: Neutral axis displacements in r direction of cantilever curved beam subjected to unit force in r direction at right end of cantilever curved beam [$\theta_0 = \pi/2$, $\zeta = 100$, $\mu = 0.01$, $\beta_{VR} = 0$, $\beta_{NVR} = 0$, $f_4^* = 1$].

ξ	$V(\xi)$				
	$P = 100$	$P = 10$	$P = 1$	$P = 0$	$P = 0^*$
0	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	0.26284	0.02928	0.00592	0.00333	0.00333
0.2	0.87395	0.09922	0.02175	0.01314	0.01314
0.3	1.75342	0.20141	0.04621	0.02896	0.02896
0.4	2.81779	0.32677	0.07767	0.05000	0.05000
0.5	3.98364	0.46603	0.11427	0.07518	0.07518
0.6	5.17105	0.61000	0.15390	0.10322	0.10322
0.7	6.30683	0.75005	0.19437	0.13263	0.13263
0.8	7.32739	0.87835	0.23344	0.16179	0.16179
0.9	8.18120	0.98824	0.26894	0.18902	0.18902
1.0	8.83071	1.07445	0.29882	0.21264	0.21264

* Lin's result [13].

TABLE 2: Neutral axis displacements in r and θ directions and angle of rotation due to bending in z direction for curved beam with linear and nonlinear boundaries subjected to uniform load in r direction [$\theta_0 = \pi/2$, $\zeta = 100$, $\mu = 0.01$, $\beta_{VR} = 1$, $\beta_{NVR} = 1$, $f_4^* = 0$, $P = 1$].

ξ	$U(\xi)$	$\Psi(\xi)$	$V(\xi)$
0	0.00000	0.00000	0.00000
0.1	-0.00037	0.03283	0.00236
0.2	-0.00004	0.05949	0.00767
0.3	0.00140	0.08033	0.15178
0.4	0.00423	0.09584	0.02411
0.5	0.00860	0.10662	0.03372
0.6	0.01455	0.11342	0.04331
0.7	0.02201	0.11705	0.05228
0.8	0.03083	0.11844	0.06011
0.9	0.04079	0.11854	0.06643
1.0	0.05161	0.11835	0.07101

where

$$\begin{aligned}
 a_3 = & -\frac{-12P + 24\beta_{NVR}V^3(1) - P\beta_{VR}}{8(3 + \beta_{VR})}, \\
 a_4 = & -\frac{24P - 24\beta_{NVR}V^3(1) + 5P\beta_{VR}}{16(3 + \beta_{VR})}.
 \end{aligned} \tag{58}$$

It is the exact nondimensional deflection of cantilevered Timoshenko and Bernoulli-Euler curved beams subjected to uniform nondimensional distributed load P . Equations (54)–(58) are exactly the same as those given by Lee et al. [18].

Table 1 shows that the exact solutions of the shifting function method are the same as Lin's results [13]. With increasing distributed load, the amount of deflection in the r direction increases at a given location on the curved beam. Table 2 shows the neutral axis displacements in the r and θ directions and the angle of rotation due to bending in

FIGURE 2: Deflections in r direction at right end of curved beam ($\theta_0 = \pi/2$) varying with the uniform load for various boundary conditions and constants [$\zeta = 100$, $\mu = 0.01$, $f_4^* = 0$].

FIGURE 3: Deflections in r direction at right end of curved beam varying with the uniform load for various center angles and constants [$\zeta = 100$, $\mu = 0.01$, $\beta_{VR} = 1$, $\beta_{NVR} = 1$, $f_4^* = 0$].

FIGURE 4: Deflections in r direction at right end of curved beam varying with center angles for various boundary conditions and constants [$\zeta = 100$, $\mu = 0.01$, $f_4^* = 0$, $P = 1$].

the z direction. The curved beam with linear and nonlinear boundaries has lower deflection in the r direction compared to that of a beam without such boundaries.

Of note, for the curved beam subjected to uniform load in the r direction, the deflections at the right end of the curved beam vary, as shown in Figures 2–4. According to (45), (49), and (53), the nonlinear spring stiffness constant is zero, and thus it is reasonable that the deflection at the right end of the curved beam is linearly related to uniform load in the r direction. When the nonlinear spring stiffness constant is nonzero value, the relationship can be curvilinear. The spring at the right end of the curved beam is strong enough to reduce the deflection variation, as shown in Figure 2. Figure 3 shows the influence of curvature on the static deflection of the beam in the r direction for a given set of boundary conditions. When the center angle is increased, the deflection will decrease. The curved beam itself can be regarded as a spring mechanism. Figure 4 shows that the center angle and boundary conditions significantly affect deflection in the r direction. If the load on the curved beam is insufficient and

the value of the deflection is less than 1, a linear spring has a greater effect than that of a nonlinear spring. On the contrary, if the value of the deflection is over 1, a nonlinear spring has a greater effect than that of a linear spring.

5. Conclusion

In this paper, the shifting function method is applied to develop the static deflection of in-plane curved Timoshenko beams with nonlinear boundary conditions. Three coupled governing differential equations are derived via Hamilton's principle and decomposed into a complete sixth-order ordinary differential characteristic equation that depends on the neutral axis displacement in the θ direction. The explicit relations among the neutral axis displacement in the r direction, the angle of rotation due to bending in the z direction, and the neutral axis displacement in the θ direction are revealed. An example was used to illustrate the analysis. It can be found that the center angle and the linear and nonlinear spring

stiffness constants will have significant influence on the static deflection of the beam.

Appendix

Coefficients of (49) and (53) are as follows:

$$\begin{aligned}
 \bar{K}_1 &= \frac{\{2(\zeta + \theta_0^2) P \beta_{VR} \cos(\theta_0) [1 - \cos(\theta_0)] - 2\zeta \theta_0^4 f_4^* \cos(\theta_0) - (\delta_1 \beta_{VR} + 2\theta_0^2) \zeta \theta_0 P \sin(\theta_0)\}}{\theta_0^4 Q}, \\
 \bar{K}_2 &= \frac{P}{\theta_0^3}, \\
 \bar{K}_3 &= -\frac{1}{2} \{2(\theta_0^2 + \zeta) P [(2\theta_0^2 + \delta_1 \beta_{VR}) \theta_0 - 2\beta_{VR} \sin(\theta_0) \cos(\theta_0)] + (\zeta \delta_1 - 2\theta_0^2) \\
 &\quad \cdot [\delta_1 P \beta_{VR} \sin(\theta_0) + \theta_0 \delta_1 P \beta_{VR} \cos(\theta_0) - 2\theta_0^4 f_4^* \sin(\theta_0) + 2\theta_0^3 P \cos(\theta_0)]\} (\theta_0^4 Q)^{-1}, \\
 \bar{K}_4 &= \frac{\{-2(\zeta + \theta_0^2) P \beta_{VR} \cos(\theta_0) [1 - \cos(\theta_0)] + 2\zeta \theta_0^4 f_4^* \cos(\theta_0) + (2\theta_0^2 + \delta_1 \beta_{VR}) \zeta \theta_0 P \sin(\theta_0)\}}{\theta_0^4 Q}, \\
 \bar{K}_5 &= \frac{(1/2) \delta_1 \{2(\zeta + \theta_0^2) P \beta_{VR} \cos(\theta_0) [\cos(\theta_0) - 1] + 2\zeta \theta_0^4 f_4^* \cos(\theta_0) + (2\theta_0^2 + \delta_1 \beta_{VR}) \zeta \theta_0 P \sin(\theta_0)\}}{\theta_0^3 Q}, \\
 \bar{K}_6 &= \frac{(1/2) \delta_1 [-2(\zeta + \theta_0^2) P \beta_{VR} \sin(\theta_0) \cos(\theta_0) + (\delta_1 P \beta_{VR} - 2\theta_0^4 f_4^*) \zeta \sin(\theta_0) + (\delta_1 \beta_{VR} + 2\theta_0^2) \zeta \theta_0 P \cos(\theta_0)]}{\theta_0^3 Q}, \quad (A.1) \\
 \bar{J}_1 &= \frac{2\zeta \cos(\theta_0)}{Q}, \\
 \bar{J}_2 &= 0, \\
 \bar{J}_3 &= \frac{(2\theta_0^2 - \zeta \delta_1) \sin(\theta_0)}{Q}, \\
 \bar{J}_4 &= -\frac{2\zeta \cos(\theta_0)}{Q}, \\
 \bar{J}_5 &= -\frac{\zeta \theta_0 \delta_1 \cos(\theta_0)}{Q}, \\
 \bar{J}_6 &= \frac{\zeta \theta_0 \delta_1 \sin(\theta_0)}{Q},
 \end{aligned}$$

where

$$\begin{aligned}
 Q &= (2\theta_0^2 + \delta_1 \beta_{VR}) \theta_0 \zeta \\
 &\quad + [\zeta (2\delta_3 - 3\delta_1) - 2\theta_0^2] \beta_{VR} \sin(\theta_0) \cos(\theta_0). \quad (A.2)
 \end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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