Research Article

A Drift-Free Left Invariant Control System on the Lie Group $SO(3) \times R^3 \times R^3$

Camelia Pop

Mathematics Department, "Politehnica" University of Timisoara, 300006 Timisoara, Romania

Correspondence should be addressed to Camelia Pop; cariesanu@yahoo.com

Received 14 December 2014; Accepted 25 March 2015

Academic Editor: Anna Pandolfi

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A controllable drift-free system on the Lie group $G = SO(3) \times R^3 \times R^3$ is considered. The dynamics and geometrical properties of the corresponding reduced Hamilton's equations on $(g^*, \{\cdot, \cdot\}_-)$ are studied, where $\{\cdot, \cdot\}_-$ is the minus Lie-Poisson structure on the dual space g^* of the Lie algebra $g = so(3) \times R^3 \times R^3$ of G . The numerical integration of this system is also discussed.

1. Introduction

The optimal control problems arise naturally in a lot of mathematics, engineering, or economics areas. An important class is the control problems on matrix Lie groups due to the possibility to study them from Poisson geometry point of view. Many dynamical systems have as configuration space a Lie group, such as $SO(3)$ for the spacecraft dynamics (see [1]), $R^2 \times SO(3)$ for the ball-plate problem (see [2]), G_5 for the dynamics of the car with two trailers (see [3]), $SL(2, R)$ for the Maxwell-Bloch system (see [4]), or $SE(2, R) \times SO(2)$ for the dynamics of a rolling penny (see [5]).

In [6] it was shown that the dynamics of the underwater vehicle problem can be viewed as Lie-Poisson dynamics, considering $G = SO(3) \times R^3 \times R^3$ and group multiplication in G defined by

$$(R_1, a_1, b_1) \cdot (R_2, a_2, b_2) = (R_1 R_2, R_1 a_2 + a_1, R_1 b_2 + b_1), \quad (1)$$

for any $(R_i, a_i, b_i) \in G, i = 1, 2$.

The underwater vehicle dynamics has Lie-Poisson form on g^* , the dual of the Lie algebra of G .

The goal of our paper is to study an optimal control problem on the Lie group G and to point out some of its dynamical and geometrical properties.

2. The Lie Group $SO(3) \times R^3 \times R^3$ and Its Lie-Poisson Structure

Let us consider the Lie group $G = SO(3) \times R^3 \times R^3$, its Lie algebra g , and the dual of the Lie algebra g^* .

A basis of the Lie algebra is given by the following matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & A_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & A_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & (2) \\ A_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, & A_6 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The Lie algebra structure of g is given by Table 1, where $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$ is the canonical basis of the Lie algebra g .

TABLE 1

$[\cdot, \cdot]$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
A_1	0	A_3	$-A_2$	0	A_6	$-A_5$	0	A_9	$-A_8$
A_2	$-A_3$	0	A_1	$-A_6$	0	A_4	$-A_9$	0	A_7
A_3	A_2	$-A_1$	0	A_5	$-A_4$	0	A_8	$-A_7$	0
A_4	0	A_6	$-A_5$	0	0	0	0	0	0
A_5	$-A_6$	0	A_4	0	0	0	0	0	0
A_6	A_5	$-A_4$	0	0	0	0	0	0	0
A_7	A_5	$-A_4$	0	0	0	0	0	0	0
A_8	A_5	$-A_4$	0	0	0	0	0	0	0
A_9	A_5	$-A_4$	0	0	0	0	0	0	0

Now, a general left invariant drift-free control system on G with fewer controls than state variables can be written in the following form:

$$\dot{X} = X \left(\sum_{i=1}^m A_i u_i \right), \quad (3)$$

where $X \in G$, u_i , $i = 1, 2, \dots, m$, are the controls, and $m < 9$.

In all the following we will concentrate on the following left invariant, drift-free control system on G with 3 controls:

$$\dot{X} = X (A_1 u_1 + A_2 u_2 + A_3 u_3 + A_5 u_5 + A_7 u_7). \quad (4)$$

Then, we have the following.

Proposition 1. *System (4) is controllable.*

Proof. Since the span of the set of Lie brackets generated by A_1, A_2, A_3, A_5, A_7 coincides with $\mathfrak{g}$, the proposition is a consequence of Chow's Theorem [7]. $\square$

3. An Optimal Control Problem for System (4)

Let J be the cost function given by

$$\begin{aligned} J(u_1, u_2, u_3, u_5, u_7) \\ = \frac{1}{2} \int_0^{t_f} [c_1 u_1^2(t) + c_2 u_2^2(t) + c_3 u_3^2(t) + c_5 u_5^2(t) \\ + c_7 u_7^2(t)] dt \\ c_1 > 0, \quad c_2 > 0, \quad c_3 > 0, \quad c_5 > 0, \quad c_7 > 0. \end{aligned} \quad (5)$$

Then we have the following.

Proposition 2. *The controls that minimize J and steer system (4) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by*

$$\begin{aligned} u_1 &= \frac{1}{c_1} x_1, & u_2 &= \frac{1}{c_2} x_2, & u_3 &= \frac{1}{c_3} x_3, \\ u_5 &= \frac{1}{c_5} x_5, & u_7 &= \frac{1}{c_7} x_7, \end{aligned} \quad (6)$$

where x_i 's are solutions of

$$\begin{aligned} \dot{x}_1 &= \left(\frac{1}{c_3} - \frac{1}{c_2} \right) x_2 x_3 - \frac{1}{c_5} x_5 x_6, \\ \dot{x}_2 &= \left(\frac{1}{c_1} - \frac{1}{c_3} \right) x_1 x_3 + \frac{1}{c_7} x_7 x_9, \\ \dot{x}_3 &= \left(\frac{1}{c_2} - \frac{1}{c_1} \right) x_1 x_2 + \frac{1}{c_5} x_4 x_5 - \frac{1}{c_7} x_7 x_8, \\ \dot{x}_4 &= -\frac{1}{c_2} x_2 x_6 + \frac{1}{c_3} x_3 x_5, \\ \dot{x}_5 &= \frac{1}{c_1} x_1 x_6 - \frac{1}{c_3} x_3 x_4, \\ \dot{x}_6 &= -\frac{1}{c_1} x_1 x_5 + \frac{1}{c_2} x_2 x_4, \\ \dot{x}_7 &= -\frac{1}{c_2} x_2 x_9 + \frac{1}{c_3} x_3 x_8, \\ \dot{x}_8 &= \frac{1}{c_1} x_1 x_9 - \frac{1}{c_3} x_3 x_7, \\ \dot{x}_9 &= \frac{1}{c_1} x_1 x_8 + \frac{1}{c_2} x_2 x_7. \end{aligned} \quad (7)$$

Proof. Let us apply Krishnaprasad's theorem (see [8]). It follows that the optimal Hamiltonian is given by

$$H(x_1, x_2, x_3, x_5, x_7) = \frac{1}{2} \left(\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_3^2}{c_3} + \frac{x_5^2}{c_5} + \frac{x_7^2}{c_7} \right). \quad (8)$$

It is in fact the controlled Hamiltonian $\bar{H}$ given by

$$\begin{aligned} \bar{H}(x_1, x_2, x_3, x_5, x_7) \\ = x_1 u_1 + x_2 u_2 + x_3 u_3 + x_5 u_5 + x_7 u_7 \\ - \frac{1}{2} (c_1 u_1^2 + c_2 u_2^2 + c_3 u_3^2 + c_5 u_5^2 + c_7 u_7^2), \end{aligned} \quad (9)$$

which is reduced to g^* via Poisson reduction. Here g_-^* is $g^* \simeq \mathbb{R}^9$ together with the minus Lie-Poisson structure generated

by the following matrix:

$$\Pi_- = \begin{bmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 & 0 & -x_9 & x_8 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 & x_9 & 0 & -x_7 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 & -x_8 & x_7 & 0 \\ 0 & -x_6 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (10)$$

Then the optimal controls are given by

$$\begin{aligned} u_1 &= \frac{1}{c_1} x_1, & u_2 &= \frac{1}{c_2} x_2, & u_3 &= \frac{1}{c_3} x_3, \\ u_5 &= \frac{1}{c_5} x_5, & u_7 &= \frac{1}{c_7} x_7, \end{aligned} \quad (11)$$

where x_i 's are solutions of the reduced Hamilton's equations given by

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6, \dot{x}_7, \dot{x}_8, \dot{x}_9]^t = \Pi_- \cdot \nabla H, \quad (12)$$

which are nothing else other than the required equations (7). $\square$

Proposition 3. Dynamics (7) has the following Hamilton-Poisson realization:

$$(\mathbb{R}^9, \Pi_-, H), \quad (13)$$

where

$$\Pi_- = \begin{bmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 & 0 & -x_9 & x_8 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 & x_9 & 0 & -x_7 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 & -x_8 & x_7 & 0 \\ 0 & -x_6 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$H(x_1, x_2, x_3, x_5, x_7) = \frac{1}{2} \left(\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_3^2}{c_3} + \frac{x_5^2}{c_5} + \frac{x_7^2}{c_7} \right). \quad (14)$$

Proof. Indeed, it is not hard to see that dynamics (7) can be put in the following equivalent form:

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6]^t = \Pi_- \cdot \nabla H, \quad (15)$$

as required. $\square$

Via Bermejo-Feiren's technique [9] we are led immediately to the following.

Proposition 4. The functions C_1, C_2 , and C_3 , given by

$$\begin{aligned} C_1 &= \frac{1}{2} (x_4^2 + x_5^2 + x_6^2), & C_2 &= \frac{1}{2} (x_7^2 + x_8^2 + x_9^2), \\ C_3 &= x_1 x_4 + x_2 x_5 + x_3 x_6, \end{aligned} \quad (16)$$

are Casimirs of our configuration.

The goal of our paper is to study some geometrical and dynamical properties for system (7).

4. Stability

Let us consider now $c_1 = c_2 = c_3 = c_5 = c_7 = 1$. Using MATHEMATICA, we can see that the equilibrium states of our dynamics (7) are

$$\begin{aligned} e_1^{MNPQ} &= (0, 0, 0, M, 0, N, 0, P, Q), & M, N, P, Q &\in \mathbb{R}, \\ e_2^{MNP} &= (0, 0, M, 0, 0, N, 0, 0, P), & M, N, P &\in \mathbb{R}, \\ e_3^{MPQ} &= (0, 0, 0, 0, M, 0, 0, P, Q), & M, P, Q &\in \mathbb{R}, \\ e_4^{MNP} &= (0, M, 0, 0, N, 0, 0, P, 0), & M, N, P &\in \mathbb{R}, \\ e_5^{MNP} &= (M, N, P, 0, 0, 0, 0, 0, 0), & M, N, P &\in \mathbb{R}, \\ e_6^{MNP} &= (M, 0, 0, N, 0, 0, P, 0, 0), & M, N, P &\in \mathbb{R}, \end{aligned}$$

$$\begin{aligned}
e_7^{MNP} &= (0, M, 0, 0, N, 0, 0, P, 0), \quad M, N, P \in \mathbb{R}, \\
e_8^{MNP} &= (0, 0, 0, M, 0, N, P, 0, 0), \quad M, N, P \in \mathbb{R}, \\
e_9^{MNP} &= \left(M, 0, N, P, 0, \frac{NP}{M}, 0, 0, 0\right), \quad M, N, P \in \mathbb{R}, \\
e_{10}^{MNP} &= \left(0, M, N, 0, 0, 0, 0, P, \frac{NP}{M}\right), \quad M, N, P \in \mathbb{R}, \\
e_{11}^{MNP} &= \left(0, 0, 0, \frac{NP}{M}, M, 0, N, P, 0\right), \quad M, N, P \in \mathbb{R}, \\
e_{12}^{MNP} &= \left(M, N, 0, \frac{NP}{M}, P, 0, -\frac{NP}{M}, -P, 0\right), \quad M, N, P \in \mathbb{R}, \\
e_{13}^{MNP} &= \left(M, N, 0, \frac{NP}{M}, P, 0, -\frac{NP}{M}, P, 0\right), \quad M, N, P \in \mathbb{R}.
\end{aligned} \tag{17}$$

Let us start to study the nonlinear stability. We have the following results.

Proposition 5. *The equilibrium states e_1^{MNPQ} , $M, N, P \in \mathbb{R}$, are nonlinearly stable for any $M, N, P, Q \in \mathbb{R}^*$.*

Proof. We will make the proof using Arnold's method (see [10]). Let $F_{\lambda, \mu, \nu} \in C^\infty(R^9, R)$ given by

$$\begin{aligned}
F_{\lambda, \mu, \nu} &= H + \frac{\lambda}{2}C_1 + \nu C_2 + \frac{\nu}{2}C_3 \\
&= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2) + \frac{\lambda}{2}(x_4^2 + x_5^2 + x_6^2) \\
&\quad + \frac{\mu}{2}(x_7^2 + x_8^2 + x_9^2) + \nu(x_7x_4 + x_8x_5 + x_9x_6).
\end{aligned} \tag{18}$$

Following Arnold's method, we have successively

$$\nabla F_{\lambda, \mu, \nu}(e_1^{MNPQ}) = 0 \quad \text{iff } \lambda = \mu = \nu = 0. \tag{19}$$

If we consider the space

$$\begin{aligned}
X &= \text{Ker}(dC_1)(e_1^{MNPQ}) \cap \text{Ker}(dC_2)(e_1^{MNPQ}) \\
&\quad \cap \text{Ker}(dC_3)(e_1^{MNPQ}) \\
&= \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right),
\end{aligned} \tag{20}$$

then

$$v^t \nabla^2 F_{0,0,0}(e_1^{MNPQ}) v = a^2 + b^2 + c^2, \tag{21}$$

for any $v \in X$; that is, $v = (a \ b \ c \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^t$, so

$$\nabla^2 F_{0,0,0}(e_1^{MNPQ})|X \times X \tag{22}$$

is positive defined. $\square$

Proposition 6. *The equilibrium states $e_3^{M,P,Q}$, $M, P, Q \in \mathbb{R}$, are nonlinearly stable if and only if $M, N \in \mathbb{R}^*$, $Q = 0$.*

Proof. We will make the proof using Arnold's method (see [10]). Let $F_{\lambda, \mu, \nu} \in C^\infty(R^9, R)$ given by

$$\begin{aligned}
F_{\lambda, \mu, \nu} &= \frac{1}{2}C_2 + \frac{\lambda}{2}H + \frac{\mu}{2}C_1 + \nu C_3 \\
&= \frac{1}{2}(x_7^2 + x_8^2 + x_9^2) + \frac{\lambda}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2) \\
&\quad + \frac{\mu}{2}(x_4^2 + x_5^2 + x_6^2) + \nu(x_7x_4 + x_8x_5 + x_9x_6).
\end{aligned} \tag{23}$$

Following Arnold's method, we have successively

$$\begin{aligned}
\nabla F_{\lambda, \mu, \nu}(e_3^{MPQ}) &= 0 \\
\text{iff } \lambda + \mu &= \frac{P^2}{M^2}, \quad \nu = -\frac{P}{M}, \quad Q = 0.
\end{aligned} \tag{24}$$

If we consider the space

$$\begin{aligned}
Y &= \text{Ker}(dH)(e_3^{MN0}) \cap \text{Ker}(dC_1)(e_3^{MN0}) \\
&\quad \cap \text{Ker}(dC_3)(e_3^{MNQ}) \\
&= \text{span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right),
\end{aligned}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (25)$$

then

$$\begin{aligned} v^t \nabla^2 F_{\lambda, \mu, \nu} (e_3^{MN0}) v &= \lambda a^2 + \lambda b^2 + \lambda c^2 + \mu d^2 + \mu e^2 \\ &+ (1 + \lambda) f^2 + g^2 + h^2 + 2\nu fd \\ &+ 2\nu he, \end{aligned} \quad (26)$$

for any $v \in X$; that is, $v = (a \ b \ c \ d \ 0 \ e \ f \ g \ h)^t$, so

$$\nabla^2 F_{\lambda, \mu, \nu} (e_3^{MN0}) \mid Y \times Y \quad (27)$$

is positive defined; so, the equilibrium states e_3^{MN0} are nonlinearly stables. $\square$

Using similar arguments, we are led to the following result.

Proposition 7. *The equilibrium states $e_5^{M,N,P}$, $M, N, P \in \mathbb{R}$, are nonlinearly stable for any $M, N, P \in \mathbb{R}^*$.*

Proof. We consider in this case the function $F_{\lambda, \mu, \nu} \in C^\infty(\mathbb{R}^9, \mathbb{R})$ given by

$$\begin{aligned} F_{\lambda, \mu, \nu} &= \frac{1}{2} C_1 + \frac{\lambda}{2} H + \frac{\mu}{2} C_2 + \nu C_3 \\ &= \frac{1}{2} (x_4^2 + x_5^2 + x_6^2) + \frac{\lambda}{2} (x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2) \\ &\quad + \frac{\mu}{2} (x_7^2 + x_8^2 + x_9^2) + \frac{\nu}{2} (x_7 x_4 + x_8 x_5 + x_9 x_6). \end{aligned} \quad (28)$$

Now, we have successively

$$\nabla F(e_5^{M,N,P}) = 0 \quad \text{iff } \lambda = 0; \quad (29)$$

if $\mu > 0$ and $\nu = 0$, then $\nabla^2 F(e_5^{M,N,P}) \mid X \times X > 0$, where

$$\begin{aligned} X = \text{span} &\left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \right. \\ &\left. \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right), \end{aligned} \quad (30)$$

so, via Arnold's method (see [10]), the equilibrium states $e_5^{M,N,P}$ are nonlinearly stables. $\square$

Remark 8. Unfortunately, for the rest of equilibrium states the energy methods do not work. The stability problem must be approached with other techniques, and it is still open.

5. Numerical Integration of Dynamics (7)

Kahan's integrator (see [11]) for (7) can be written in the following form:

$$\begin{aligned} x_1^{n+1} - x_1^n &= \frac{h}{2} \left(\frac{1}{c_3} - \frac{1}{c_2} \right) (x_3^{n+1} x_2^n + x_2^{n+1} x_3^n) \\ &\quad - \frac{h}{2c_5} (x_5^{n+1} x_6^n + x_6^{n+1} x_5^n), \\ x_2^{n+1} - x_2^n &= \frac{h}{2} \left(\frac{1}{c_1} - \frac{1}{c_3} \right) (x_1^{n+1} x_3^n + x_3^{n+1} x_1^n) \\ &\quad + \frac{h}{2c_7} (x_7^{n+1} x_9^n + x_9^{n+1} x_7^n), \end{aligned}$$

$$\begin{aligned}
x_3^{n+1} - x_3^n &= \frac{h}{2} \left(\frac{1}{c_2} - \frac{1}{c_1} \right) (x_1^{n+1} x_2^n + x_2^{n+1} x_1^n) \\
&\quad + \frac{h}{2c_5} (x_4^{n+1} x_5^n + x_5^{n+1} x_4^n) \\
&\quad - \frac{h}{2c_7} (x_7^{n+1} x_8^n + x_8^{n+1} x_7^n), \\
x_4^{n+1} - x_4^n &= -\frac{h}{2c_2} (x_2^{n+1} x_6^n + x_6^{n+1} x_2^n) \\
&\quad + \frac{h}{2c_3} (x_3^{n+1} x_5^n + x_5^{n+1} x_3^n), \\
x_5^{n+1} - x_5^n &= \frac{h}{2c_1} (x_1^{n+1} x_6^n + x_6^{n+1} x_1^n) \\
&\quad - \frac{h}{2c_3} (x_4^{n+1} x_3^n + x_3^{n+1} x_4^n), \\
x_6^{n+1} - x_6^n &= -\frac{h}{2c_1} (x_1^{n+1} x_5^n + x_5^{n+1} x_1^n) \\
&\quad + \frac{h}{2c_2} (x_2^{n+1} x_4^n + x_4^{n+1} x_2^n), \\
x_7^{n+1} - x_7^n &= -\frac{h}{2c_2} (x_2^{n+1} x_9^n + x_9^{n+1} x_2^n) \\
&\quad + \frac{h}{2c_3} (x_3^{n+1} x_8^n + x_8^{n+1} x_3^n), \\
x_8^{n+1} - x_8^n &= \frac{h}{2c_1} (x_1^{n+1} x_9^n + x_9^{n+1} x_1^n) \\
&\quad - \frac{h}{2c_3} (x_3^{n+1} x_7^n + x_7^{n+1} x_3^n), \\
x_9^{n+1} - x_9^n &= \frac{h}{2c_1} (x_1^{n+1} x_8^n + x_8^{n+1} x_1^n) \\
&\quad + \frac{h}{2c_2} (x_2^{n+1} x_7^n + x_7^{n+1} x_2^n).
\end{aligned} \tag{31}$$

A long but straightforward computation or using eventually MATHEMATICA leads us to the following.

Proposition 9. *Kahan's integrator (31) has the following properties:*

- (i) *It is not Poisson preserving.*
- (ii) *It does not preserve the Casimirs C_1, C_2, C_3 of our Poisson configuration (g, Π_-) .*

(iii) *It does not preserve the Hamiltonian H of our system (7).*

We will discuss now the numerical integration of dynamics (7) via the Lie-Trotter integrator (see [12]). In the beginning, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3} + X_{H_5} + X_{H_7}, \tag{32}$$

where

$$\begin{aligned}
H_1 &= \frac{1}{2c_1} x_1^2, & H_2 &= \frac{1}{2c_2} x_2^3, & H_3 &= \frac{1}{2c_3} x_3^2, \\
H_5 &= \frac{1}{2c_5} x_5^2, & H_7 &= \frac{1}{2c_7} x_7^2.
\end{aligned} \tag{33}$$

Their corresponding integral curves are, respectively, given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \\ x_7(t) \\ x_8(t) \\ x_9(t) \end{bmatrix} = A_i \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \\ x_6(0) \\ x_7(0) \\ x_8(0) \\ x_9(0) \end{bmatrix}, \quad i = 1, 2, 3, 5, 7, \tag{34}$$

where

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos at & \sin at & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin at & \cos at & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos at & \sin at & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin at & \cos at & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos at & \sin at \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin at & \cos at \end{bmatrix},$$

$$a = \frac{x_1(0)}{c_1},$$

A_2

$$= \begin{bmatrix} \cos bt & 0 & -\sin bt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin bt & 0 & \cos bt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos bt & 0 & -\sin bt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin bt & 0 & \cos bt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos bt & 0 & -\sin bt \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin bt & 0 & \cos bt \end{bmatrix},$$

$$b = \frac{x_2(0)}{c_2},$$

 A_3

$$= \begin{bmatrix} \cos ct & \sin ct & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin ct & \cos ct & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos ct & \sin ct & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin ct & \cos ct & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos ct & \sin ct & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin ct & \cos ct & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$c = \frac{x_3(0)}{c_3},$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -dt & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & dt & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad d = \frac{x_5(0)}{c_5},$$

$$A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & et \\ 0 & 0 & 1 & 0 & 0 & 0 & -et & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad e = \frac{x_7(0)}{c_7}.$$

(35)

Then the Lie-Trotter integrator is given by

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \\ x_4^{n+1} \\ x_5^{n+1} \\ x_6^{n+1} \end{bmatrix} = A_1 A_2 A_3 A_4 A_5 \begin{bmatrix} x_1^n \\ x_2^n \\ x_3^n \\ x_4^n \\ x_5^n \\ x_6^n \end{bmatrix}; \quad (36)$$

that is,

$$\begin{aligned} x_1^{n+1} &= x_1^n \cos bt \cos ct - dt x_6^n \cos bt \cos ct \\ &\quad + x_3^n (-\sin bt + \cos bt \sin ct) \\ &\quad + dt x_4^n (-\sin bt + \cos bt \sin ct) \\ &\quad - et x_8^n (-\sin bt + \cos bt \sin ct), \\ x_2^{n+1} &= x_1^n (\cos ct \sin at \sin bt - \cos at \sin ct) \\ &\quad - dt x_6^n (\cos ct \sin at \sin bt - \cos at \sin ct) \\ &\quad + x_3^n (\cos at \cos ct + \cos bt \sin at + \sin at \sin bt \sin ct) \\ &\quad + dt x_4^n (\cos at \cos ct + \cos bt \sin at \\ &\quad \quad + \sin at \sin bt \sin ct) \\ &\quad - et x_8^n (\cos at \cos ct + \cos bt \sin at \\ &\quad \quad + \sin at \sin bt \sin ct), \\ x_3^{n+1} &= x_1^n (\cos at \cos ct \sin bt + \sin at \sin ct) \\ &\quad - dt x_6^n (\cos at \cos ct \sin bt + \sin at \sin ct) \\ &\quad + x_3^n (\cos at \cos bt - \cos ct \sin at + \cos at \sin bt \sin ct) \\ &\quad + dt x_4^n (\cos at \cos bt - \cos ct \sin at \\ &\quad \quad + \cos at \sin bt \sin ct) \\ &\quad - et x_8^n (\cos at \cos bt - \cos ct \sin at \\ &\quad \quad + \cos at \sin bt \sin ct), \\ x_4^{n+1} &= x_4^n \cos bt \cos ct - x_6^n \sin bt + x_5^n \cos bt \sin ct, \\ x_5^{n+1} &= x_6^n \cos bt \sin at \\ &\quad + x_4^n (\cos ct \sin at \sin bt - \cos at \sin ct) \\ &\quad + x_5^n (\cos at \cos ct + \sin at \sin bt \sin ct), \\ x_6^{n+1} &= x_6^n \cos at \cos bt \\ &\quad + x_4^n (\cos at \cos ct \sin bt + \sin at \sin ct) \\ &\quad + x_5^n (-\cos ct \sin at + \cos at \sin bt \sin ct), \end{aligned}$$

$$\begin{aligned}
x_7^{n+1} &= x_7^n \cos bt \cos ct - x_9^n \sin bt + x_8^n \cos bt \sin ct, \\
x_8^{n+1} &= x_9^n \cos bt \sin at \\
&\quad + x_7^n (\cos ct \sin at \sin bt - \cos at \sin ct) \\
&\quad + x_8^n (\cos at \cos ct + \sin at \sin bt \sin ct), \\
x_9^{n+1} &= x_9^n \cos at \cos bt \\
&\quad + x_7^n (\cos at \cos ct \sin bt + \sin at \sin ct) \\
&\quad + x_8^n (-\cos ct \sin at + \cos at \sin bt \sin ct).
\end{aligned} \tag{37}$$

Now, using MATHEMATICA we obtain the following properties.

Proposition 10. *The Lie-Trotter integrator (37) has the following properties:*

- (i) *It preserves the Poisson structure Π_- .*
- (ii) *It preserves the Casimirs C_1, C_2, C_3 of our Poisson configuration (g, Π_-) .*
- (iii) *It does not preserve the Hamiltonian H of our system (7).*
- (iv) *Its restriction to the coadjoint orbit $(\mathcal{O}_k, \omega_k)$, where*

$$\begin{aligned}
\mathcal{O}_k &= \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \mathbb{R}^9 / \\
&\quad x_4^2 + x_5^2 + x_6^2 = \text{const}, \\
&\quad x_1 x_4 + x_2 x_5 + x_3 x_6 = \text{const}, \\
&\quad x_7^2 + x_8^2 + x_9^2 = \text{const}\}
\end{aligned} \tag{38}$$

and ω_k is the Kirilov-Kostant-Souriau symplectic structure on $\mathcal{O}_k$ gives rise to a symplectic integrator.

6. Conclusion

The paper presents a left invariant controllable system on the Lie group $G = SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$; this arises naturally from the study of the buoyancy's dynamics for which the Lie group represents the phase space [6], as well as for the charged top dynamics (see [13]). Similar problems have been studied for a lot of Lie groups: for $SE(2, \mathbb{R}) \times SO(2)$ in [5], for $SE(1, 1)$ in [14], for $SO(4)$ in [15], for $SE(3, \mathbb{R})$ in [16], for a specific Lie group G_4 in [17], and so on. For all these examples, the Poisson geometry approach gives the geometric frame of the study and provides specific methods to obtain stability results, numerical integration using Poisson or non-Poisson integrators, or the existence of different type of periodic orbits (see [18, 19]).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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