Research Article

Finite-Time Sliding Mode Control Design for Unknown Nonaffine Pure-Feedback Systems

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A class of unknown nonaffine pure-feedback nonlinear systems is investigated and a novel output feedback control scheme with low complexity is proposed, based on the sliding mode control theory. The scheme is capable of guaranteeing output tracking error with finite-time convergence and bounded closed loop signals. In this scheme, a novel transformation method is included, which can easily transform the state-feedback control of nonaffine systems into output feedback control of strict-feedback affine systems. Based on the transformed affine systems, a novel finite-time sliding mode control is designed, which is continuous and nonsingular. The control scheme proposed in this work is simple and easy to implement, in which the "explosion of complexity" caused by backstepping-like scheme is completely avoided. And the finite-time convergence is successfully achieved. In addition, the scheme is designed based on output feedback control. And the dynamics of the nonaffine nonlinear systems is unknown in the design process. Thus, the system knowledge needed is reduced.

1. Introduction

In the past decade, control design for complex nonlinear systems has attracted a lot of attention. And many breakthrough results have been obtained, such as feedback linearization technical [1] and adaptive backstepping technical [2]. The aforementioned results assumed the input is affine in the systems considered. In practice, there are many nonlinear systems with nonaffine structure, such as aircraft dynamics [3] and mechanical systems [4]. Explicit inverting control design is impossible for nonaffine systems, because the input does not appear linearly in the systems. Therefore, the control design for nonaffine systems is more complex and more challenging.

Many researches have been done for nonaffine pure-feedback nonlinear systems. In the literatures [5–10], non-affine pure-feedback systems are studied, in which the model is known. In practice, it is difficult to obtain an exact dynamic model of the system. And in the literature [11–14], the controller design for unknown nonaffine pure-feedback systems is studied. However, the proposed control methods above are all designed based on the backstepping technical.

A drawback with the back-stepping technique is the problem of "explosion of complexity," which is caused by the repeated differentiations of certain nonlinear functions such as virtual controls [15, 16]. To overcome the problem, dynamic surface control (DSC) was proposed in the controller design for nonaffine system [17], by employing first-order filtering of the synthetic virtual control input at each step of backstepping approach. However, this method will produce an algebraic loop, because the controller is employed in the approximation algorithm, whose output is simultaneously utilized in the controller. The problem of algebraic loop is solved in [18] by combining the DSC technique and input-to-state stability-(ISS-) modular design method [6]. However, in [18] the corresponding neural network (NN) and filter are needed to be designed in each recursive step. Thus, the method is still complex. Recently, a low-complexity global approximation-free backstepping-like approach is proposed for pure-feedback nonaffine systems in [19], under the assumption that the full states are known. And the complexity explosion issue is avoided. However, it is still a recursive method. And it is usually impossible that all the states are available in the actual process. Hence, for nonaffine systems, it needs to

further study the low-complexity controller design that is not based on the backstepping technique and needs less system knowledge.

It has been recognized that sliding mode control is efficient to design a robust controller. In most of the sliding mode controllers, the sliding mode surface is linear, which can only guarantee the infinite-time convergence of the states [20, 21]. To implement finite-time convergence, nonlinear sliding mode surfaces are employed. One of such sliding mode surfaces is the terminal sliding mode (TSM) [22]. However, TSM controller may cause the singularity problem in the closed-loop system [23]. To overcome the singularity problem, the nonsingular terminal sliding mode control (NTSM) is proposed in [24, 25]. However, NTSM can only be used in the second-order system. For high-order systems, a finite-time sliding mode controller is designed in [26]. However, the sliding mode controllers mentioned above are all designed for affine systems and are difficult to directly be applied to nonaffine systems. According to the authors' best knowledge, all the sliding mode control designed for nonaffine systems [27–32] can only guarantee the infinite-time convergence of the states. The finite-time controllers have faster convergence rate than that of infinite-time controllers. Furthermore, they possess better robustness and better disturbance rejection performance, because fractional power items are contained in them [33, 34]. Thus, research on finite-time control of nonaffine systems has important practical and theoretical significance.

In this paper, a novel finite-time convergence control scheme is proposed for unknown nonaffine pure-feedback system, based on sliding mode control theory. In the design procedure, the input of the system is augmented with a low-pass filter firstly. Then the nonaffine system is transformed into an affine strict-feedback system through states transformation. Because the states in the transformed system are unavailable for controller design, a finite-time differentiator is introduced to obtain the unavailable states. Finally, a finite-time sliding mode controller is designed. Main contributions of the paper are the following:

(1) Complexity of the control for nonaffine system is reduced dramatically. A novel transformation method is proposed, capable of completely avoiding "explosion of complexity" caused by backstepping-like scheme. Moreover, in the proposed scheme, only one estimator is needed to obtain the states used in the controller.

(2) The finite-time convergence of the closed system is achieved for the first time. For nonaffine pure-feedback nonlinear systems, a nonsingular, finite-time sliding mode control is designed for the first time. And the controller is continuous; thus, the chattering of actuators is alleviated.

(3) The system knowledge needed is reduced. In the proposed scheme, only the outputs of the system, other than all the states, are used. Furthermore, the control is designed based on the unknown nonaffine pure-feedback systems; thus, accurate model of the system is not required.

2. Model and Problem Formulation

Consider a class of nonaffine pure-feedback systems as follows:

$$\dot{x}_i(t) = f_i(x_1, \ldots, x_i, x_{i+1}), \quad i = 1, \ldots, n-1,$$
$$\dot{x}_n(t) = f_n(x_1, \ldots, x_n, u), \tag{1}$$
$$y = x_1$$

in which $x_i(t) \in \mathfrak{R}$, $i = 1, \ldots, n$, are the states, $u \in \mathfrak{R}$ is the input, y denotes the output, and $f_i \in \mathfrak{R}$, $i = 1, \ldots, n$, are unknown smooth nonlinear functions.

Assumption 1. In (1), f_i, $i = 1, \ldots, n$, are continuous and differentiable, satisfying

$$b_{i0} \leq \frac{\partial f_i(x_1, \ldots, x_i, x_{i+1})}{\partial x_{i+1}} \leq b_{i1}, \quad i = 1, \ldots, n-1,$$
$$b_{n0} \leq \frac{\partial f_n(x_1, \ldots, x_n, u)}{\partial u} \leq b_{n1} \tag{2}$$

in which b_{i0} and b_{i1}, $i = 1, \ldots, n$, are positive constant value (i.e., $b_{i0} > 0$, $b_{i1} > 0$, $i = 1, \ldots, n$).

Assumption 1 is a commonly used sufficient condition for controllability of nonaffine system [19].

Assumption 2. The desired trajectory $y_d : \mathfrak{R}_+ \rightarrow \mathfrak{R}$ is known. It is sufficiently smooth, and there are constants c_i, $i = 1, \ldots, n+1$, satisfying $|y_d^{(i)}| \leq c_i$.

In this paper, we focus on the control of unknown nonaffine pure-feedback system. The objective is to obtain a low-complexity controller, such that the tracking error with respect to the desired trajectory can converge to the equilibrium point with finite time, and all the states are bounded.

3. Finite-Time Output Feedback Control Based on Sliding Mode Control Theory

3.1. Design of a Novel System Transformation Method. Recently, in [21], a low-pass filter used to attenuate the chattering is introduced in the input channel (i.e., $\dot{u} = Tu + v$), when designing the sliding mode control for affine systems. Inspired by this, we proposed a novel transformation method, in which we will show that the low-pass filter can also be used to transform the nonaffine systems into affine systems. The transformation procedure is given as follows.

Augment the nonaffine system (1) through introducing a low-pass filter in the input channel. And the augmented system can be denoted by the following:

$$\dot{x}_i(t) = f_i(x_1, \ldots, x_i, x_{i+1}), \quad i = 1, \ldots, n-1,$$
$$\dot{x}_n(t) = f_n(x_1, \ldots, x_n, u),$$
$$\dot{u} = Tu + v, \tag{3}$$
$$y = x_1.$$

In (3), $T > 0$, u becomes a new state, and v denotes the new control input.

Let $e_1 = x_1 - y_d$. And definite new states $[q_1 \ q_2 \ \cdots \ q_{n+1}]^T$, in which $q_1 = e_1$, $q_i = e^{(i-1)}$, $i = 2, \ldots, n+1$. Based on (3), one can obtain

$$\dot{q}_1 = \dot{x}_1 - \dot{y}_d,$$

$$\dot{q}_{i+1} = f_1^{(i)} - y_d^{(i+1)} = \sum_{j=1}^{i+1} \frac{\partial q_{i+1}}{\partial x_j} f_j - y_d^{(i+1)},$$

$$i = 1, \ldots, n-1, \quad (4)$$

$$\dot{q}_{n+1} = f_1^{(n)} - y_d^{(n+1)}$$

$$= \sum_{j=1}^{n} \frac{\partial q_{n+1}}{\partial x_j} f_j + \frac{\partial q_{n+1}}{\partial u} (Tu + v) - y_d^{(n+1)}.$$

Now, the initial nonaffine system denoted by (1) is transformed into an affine strict-feedback integral chain system denoted by the following:

$$\dot{q}_i = q_{i+1}, \quad i = 1, \ldots, n,$$

$$\dot{q}_{n+1} = F + Bv \quad (5)$$

in which

$$F = \sum_{j=1}^{n} \frac{\partial q_{n+1}}{\partial x_j} f_j + \frac{\partial q_{n+1}}{\partial u} Tu - y_d^{(n+1)},$$

$$B = \frac{\partial q_{n+1}}{\partial u} = \frac{\partial f_n}{\partial u} \prod_{i=1}^{n-1} \frac{\partial f_i}{\partial x_{i+1}}. \quad (6)$$

Base on Assumption 1, one can obtain $0 < \prod_{i=1}^{n} b_{i0} \le B \le \prod_{i=1}^{n} b_{i1}$. Moreover, if $|F| \le \varpi$, combing (5) and (6), the following differential inclusion can be concluded:

$$\dot{q}_{n+1} \in [-\varpi, \varpi] + [K_m, K_M] v \quad (7)$$

in which $K_m = \prod_{i=1}^{n} b_{i0}$, $K_M = \prod_{i=1}^{n} b_{i1}$.

3.2. Design of a Finite-Time Differentiator.

In transformed system (5), only the states $q_1 = e_1$ are measurable, and the states q_i, $i = 2, \ldots, n+1$, are not available. All the unmeasurable states can be obtained through solving the derivatives of q_1. In this section, a finite-time differentiator is given by

$$\dot{\varsigma}_1 = \nu_1 = -\lambda_{n+1} L^{1/(n+1)} \left| \varsigma_1 - q_1 \right|^{n/(n+1)} \mathrm{sign} (\varsigma_1 - q_1)$$

$$+ \varsigma_2$$

$$\vdots \quad (8)$$

$$\dot{\varsigma}_n = \nu_n = -\lambda_2 L^{1/2} \left| \varsigma_n - \nu_{n-1} \right|^{1/2} \mathrm{sign} (\varsigma_n - \nu_{n-1}) + \varsigma_{n+1}$$

$$\dot{\varsigma}_{n+1} = \nu_{n+1} = -\lambda_1 L \, \mathrm{sign} (\varsigma_{n+1} - \nu_n),$$

where L is a Lipschitz constant of $q_1^{(n+1)}$ and ς_i is the estimate of $q_1^{(i-1)}$.

Define the estimated errors and their derivatives as $e_i = \varsigma_i - q_1^{(i-1)} = \varsigma_i - q_i$, $i = 1, \ldots, n+1$. The following dynamic equations can be obtained:

$$\dot{e}_1 = -\lambda_n L^{1/(n+1)} \left| e_1 \right|^{n/(n+1)} \mathrm{sign} (e_1) + e_2$$

$$\vdots \quad (9)$$

$$\dot{e}_n = -\lambda_1 L^{1/2} \left| e_n - \dot{e}_{n-1} \right|^{1/2} \mathrm{sign} (e_n - \dot{e}_{n-1}) + e_n$$

$$\dot{e}_{n+1} \in -\lambda_0 L \mathrm{sign} (e_{n+1} - \dot{e}_n) + [-L, L].$$

It follows from [35] that the estimated errors can converge to zero in finite time.

3.3. Design of a Nonsingular Finite-Time Sliding Mode Control.

A nonlinear sliding mode surface is given by

$$s = \varsigma_{n+1} + \int_0^t \left[\sum_{i=1}^{n+1} k_i \, \mathrm{sign} (\varsigma_i) \left| \varsigma_i \right|^{\alpha_i} \right] dt \quad (10)$$

in which k_i, $i = 1, \ldots, n+1$, can ensure that the polynomial $p^{n+1} + k_{n+1} p^n + \cdots + k_2 p + k_1 = 0$ is Hurwitz, $\alpha_i \in (0, 1)$, $i = 1, \ldots, n+1$, satisfying $\alpha_{i-1} = \alpha_i \alpha_{i+1}/(2\alpha_{i+1} - \alpha_i)$, $i = 2, \ldots, n+1$, $\alpha_{n+2} = 1$, $\alpha_{n+1} = \alpha_0 \in (0, 1)$.

Theorem 3. *Consider nonaffine pure-feedback system (1), which satisfies Assumptions 1 and 2. All the initial states $(x_1(t_0), x_2(t_0), \ldots, x_n(t_0)) \in \varphi_0$ and $\varphi_0 \in \mathfrak{R}^n$ is a compact set. When choosing the control input denoted by*

$$\dot{u} = Tu + v,$$

$$v = v_1 + v_2,$$

$$v_1 = -B_0^{-1} \sum_{i=1}^{n+1} k_i \, \mathrm{sign} (\varsigma_i) \left| \varsigma_i \right|^{\alpha_i}, \quad (11)$$

$$v_2 = -B_0^{-1} \lambda s - B_0^{-1} \psi \, \mathrm{sign} (s),$$

$$\psi = \left(\eta + B_0 \left(\rho + \varepsilon_1 \left| v_1 \right| \right) \right) (1 - \varepsilon_0)^{-1}$$

in which the sliding mode surface s is designed as (10), and ς_i, $(i = 0, 1, \ldots, n)$, are obtained by differentiator (8), $T > 0$, $B_0 > 0$, satisfying $\prod_{i=1}^{n} b_{i1} - B_0 > B_0 - \prod_{i=1}^{n} b_{i0} > 0$, $0 < \varepsilon_0 \le B_0^{-1}(B_0 - \prod_{i=1}^{n} b_{i0})$, $\lambda > 0$, $\varepsilon_1 \ge B_0^{-1}(\prod_{i=1}^{n} b_{i1} - B_0)$, $\eta > 0$, ρ, satisfying $B_0 \rho > F$, then the output tracking error $q_1 = x_1 - y_d$ will converge to zero in finite time, and all the signals in the close-loop system are bounded.

Proof. According to $e_i = \varsigma_i - q_1^{(i-1)} = \varsigma_i - q_i$, $i = 1, \ldots, n+1$, one can obtain $\zeta_{n+1} = q_{n+1} + e_{n+1}$. Then combing (5) and (10) yields

$$\dot{s} = \dot{\varsigma}_{n+1} + \sum_{i=1}^{n+1} k_i \operatorname{sign}(\varsigma_i) |\varsigma_i|^{\alpha_i}$$

$$= F + Bv + \dot{e}_{n+1} + \sum_{i=1}^{n+1} k_i \operatorname{sign}(\varsigma_i) |\varsigma_i|^{\alpha_i} \tag{12}$$

$$= F + B_0 (1 + \varepsilon) v + \dot{e}_{n+1} + \sum_{i=1}^{n+1} k_i \operatorname{sign}(\varsigma_i) |\varsigma_i|^{\alpha_i}.$$

Because $B_0 > 0$, satisfying $\prod_{i=1}^{n} b_{i1} - B_0 > B_0 - \prod_{i=1}^{n} b_{i0} > 0$, $0 < \varepsilon_0 \le B_0^{-1}(B_0 - \prod_{i=1}^{n} b_{i0})$, $\varepsilon_1 \ge B_0^{-1}(\prod_{i=1}^{n} b_{i1} - B_0)$, it implies that $\varepsilon \in [-\varepsilon_0, \varepsilon_1]$, $0 < \varepsilon_0 < 1$, $\varepsilon_1 > 0$, $\varepsilon_1 > \varepsilon_0$.

Noting $e_i = \varsigma_i - q_1^{(i-1)} = \varsigma_i - q_i$, $i = 1, \ldots, n+1$, and combing (10), the dynamic of ς_i, $(i = 1, \ldots, n+1)$ can be obtained as follows:

$$\dot{\varsigma}_i = \varsigma_{i+1} + \widetilde{e}_i, \quad i = 1, \ldots, n,$$

$$\dot{\varsigma}_{n+1} = \dot{s} - \sum_{i=1}^{n+1} k_i \operatorname{sign}(\varsigma_i) |\varsigma_i|^{\alpha_i}, \tag{13}$$

where $\widetilde{e}_i = \dot{e}_i - e_{i+1}$, $i = 1, \ldots, n$.

It denotes that dynamic (12) and dynamic (13) are affected by the estimated error dynamic (9). In the following, it will show that dynamic (12) and dynamic (13) will not be driven into infinity in finite time by dynamic (9).

Define a finite-time bounded function as follows:

$$V = \frac{1}{2} \sum_{i=1}^{n+1} \varsigma_i^2 + \frac{1}{2} s^2. \tag{14}$$

Taking derivative of V with respect to time, then substituting (13), yields

$$\dot{V} = \sum_{i=1}^{n} \varsigma_i (\varsigma_{i+1} + \widetilde{e}_i) + \varsigma_{n+1} \left(\dot{s} - \sum_{i=1}^{n+1} k_i \operatorname{sign}(\varsigma_i) |\varsigma_i|^{\alpha_i} \right) \tag{15}$$

$$+ s\dot{s}.$$

Noting that when $\alpha_i \in (0, 1)$ $(i = 1, \ldots, n+1)$, it implies that $|\varsigma_i|^{\alpha_i} \le 1 + |\varsigma_i|$, and combing (15), one can obtain

$$\dot{V} \le \sum_{i=1}^{n} |\varsigma_i| (|\varsigma_{i+1}| + |\widetilde{e}_i|)$$

$$+ |\varsigma_{n+1}| \left(|\dot{s}| + \sum_{i=1}^{n+1} k_i (1 + |\varsigma_i|) \right) + |s| |\dot{s}|$$

$$\le \frac{1}{2} \sum_{i=1}^{n} (\varsigma_i^2 + \varsigma_{i+1}^2) + \frac{1}{2} \sum_{i=1}^{n} (\varsigma_i^2 + \widetilde{e}_i^2) + |\varsigma_{n+1}| |\dot{s}| \tag{16}$$

$$+ \frac{1}{2} \varsigma_{n+1}^2 + \frac{1}{2} \left(\sum_{i=1}^{n+1} k_i \right)^2 + \frac{1}{2} \sum_{i=1}^{n+1} k_i (\varsigma_{n+1}^2 + \varsigma_i^2)$$

$$+ |s| |\dot{s}| \le l_1 V + l_2 + (|\varsigma_{n+1}| + |s|) |\dot{s}|.$$

in which

$$l_1 = 3 + 2 \left(\sum_{i=1}^{n+1} k_i \right),$$

$$l_2 = \sum_{i=1}^{n} \frac{\widetilde{e}_i^2}{2} + \frac{1}{2} \left(\sum_{i=1}^{n+1} k_i \right)^2. \tag{17}$$

Substituting (11) into (12) yields

$$\dot{s} = - (1 - \varepsilon_0)^{-1} (1 + \varepsilon)$$

$$\cdot \left(\eta + B_0 \rho + \varepsilon_1 \left| \sum_{i=1}^{n+1} k_i |\varsigma_i|^{\alpha_i} \operatorname{sign}(\varsigma_i) \right| \right) \operatorname{sign}(s) \tag{18}$$

$$- \varepsilon \sum_{i=1}^{n+1} k_i \operatorname{sign}(\varsigma_i) |\varsigma_i|^{\alpha_i} - (1 + \varepsilon) \lambda s + F + \dot{e}_{n+1}.$$

Note that $\varepsilon \in [-\varepsilon_0, \varepsilon_1]$, $0 < \varepsilon_0 < 1$, $\varepsilon_1 > \varepsilon_0 > 0$, and $|F| < B_0 \rho$ and $|\varsigma_i|^{\alpha_i} \le 1 + |\varsigma_i|$. And from (18), one can obtain

$$|\dot{s}|$$

$$\le (1 - \varepsilon_0)^{-1} (1 + \varepsilon) \left(\eta + B_0 \rho + \varepsilon_1 \sum_{i=1}^{n+1} k_i |\varsigma_i|^{\alpha_i} \right)$$

$$+ \varepsilon \sum_{i=1}^{n+1} k_i |\varsigma_i|^{\alpha_i} + (1 + \varepsilon) \lambda |s| + |F| + |\dot{e}_{n+1}|$$

$$\le (1 - \varepsilon_0)^{-1} (1 + \varepsilon_1) \left(\eta + B_0 \rho + \varepsilon_1 \sum_{i=1}^{n+1} k_i (1 + |\varsigma_i|) \right) \tag{19}$$

$$+ \varepsilon_1 \sum_{i=1}^{n+1} k_i (1 + |\varsigma_i|) + (1 + \varepsilon_1) \lambda |s| + B_0 \rho + |\dot{e}_{n+1}|$$

$$\le \vartheta_1 + \vartheta_2 \sum_{i=1}^{n+1} k_i |\varsigma_i| + (1 + \varepsilon_1) \lambda |s|.$$

In (19),

$$\vartheta_1 = (1 - \varepsilon_0)^{-1} (1 + \varepsilon_1) (\eta + B_0 \rho)$$

$$+ \frac{(2 + \varepsilon_1 - \varepsilon_0)}{(1 - \varepsilon_0)} \varepsilon_1 \left(\sum_{i=1}^{n+1} k_i \right) + B_0 \rho + |\dot{e}_{n+1}|, \tag{20}$$

$$\vartheta_2 = \frac{(2 + \varepsilon_1 - \varepsilon_0)}{(1 - \varepsilon_0)} \varepsilon_1 > 0.$$

Substituting (19) into (16) yields

$$\dot{V} \le l_1 V + l_2 + \left(|\varsigma_{n+1}| + |s| \right) \left(\vartheta_1 + \vartheta_2 \sum_{i=1}^{n+1} k_i |\varsigma_i| \right)$$

$$+ \left(|\varsigma_{n+1}| + |s| \right) (1 + \varepsilon_1) \lambda |s|$$

$$\le l_1 V + l_2 + \left(|\varsigma_{n+1}| \vartheta_1 + |s| \vartheta_1 \right)$$

$$+ \vartheta_2 \sum_{i=1}^{n+1} k_i \left(|\varsigma_i| |\varsigma_{n+1}| + |\varsigma_i| |s| \right)$$

$$+ (1 + \varepsilon_1) \lambda \frac{\varsigma_{n+1}^2 + 3s^2}{2} \tag{21}$$

$$\le l_1 V + l_2 + \vartheta_1^2 + \frac{\varsigma_{n+1}^2 + s^2}{2}$$

$$+ \vartheta_2 \left(\sum_{i=1}^{n+1} k_i \frac{\varsigma_{n+1}^2 + 2\varsigma_i^2 + s^2}{2} \right)$$

$$+ (1 + \varepsilon_1) \lambda \frac{\varsigma_{n+1}^2 + 3s^2}{2} \le (l_1 + l_3) V + (l_2 + l_4).$$

Noting (21) and (17), one can obtain

$$l_1 + l_3 = 4 + (2 + 3\vartheta_2) \left(\sum_{i=1}^{n+1} k_i \right) + 3(1 + \varepsilon_1) \lambda,$$

$$l_2 + l_4 = \vartheta_1^2 + \sum_{i=1}^{n} \frac{\widetilde{e}_i^2}{2} + \frac{1}{2} \left(\sum_{i=1}^{n+1} k_i \right)^2, \tag{22}$$

where e_i, $i = 1, \ldots, n + 1$, in (9) can converge to zero in finite time; thus, $\dot{e}_{n+1}$ and $\widetilde{e}_i$, $i = 1, \ldots, n$, are bounded. Consequently, ϑ_1 and ϑ_2 denoted by (20) are bounded. And $l_1 + l_3$ and $l_2 + l_4$ in (22) are bounded. Thus, it can be concluded from (21) and (14) that V, ς_i $(i = 1, \ldots, n + 1)$ and s will not escape to infinity in finite time before $\widetilde{e}_i$, $i = 1, \ldots, n$, converge to zero.

Because $\widetilde{e}_i$, $i = 1, \ldots, n$, will converge to zero in finite time, system (12) will then reduce to

$$\dot{s} = F + B_0 (1 + \varepsilon) v + \sum_{i=1}^{n+1} k_i \operatorname{sign} (\varsigma_i) |\varsigma_i|^{\alpha_i}. \tag{23}$$

Substituting (11) into (23) yields

$$\dot{s} = B_0 \varepsilon v_1 + B_0 (1 + \varepsilon) v_2 + F. \tag{24}$$

Noting that $\varepsilon \in [-\varepsilon_0, \varepsilon_1]$, $\varepsilon_1 > \varepsilon_0$, $0 < \varepsilon_0 < 1$, $\varepsilon_1 > \varepsilon_0 > 0$ and $\lambda > 0$, and further substituting (11) into (24), it follows that

$$s\dot{s} = s B_0 \varepsilon v_1 + s B_0 (1 + \varepsilon) v_2 + s F$$

$$\le B_0 |s| |\varepsilon| |v_1| - |s| (1 + \varepsilon) \psi - (1 + \varepsilon) \lambda s^2 + |s| |F| \tag{25}$$

$$\le B_0 \varepsilon_1 |s| |v_1| - |s| (1 - \varepsilon_0) \psi + |s| |F|.$$

Substituting ψ denoted by (11) into (25), and combing with $B_0 \rho > F$, yields

$$s\dot{s} \le B_0 \varepsilon_1 |s| |v_1| - |s| (\eta + B_0 \rho + B_0 \varepsilon_1 |v_1|) + |s| |F|$$

$$\le -\eta |s| - |s| (B_0 \rho - |F|) \le -\eta |s|. \tag{26}$$

According to (26), we know that the sliding mode surfaces s and $\dot{s}$ will converge to zero in finite time. And (9) implies that $\widetilde{e}_i$ will converge to zero with finite time. So dynamic (13) will reduce to

$$\dot{\varsigma}_i = \varsigma_{i+1}, \quad i = 1, \ldots, n,$$

$$\dot{\varsigma}_{n+1} = -\sum_{i=1}^{n+1} k_i \operatorname{sign} (\varsigma_i) |\varsigma_i|^{\alpha_i}. \tag{27}$$

From (27), it can be concluded that ς_i $(i = 1, \ldots, n + 1)$ and $\dot{\varsigma}_{n+1}$ will converge to zero in finite time [21]. And at this time $q_i = \varsigma_i$ $(i = 1, \ldots, n + 1)$ denote the tracking error and its derivatives, so the tracking error and its derivatives are bounded and can converge to zero in finite time. Because ς_i, $i = 1, \ldots, n + 1$, are all bounded, the new input v in (11) is bounded. And according to the bounded input and bounded output properties of one-order linear system [36], u produced by $\dot{u} = Tu + v$ in (11) is bounded too. Because $q_1 = x_1 - y_d$ is bounded, state x_1 is bounded. Furthermore, because $q_2 = f_1 - \dot{y}_d$ is bounded, thus f_1 is bounded. Then it can be concluded that x_2 is bounded according to the fact that $\partial f_1 / \partial x_2$ is bounded. Through using the procedure recursively, it can be concluded that all the other states x_i, $i = 2, \ldots, n$, are bounded too. □

Remark 4. Compared with the commonly used backstepping-like controller, the complexity is reduced dramatically in the proposed controller through using a novel system transformation. And the finite-time convergence of the tracking error can also be insured. The system knowledge needed is reduced, and only the output state is used in the proposed controller, while all the states are usually needed in the backstepping-like controller.

4. Simulation

In this section, the feasibility of the scheme proposed in this paper will be demonstrated. Consider the following three-order system:

$$\dot{x}_1 = 0.2x_1 + (1 + \exp(-x_1)) x_2 + 0.3 \sin x_1$$

$$+ 0.1 \sin t,$$

$$\dot{x}_2 = \frac{1 - \exp(x_1 x_2)}{1 + \exp(x_1 x_2)} + (1 + 0.2 \sin x_1) x_3$$

$$+ 0.02 \sin(x_2 t), \tag{28}$$

$$\dot{x}_3 = 0.5 \sin(x_1 x_2 x_3) + (0.42 + 0.1 \cos(x_2 t)) u$$

$$+ 0.1 \sin(0.5u),$$

$$y = x_1$$
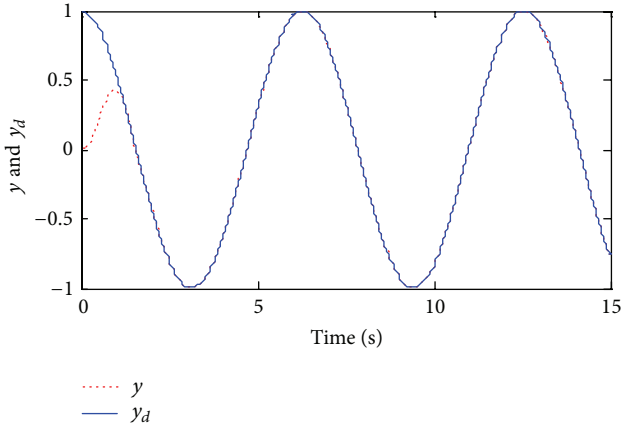
FIGURE 1: Actual output y and desired output y_d.
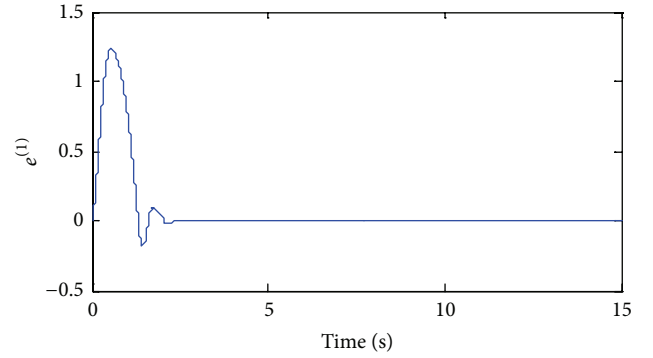

FIGURE 2: Tracking error.

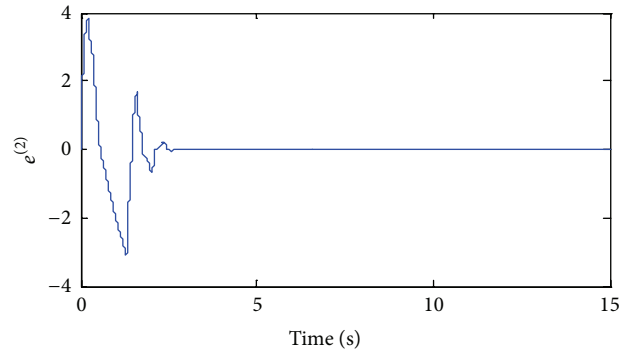FIGURE 3: The first-order derivative of tracking error.

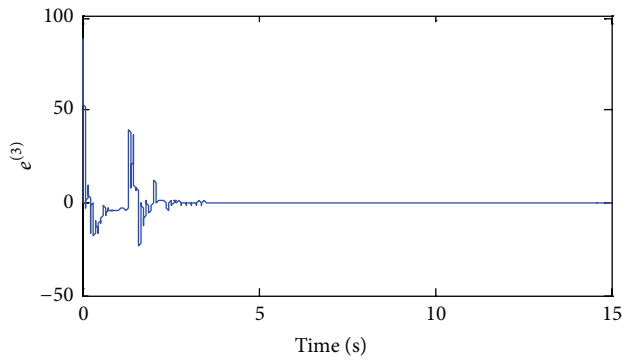

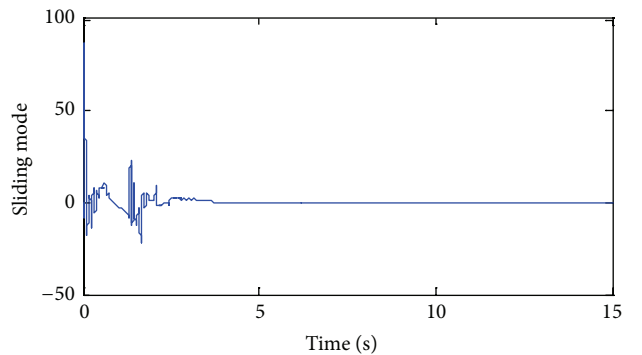FIGURE 4: The second-order derivative of tracking error.

FIGURE 5: The third-order derivative of tracking error.
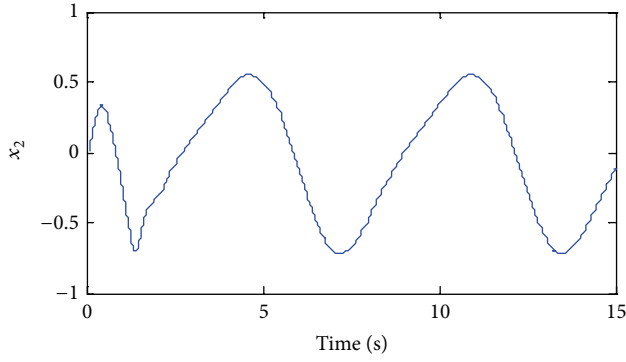

FIGURE 6: The sliding mode surface.

with the desired trajectory $y_d = \sin(t + \pi/2)$ and initial conditions $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 0$.

It can be verified that Assumptions 1 and 2 are satisfied. In the controller design, the differentiator is firstly designed as (8) with the parameters: $L = 300$, $\lambda_0 = 1.1$, $\lambda_1 = 30$, $\lambda_2 = 50$, and $\lambda_2 = 80$. The sliding mode surface is given by (10) with $k_1 = 625$, $k_2 = 500$, $k_3 = 150$, $k_4 = 40$, $\alpha_1 = 1/2$, $\alpha_2 = 4/7$, $\alpha_3 = 2/3$, $\alpha_4 = 4/5$. And the initial value of the integral part is 0. The controller is given by (11); in (11), $T = -5$, $\lambda = 20$, $B_0 = 0.6$, $\eta = 1$, $\rho = 3$, $\varepsilon_1 = 6$, $\varepsilon_0 = 0.8$. The simulation results are shown as in Figures 1–10.

As a comparison, an improved backstepping-like controller designed in [18] is also simulated. In [18] the complexity issue is alleviated through using the dynamic surface technique, which can avoid calculating the derivative of the virtual control. The controller is designed as $u = -k_3 s_3 + \widehat{\theta}_3^T \xi_3(x_1, x_2, x_3, z_3)$, in which $s_3 = x_3 - z_3$ and z_3 is tuned by one-order filter $\tau_2 \dot{z}_3 + z_3 = \alpha_3$. $\alpha_3 = -k_2 s_2 + \widehat{\theta}_2^T \xi_2(x_1, x_2, \dot{z}_2)$ is a virtual control, in which $s_2 = x_2 - z_2$, and z_2 is tuned by one-order filter $\tau_2 \dot{z}_2 + z_2 = \alpha_2$. $\alpha_2 = -k_1 s_1 + \widehat{\theta}_1^T \xi_1(x_1, \dot{y}_d)$ is a virtual control, in which $s_1 = x_1 - y_d$. $\theta_i^T \xi_i$, $i = 1, 2, 3$, are three NN units. And the NN weights are determined as $\dot{\widehat{\theta}}_i = \Gamma_i[-\xi_i s_i - \eta_i \widehat{\theta}_i]$, $i = 1, 2, 3$. The comparison of the two methods is illustrated in Figures 11-12.
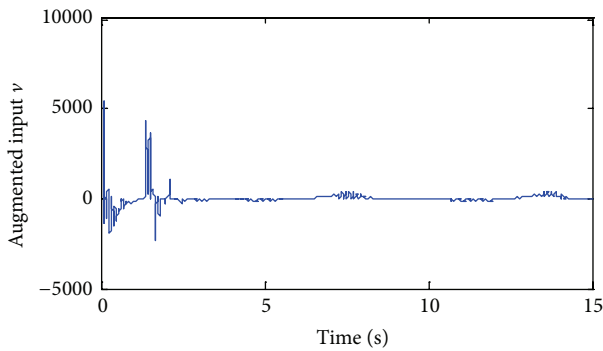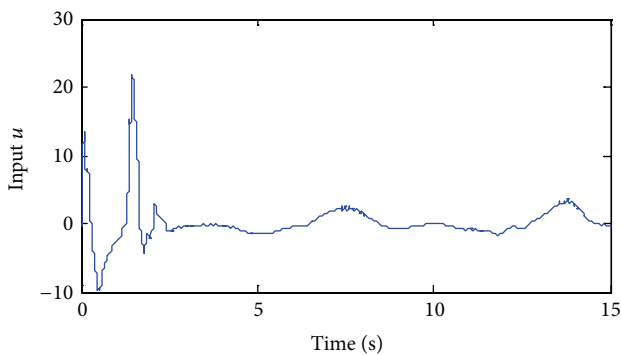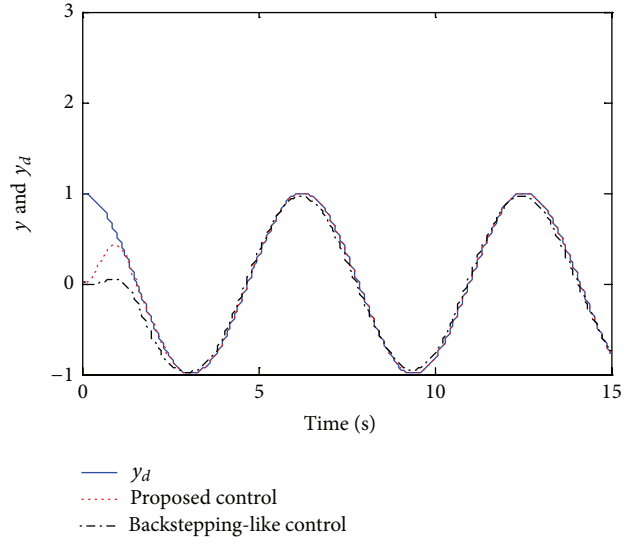
FIGURE 7: The closed loop system state x_2.
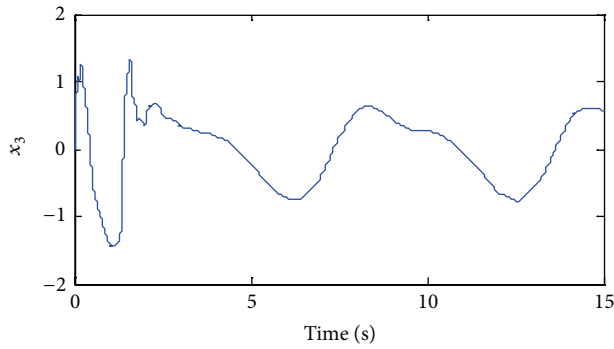
FIGURE 8: The closed loop system state x_3.

FIGURE 9: Augmented input v.

FIGURE 10: Actual input u.

— y_d
···· Proposed control
-·-· Backstepping-like control

FIGURE 11: Tracking performance of the two controllers.
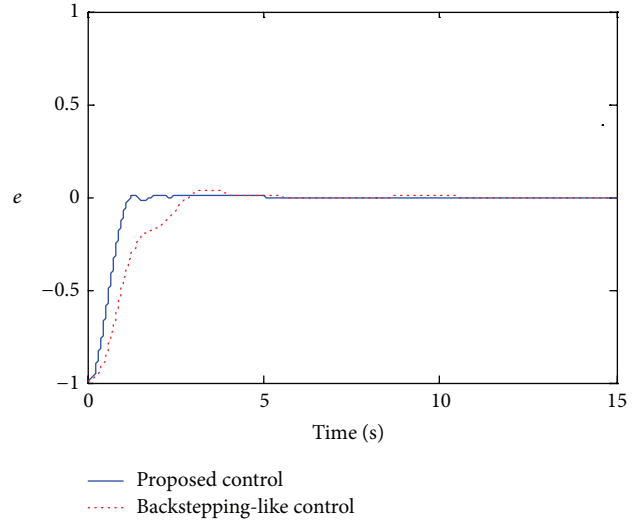

— Proposed control
···· Backstepping-like control

FIGURE 12: Tracking error of the two controllers.

According to the simulation results, the following can be concluded:

(1) The proposed controller can insure the tracking error and its derivatives have a finite-time convergence property (Figures 2–5). This is because the proposed controller can insure a finite-time convergence of the sliding mode surface (Figure 6); moreover, the sliding mode surface in this paper is a nonsingular finite-time terminal sliding mode surface, so the states on the sliding mode surface can converge to zero in finite time.

(2) The proposed sliding mode controller is chattering-free. Though the augmented input is desecrated (Figure 9), the actual input u is continuous and smooth, because u is produced by $\dot{u} = Tu + v$.

Thus, the chattering of the conventional sliding mode control is avoided.

(3) The proposed controller is simple and easy to implement. And the complexity of the current controllers designed for nonaffine systems is greatly avoided. It is noted that three NN units, two filters, and all the three states are needed in the backstepping-like controller proposed in [18]. And the number of NN units and filters will increase with the increase of the system order. However, only one output state, one differentiator, and one filter are needed in the controller proposed in this paper. In addition, the number of the needed states, differentiators, and filters will not increase with the increase of the system order. So the complexity of the controller for nonaffine pure-feedback systems is greatly avoided.

(4) Compared to the controller in [18], the needed knowledge of system is reduced. In the proposed controller, only the output and the order of the system, other than all the states and the accurate system model, are needed.

(5) According to Figures 11-12, it is shown that the control performance of the proposed controller in this paper is as good as that of the improved backstepping-like controller proposed in [18].

5. Conclusion

In the paper, a novel control scheme with low complexity is proposed for unknown nonaffine pure-feedback systems. The scheme involves a novel transformed method, which can transform the nonaffine systems into affine strict-feedback system. And a nonsingular finite-time sliding mode controller is also implemented. Compared with the related article, the proposed scheme not only can achieve finite-time convergence, but also is much simpler and easy to implement and needs less system knowledge. And the simulation results demonstrate that the tracking performance is also very good.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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