Research Article

Robust $H_\infty$ Filtering for Discrete-Time Markov Jump Linear System with Missing Measurements

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The problem of robust $H_\infty$ filtering is investigated for discrete-time Markov jump linear system (DMJLS) with uncertain parameters and missing measurements. The missing measurements process is modelled as a Bernoulli distributed sequence. A robust $H_\infty$ filter is designed and sufficient conditions are established in terms of linear matrix inequalities via a mode-dependent Lyapunov function approach, such that, for all admissible uncertain parameters and missing measurements, the resulting filtering error system is robustly stochastically stable and a guaranteed $H_\infty$ performance constraint is achieved. Furthermore, the optimal $H_\infty$ performance index is subsequently obtained by solving a convex optimisation problem and the missing measurements effects on the $H_\infty$ performance are evaluated. A numerical example is given to illustrate the feasibility and effectiveness of the proposed filter.

1. Introduction

Discrete-time Markov jump linear system (DMJLS) is an important type of stochastic hybrid system, of which the parameters jumping is governed according to a finite state Markov chain. Therefore, DMJLS has both obvious continuous and discrete dynamics and is commonly used to model problems with abrupt variations in practical engineering system structures and parameters, which may be caused by random failures or sudden violent environments changes in a wide variety of areas, such as electrical engineering, signal processing, target tracking, and multifault diagnosis (see [1–5] and references therein). During the past decades, DMJLS has attracted more and more increasing research interest, among which the state estimation comprises an important research issue and has found many practical applications. Take a multifault diagnosis problem, for example. The system parameters may change under different fault condition and the fault switching always presents Markovian characteristic. Thus, the problem of state estimation for a multifault system can be modelled as filtering for DMJLS.

Among the DMJLS state estimation methods, the interacting multiple model (IMM) method, proposed in [6], is considered to be one of the most representative suboptimal estimators and performs much more cost-effectively than the other recursive methods. However, the parametric uncertainties often occur in many practical engineering systems. Unfortunately, IMM has some considerable drawbacks lying in its inevitable dependence on exact system model and noise statistic restriction with known Gaussian distribution. Nowadays, more and more attention has been focused on the $H_\infty$ filter, which can deal with the state estimation problem for DMJLS with uncertain models and unknown but energy bounded external noises [7]. The $H_\infty$ filter is briefly described as the design of a filter for a given system such that the $L_2$-induced gain from the exogenous disturbance to the filter error is below a prescribed level. Due to its loose requirement on the system model and noise statistics, more and more attention has been attracted and significant advances have been made in the $H_\infty$ filtering method.

On the other hand, besides the parametric uncertainties, in practical applications, the measured outputs are also usually subject to the uncertainties of randomly occurring missing phenomenon due to kinds of reasons, such as the sensor fault, external disturbance, or network data transmission loss [8, 9], where only noise is regarded as the
original measurements to motivate the estimator, without the prior knowledge of whether the true signals are contained or not. As to the problem of state estimation with missing measurements, a Bernoulli distribution is usually adopted to describe the missing behavior, which is considered to be firstly proposed in [10], wherein an optimal recursive filter is presented for systems with missing measurements. Another model for the missing measurements phenomenon is a Markovian jumping process. A jump linear estimator in terms of a predictor is proposed in [11], which can select a corrector gain at each time. In [12], an IMM based distributed filter for DMJLS with missing measurements is derived by describing the switching of system mode and missing measurements state with two independent Markov chains, and thus, a new overall Markov chain can be obtained by the product of the above two mode sets.

Although more and more efforts have been tried on the problem of filtering with missing measurements, almost all existing results are concerned with linear system [13–16]. The issues involved with DMJLS have not been fully investigated. While taking the parameter uncertainties into account simultaneously, the results of filtering for DMJLS are still scarce up to now, to the best of the authors’ knowledge. The main difficulty to deal with such a problem lies in how to incorporate the probabilistic missing measurement into a robust estimation framework. Thus, this intuition motivates this paper to initiate the research on considering parametric uncertainties, missing measurements in the state estimation for DMJLS.

This paper is concerned with the problem of $H_{\infty}$ filtering for DMJLS with parametric uncertainties and missing measurements. The missing measurements considered in this paper follow a Bernoulli switching process. A robust $H_{\infty}$ filter is designed and sufficient conditions for the existence of a feasible solution to the problem are discussed by using a mode-dependent Lyapunov function approach, such that for all possible uncertain parameters and missing measurements, the resulting filtering error system is robustly stochastically stable and the estimation error bound is with a guaranteed $H_{\infty}$ attenuation level. Furthermore, the optimal $H_{\infty}$ performance index is subsequently obtained by solving a convex optimisation problem and the missing measurements effects on $H_{\infty}$ performance are evaluated. The main contribution of this paper lies in considering the impacts of uncertain parameters and missing measurements simultaneously in the problem of state estimation for DMJLS and both the stochastic and deterministic problem are dealt with in a unified framework, which is important and challenging in both practice and theory.

The rest of the paper is organized as follows. In Section 2, the main problem is formulated and some preliminaries are given. In Section 3, the performance of the robust mode-dependent $H_{\infty}$ filter is analyzed. The filter design problem is tackled in Section 4. Some illustrative simulation results are given in Section 5. Lastly, the conclusion is drawn in Section 6.

Notation 1. In this paper, the notations used are standard. The subscript "T" denotes matrix transposition. Both $\mathbb{R}^{m \times 1}$ and $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space. $\| \cdot \|$ represents the Euclidean norm. $I$ is the identity matrix with appropriate dimensions. $L_2[0, +\infty)$ is the space of square-summable vector functions over $[0, +\infty)$. $*$ means the symmetric terms in a symmetric matrix. For a real matrix $P$, $P > 0$ means that $P$ is symmetric and positive definite. $E[x]$ denotes the mathematical statistical expectation of a stochastic variable $x$ and $E[x \mid y]$ means the conditional expectation of $x$ given $y$.

2. Problem Formulation and Preliminary

For a given probability space $(\Omega, F, P)$, where $\Omega$ represents the sample space, $F$ is the algebra of events, and $P$ is the probability measure defined on $F$, consider the following DMJLS with parametric uncertainties and missing measurements:

$$
\begin{align*}
    x_{k+1} &= [A(\theta_k) + \Delta A(k, \theta_k)] x_k \\
             &+ [B(\theta_k) + \Delta B(k, \theta_k)] w_k, \\
    y_k &= \xi_k C(\theta_k) x_k + D(\theta_k) w_k, \\
    z_k &= L(\theta_k) x_k,
\end{align*}
$$

(1)

where $x_k \in \mathbb{R}^{m \times 1}$ is the system state vector; $y_k \in \mathbb{R}^{m \times 1}$ denotes the measured output; $w_k$ is the external noise which belongs to $L_2[0, +\infty)$; $z_k \in \mathbb{R}^{p \times 1}$ is the objective signal to be estimated; and $\{\theta_k, k \in N\}$ is a discrete-time homogeneous Markov chain taking values in a finite set $S = \{1, \ldots, s\}$ with transition probability $\pi_{ij} = \text{Prob}(\theta_k = j \mid \theta_{k-1} = i)$ for all $i, j \in S$.

$\Delta A(k, \theta_k)$ and $\Delta B(k, \theta_k)$, denoting the norm-bounded parameter uncertainties, are unknown real-valued matrices with appropriate dimensions corresponding to each mode satisfying

$$
[\Delta A(k, \theta_k) \quad \Delta B(k, \theta_k)] = M(\theta_k) F(k, \theta_k) \begin{bmatrix} N_a(\theta_k) & N_b(\theta_k) \end{bmatrix},
$$

(2)

$$
F(k, \theta_k) F^T(k, \theta_k) \leq I,
$$

\[ \forall k > 0, \]

where $M(\theta_k), N_a(\theta_k),$ and $N_b(\theta_k)$ are known real constant matrices of appropriate dimensions corresponding to each mode and $F(k, \theta_k)$ is unknown matrix representing the mode-dependent uncertainties. The model of parametric uncertainties actually comes from the tolerance of physical system parameters, which can be modelled in norm-bounded form. This form of uncertainties is commonly used in robust filtering and control [17, 18].

When the system is in mode $i \in S$, that is, $\theta_k = i$, the mode-dependent system matrices are notated for simplification as follows:

$$
\begin{align*}
    A_i &:= A(\theta_k = i), \\
    B_i &:= B(\theta_k = i), \\
    C_i &:= C(\theta_k = i),
\end{align*}
$$
Different from the Markov process model in [11, 12], \( \xi_k \), denoting the measurements state (missing or normal), is a Bernoulli distributed sequence [10] taking the values of 1 and 0 with

\[
\text{Prob}\{\xi_k = 1\} = E\{\xi_k\} = \beta, \\
\text{Prob}\{\xi_k = 0\} = 1 - E\{\xi_k\} = 1 - \beta,
\]

and \( \beta \in [0, 1] \) is a known positive scalar, noting that [15]

\[
E\{\xi_k - \beta\} = 0, \\
E\{(\xi_k - \beta)^2\} = \beta (1 - \beta).
\]

It is obvious that \( \xi_k = 0 \) directly describes the missing measurements phenomenon. It is worth mentioning that the filter cannot distinguish whether the valid measurement is missing or not.

Consider the following mode-dependent \( H_{\infty} \) filtering system, which is a DMJLS adaptation for the filter model in [15]:

\[
\tilde{x}_{k+1} = G_i \tilde{x}_k + K_i (y_k - \beta C_i \tilde{x}_k), \\
\tilde{z}_k = H_i \tilde{x}_k,
\]

where \( \tilde{x}_k \in \mathbb{R}^{m \times 1} \) denotes the filter state vector, \( y_k \in \mathbb{R}^{m \times 1} \) is the input of the filter, and \( \tilde{z}_k \) is the estimation of \( z_k \) and it is mentioned that \( G_i, K_i, \) and \( H_i \) are the mode-dependent \( H_{\infty} \) filter parameters to be determined.

Augment the model of (1) to include the states of the filter (6)-(7), and the filtering error system is obtained below:

\[
\bar{x}_{k+1} = \bar{A}_i \bar{x}_k + (\xi_k - \beta) \bar{A}_{ii} \bar{x}_k + \bar{B}_i w_k, \\
e_k = \bar{L}_i \bar{x}_k,
\]

where \( \bar{x}_k = [x_k^T \; \tilde{x}_k^T]^T, \; \bar{z}_k = z_k - \tilde{z}_k \), and the parameters of the augmented systems are given by

\[
\bar{A}_i = \begin{bmatrix} A_i + \Delta A_{ik} & 0 \\ \beta K_i C_i & G_i - \beta K_i C_i \end{bmatrix}, \\
\bar{B}_i = \begin{bmatrix} B_i \\ \beta K_i D_i \end{bmatrix}, \\
\bar{M}_i = \begin{bmatrix} M_i \\ 0 \end{bmatrix}, \\
\bar{N}_{ai} = \begin{bmatrix} N_{ai} \\ 0 \end{bmatrix}, \\
\bar{N}_{bi} = N_{bi}.
\]

\( \Delta A_{ik} = \Delta A(k,\theta_k = i), \Delta B_{ik} = \Delta B(k,\theta_k = i), F_{ik} = F(k,\theta_k = i) \), \( M_i = M(\theta_k = i) \), \( N_{ai} = N_a(\theta_k = i) \), \( N_{bi} = N_b(\theta_k = i) \).

\begin{equation}
E\left\{ \sum_{k=0}^{\infty} \| \bar{z}_k \|^2 \mid \mathcal{F}_0, \theta_0 \right\} < \infty. \tag{11}
\end{equation}

\textbf{Definition 1} (see [7]). System (8)-(9) is said to have robustly stochastic stability if in the case of \( u_k \equiv 0 \) for every initial condition \( x_0 \in \mathbb{R}^2 \) and \( \theta_0 \in \mathcal{S} \), the following inequality holds:

\begin{equation}
E\left\{ \sum_{k=0}^{\infty} \| \bar{z}_k \|^2 \mid \mathcal{F}_0, \theta_0 \right\} < \infty. \tag{11}
\end{equation}

\textbf{Definition 2} (see [7]). Given a scalar \( \gamma > 0 \), system (8)-(9) satisfies an \( H_{\infty} \) noise attenuation performance index \( \gamma \) under the robustly stochastic stability condition, if it is robustly stochastically stable and, under zero initial condition, for all nonzero \( u_k \in L_2[0, +\infty) \) the following inequality holds:

\begin{equation}
E\left\{ \sum_{k=0}^{\infty} \| \bar{z}_k \|^2 \right\} < \gamma^2 \sum_{k=0}^{\infty} \| w_k \|^2. \tag{12}
\end{equation}
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To proceed further, a well-known lemma is introduced to tackle the uncertainties as follows.

**Lemma 3** (see [19]). For real matrices $Q = Q^T$, $M$, $F$, and $N$ with appropriate dimensions. Suppose that $F$ satisfies $FT \leq I$, and the following inequality holds:

$$Q + MFN + N^TFM^T < 0.$$ (13)

If and only if there exists a positive scalar $\varepsilon > 0$ holding

$$Q + \varepsilon^{-1}MM^T + \varepsilon N^TN < 0,$$ (14)

then the problem in this paper is addressed below.

**Problem 4.** With Definitions 1 and 2, the objective in this paper is to design a robust mode-dependent $H_\infty$ filter of the form (7)-(8) for system (1), such that given a prescribed level of noise attenuation $\gamma > 0$, for all possible missing measurements, the filtering error system (8)-(9) is robustly stochastically stable for the whole uncertain domain and satisfies the $H_\infty$ robustness performance (12).

### 3. $H_\infty$ Error Performance Analysis

In this section, the performance analysis of the mode-dependent $H_\infty$ filter approach is given. Theorem 5 presents the sufficient conditions for the existence of the $H_\infty$ filter.

**Theorem 5.** Consider the DMJLS (1). Given a constant $\gamma > 0$ as the $H_\infty$ performance index, the filtering error system (8)-(9) is robustly stochastically stable and satisfies the $H_\infty$ robustness performance, if there exists a matrix $P_i > 0$ satisfying the following LMIs:

$$\Psi_i = \begin{bmatrix}
-P_i & 0 & 0 & P_i \bar{A}_i & P_i \bar{B}_i \\
-P_i & 0 & \alpha P_i \bar{A}_i & 0 & 0 \\
* & -P_i & -I & T_i & 0 \\
* & * & -P_i & 0 & 0 \\
* & * & * & -P_i & 0 \\
* & * & * & * & -\gamma^2 I
\end{bmatrix} < 0,$$ (15)

for $i = 1, 2, \ldots, s$,

where $\alpha = [(1 - \beta)\beta]^{1/2}$, $\bar{P}_i = \sum_{j=1}^s \lambda_{ij}P_j$, and $P_i$ is a stochastic mode-dependent Lyapunov matrix for the filtering error system.

**Proof.** For the filtering error system, construct a stochastic mode-dependent Lyapunov function as

$$V(\bar{x}_k, i) = \bar{x}_k^T \bar{P}_i \bar{x}_k, \quad \text{for } i = 1, 2, \ldots, s.$$ (16)

Then

$$E\{\Delta V(\bar{x}_k, i)\} = E[V(\bar{x}_{k+1}, i) | \bar{x}_k, i] - V(\bar{x}_k, i)
=E\left\{\bar{x}_{k+1}^T \sum_{j=1}^s \pi_{ij} P_j \bar{x}_{k+1}\right\} - \bar{x}_k^T \bar{P}_i \bar{x}_k
=E\left\{\bar{x}_{k+1}^T \bar{P}_i \bar{x}_k - \bar{x}_k^T \bar{P}_i \bar{x}_k\right\}
=E\left\{\bar{x}_k^T \left(\bar{A}_i^T \bar{P}_i \bar{A}_i - \bar{P}_i\right) \bar{x}_k\right\}
+E\left\{\left(\bar{A}_i^T \bar{P}_i \bar{A}_i - \bar{P}_i\right) \bar{x}_k\right\}
+E\left\{\bar{A}_i^T \bar{P}_i \bar{A}_i \bar{x}_k\right\}
+2\bar{x}_k^T \bar{A}_i^T \bar{P}_i \bar{B}_i w_k + \omega_k^T \bar{B}_i^T \bar{B}_i w_k
=E\left\{\bar{x}_k^T \left(\bar{A}_i^T \bar{P}_i \bar{A}_i - \bar{P}_i + \alpha^2 \bar{A}_i^T \bar{P}_i \bar{A}_i\right) \bar{x}_k\right\}
+2\bar{x}_k^T \bar{A}_i^T \bar{P}_i \bar{B}_i w_k + \omega_k^T \bar{B}_i^T \bar{B}_i w_k.
$$

From (15), according to the basic matrix manipulations,

$$\begin{bmatrix}
-P_i & 0 & P_i \bar{A}_i & P_i \bar{B}_i \\
-P_i & 0 & \alpha P_i \bar{A}_i & 0 & 0 \\
* & -P_i & -I & T_i & 0 \\
* & * & -P_i & 0 & 0 \\
* & * & * & -P_i & 0 \\
* & * & * & * & -\gamma^2 I
\end{bmatrix} < 0.
$$

Thus,

$$\begin{aligned}
\Delta V(\bar{x}_k, i) &= \bar{x}_k^T \left(\bar{A}_i^T \bar{P}_i \bar{A}_i - \bar{P}_i + \alpha^2 \bar{A}_i^T \bar{P}_i \bar{A}_i\right) \bar{x}_k \\
&\leq -\mu \bar{x}_k^T \bar{x}_k < 0,
\end{aligned}$$

with $\mu = \lambda_{\min}\left(-\bar{A}_i^T \bar{P}_i \bar{A}_i - P_i + \alpha^2 \bar{A}_i^T \bar{P}_i \bar{A}_i\right)$ as the minimum eigenvalue of $-\bar{A}_i^T \bar{P}_i \bar{A}_i - P_i + \alpha^2 \bar{A}_i^T \bar{P}_i \bar{A}_i$.

Then,

$$E\left\{\sum_{k=0}^M \Delta V(\bar{x}_k, i)\right\} = E\left[V(\bar{x}_{M+1}, i) | \bar{x}_k, i\right] - E\left[V(\bar{x}_0, 0)\right] \leq -\mu E\left[\sum_{k=0}^M \bar{x}_k^T \bar{x}_k\right].$$

Thus,

$$E\left\{\sum_{k=0}^M \|\bar{x}_k\|^2\right\} \leq \frac{1}{\mu} \left[E\left[V(\bar{x}_0, 0)\right] - E\left[V(\bar{x}_{k+1}, k+1) | \bar{x}_k, i\right]\right] \leq \frac{1}{\mu} E\left[V(\bar{x}_0, i)\right] \leq \infty.$$

The filtering error system (8)-(9) is robustly stochastically stable according to Definition 1.

Assume the zero initial condition $V(\bar{x}_0, i) = 0$, and one has that

$$E\left[\sum_{k=0}^M \Delta V(\bar{x}_k, i)\right] = V(\bar{x}_{M+1}, i) \geq 0.$$ (21)
Consider the performance index:

\[
J = E \left\{ \sum_{k=0}^{\infty} \left[ \|z_k\|^2 - \gamma^2 \|w_k\|^2 \right] \right\}.
\]

(22)

Therefore,

\[
J \leq E \left\{ \sum_{k=0}^{\infty} \left[ \|z_k\|^2 - \gamma^2 \|w_k\|^2 \right] + \Delta V (x_k, i) \right\}
\]

\[
= \sum_{k=0}^{\infty} \left[ \|z_k\|^2 - \gamma^2 \|w_k\|^2 \right] \]

\[
= \sum_{k=0}^{\infty} \left[ ||L(\theta_k) x_k||^2 - \gamma^2 ||w_k||^2 \right]
\]

\[
= \sum_{k=0}^{\infty} \left[ \|L(\theta_k) x_k\|^2 - \gamma^2 \|w_k\|^2 \right] \]

\[
+ x_k^T (A_i^T I - P_i + \alpha^2 A_i^T P_i A_i) x_k
\]

\[
+ 2 \lambda_k^T A_i^T P_i B_i w_k + w_k^T B_i^T P_i B_i w_k \]

\[
= \sum_{k=0}^{\infty} \eta_k^T \Phi_k \eta_k, \]

where

\[
\eta_k = \left[ x_k^T \quad w_k^T \right]^T,
\]

\[
\Phi_k
\]

\[
\begin{bmatrix}
A_i^T P_i A_i - P_i + \alpha^2 A_i^T P_i A_i + L_i^T \Lambda_i + A_i^T P_i B_i \\
* & B_i^T P_i B_i - \gamma^2 I
\end{bmatrix}.
\]

(24)

According to Schur complement, (15) guarantees \( \Phi_i < 0 \), which means \( \|z_k\|^2 < \gamma^2 \|w_k\|^2 \) and \( J < 0 \). The proof is completed.

In order to design an admissible \( H_\infty \) filter to eliminate the cross coupling of matrix product terms between the Lyapunov matrix and system matrix among different subsystems, when solving (15), the following theorem is presented inspired by [20].

**Theorem 6.** Consider the DMJLS (1). The filtering error system (8)-(9) is robustly stochastically stable and satisfies the given \( H_\infty \) performance index \( \gamma > 0 \), if there exist matrices \( P_i > 0, X_i \) satisfying the following LMIs:

\[
\begin{bmatrix}
-X_i - X_i^T + \bar{P} & 0 & 0 & X_i \bar{A}_i & X_i \bar{B}_i & X_i \bar{M}_i \\
* & -X_i - X_i^T + \bar{P} & 0 & \alpha X_i \bar{A}_{ii} & 0 & 0 \\
* & * & -I & L_i & 0 & 0 \\
* & * & * & -P_i + \varepsilon_i \bar{N}_{ai} \bar{N}_{bi} & \varepsilon_i \bar{N}_{ai} \bar{N}_{bi} & 0 \\
* & * & * & * & -\gamma^2 I + \varepsilon_i \bar{N}_{bi} \bar{N}_{bi} & 0 \\
* & * & * & * & * & -\varepsilon_i I
\end{bmatrix} < 0,
\]

(25)

where \( \alpha = [(1 - \beta) \beta]^{1/2} \).
Proof. Note that (25) can be written in the form of (13) as

\[ Q_\ell + \Gamma_\ell F_\ell \Gamma_\ell^T + \Gamma_\ell^T F_\ell^T \Gamma_\ell^T < 0, \]

with

\[
Q_\ell = \begin{bmatrix}
-X_\ell - X_\ell^T + \mathcal{P}_\ell & 0 & 0 & X_\ell \mathcal{A}_\ell & X_\ell \mathcal{B}_\ell \\
* & -X_\ell - X_\ell^T + \mathcal{P}_\ell & 0 & \alpha X_\ell \mathcal{A}_\ell & 0 \\
* & * & -I & \mathcal{T}_\ell & 0 \\
* & * & * & -P_\ell & 0 \\
* & * & * & * & -\gamma^2 I
\end{bmatrix},
\]

\[
\Gamma_\ell = \begin{bmatrix}
\mathcal{M}_\ell^T & X_\ell^T & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}^T,
\]

\[
\Gamma_\ell = \begin{bmatrix}
0 & 0 & \bar{N}_{\ell m} & \bar{N}_{\ell b}
\end{bmatrix}^T.
\]

By applying Schur complement and Lemma 3 to (29), the LMIs (25) hold, if and only if there exists a positive scalar \( \epsilon \), satisfying the LMIs (28), which completes the proof. \( \square \)

4. \( H_{\infty} \) Filter Design

Theorem 8 below presents the solvability for the addressed robust \( H_{\infty} \) filter design problem in Theorem 7 and whether the LMIs (28) are feasible for all the possible parametric uncertainties and missing measurements.

**Theorem 8.** Consider the DMJLS (1) with parameter uncertainties and missing measurements. The filtering error system (8)-(9) is robustly stochastically stable and satisfies a prescribed \( H_{\infty} \) performance index \( \gamma > 0 \), if there exist matrices \( P_\ell, P_{\ell m}, P_{\ell 3}, G_\ell, K_\ell, H_\ell, U_\ell, S_\ell \), and \( R_\ell \) and scalar \( \epsilon > 0 \) satisfying the following LMIs:

\[
\begin{bmatrix}
\Pi_{11} & \Pi_{12} & 0 & 0 & \Pi_{16} & \Pi_{17} & \Pi_{18} & U_\ell & M_i \\
\Pi_{21} & 0 & 0 & 0 & \Pi_{26} & \Pi_{27} & \Pi_{28} & S_\ell & M_i \\
* & * & \Pi_{33} & \Pi_{36} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Pi_{44} & 0 & \alpha \mathcal{K}_\ell C_i & 0 & 0 & 0 \\
* & * & * & * & * & -I & L_\ell & -H_\ell & 0 \\
* & * & * & * & \Pi_{66} & -P_\ell & \epsilon N_{\ell m}^T N_{\ell b} & 0 & 0 \\
* & * & * & * & * & -P_\ell & 0 & 0 & 0 \\
* & * & * & * & * & * & \Pi_{88} & 0 & 0 \\
* & * & * & * & * & * & * & * & -\epsilon I
\end{bmatrix},
\]

where \( \alpha = [(1 - \beta) \beta]^{1/2} \) and the elements

\[
\Pi_{11} = \Pi_{33} = -U_\ell - U_\ell^T + \sum_{j=1}^i \lambda_{ij} P_{1j},
\]

\[
\Pi_{12} = \Pi_{34} = -R_\ell - S_\ell^T + \sum_{j=1}^i \lambda_{ij} P_{2j},
\]

\[
\Pi_{16} = U_\ell A_\ell + \beta \mathcal{K}_\ell C_i,
\]

\[
\Pi_{17} = \Pi_{27} = \mathcal{G}_\ell - \beta \mathcal{K}_\ell C_i,
\]

\[
\Pi_{18} = U_\ell B_\ell + \mathcal{K}_\ell D_\ell,
\]

\[
\Pi_{22} = \Pi_{44} = -R_\ell - R_\ell^T + \sum_{j=1}^i \lambda_{ij} P_{3j},
\]

\[
\Pi_{26} = S_\ell A_\ell + \beta \mathcal{K}_\ell C_i,
\]

\[
\Pi_{28} = S_\ell B_\ell + \mathcal{K}_\ell D_\ell,
\]

\[
\Pi_{66} = -P_{1i} + \epsilon N_{\ell m}^T N_{\ell b},
\]

\[
\Pi_{88} = -\gamma^2 I + \epsilon N_{\ell m}^T N_{\ell b}.
\]

Moreover, if the LMIs (31)-(32) have a feasible solution, the parameters of an admissible filter can be given by

\[
G_\ell = R_\ell^{-1} \mathcal{G}_\ell,
\]

\[
K_\ell = R_\ell^{-1} \mathcal{K}_\ell,
\]

\[
H_\ell = \mathcal{H}_\ell.
\]

**Proof.** Assume the matrices \( P_\ell \) and \( X_\ell \) in Theorem 7 in the form of

\[
P_\ell = \begin{bmatrix}
P_{11} & P_{12} \\
* & P_{33}
\end{bmatrix},
\]

\[
X_\ell = \begin{bmatrix}
U_\ell & R_\ell \\
S_\ell & R_\ell
\end{bmatrix},
\]

where \( R_\ell \) is assumed to be nonsingular. After substituting the above matrices into (28) and replacing with \( \mathcal{G}_\ell = R_\ell \mathcal{G}_\ell, \mathcal{K}_\ell = R_\ell \mathcal{K}_\ell, \) and \( \mathcal{H}_\ell = H_\ell \), it is easily shown that (31)-(32) are equivalent to (28). Therefore, the filtering error system is guaranteed to be robustly stochastically stable and achieves the prescribed \( H_{\infty} \) performance constraint. The proof is finally concluded. \( \square \)

**Remark 9.** By denoting \( \zeta = \gamma^2 \), the minimum \( H_{\infty} \) noise attenuation performance bound and the corresponding filter
parameter design can be obtained by solving the following convex optimisation problem [21]:

\[
\min_{P_{s}, P_{b}, G_{s}, R_{s}} \zeta
\]

subject to \( LMI s (33) \) over \( \begin{bmatrix} P_{s} & P_{b} \\ * & P_{b} \end{bmatrix} > 0, \quad \epsilon_{i} > 0 \).

Therefore, the optimal performance of the admissible \( H_{\infty} \) filter can be obtained with the minimum \( H_{\infty} \) performance constraint.

5. Numerical Example

In this section, a numerical example is presented to demonstrate the effectiveness and the feasibility of the proposed filter. Consider the DMJLS with two subsystems and the following parameters:

\[
A_{1} = \begin{bmatrix} 0 & -0.45 \\ 0.81 & 0.81 \end{bmatrix},
\]

\[
A_{2} = \begin{bmatrix} 0 & -0.27 \\ 0.81 & 1.23 \end{bmatrix};
\]

\[
B_{1} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},
\]

\[
B_{2} = \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix};
\]

\[
C_{1} = \begin{bmatrix} 0.4 & 0.5 \end{bmatrix},
\]

\[
C_{2} = \begin{bmatrix} 0.3 & 0.2 \end{bmatrix};
\]

\[
D_{1} = 0.8,
\]

\[
D_{2} = 0.9;
\]

\[
L_{1} = L_{2} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix};
\]

\[
M_{1} = M_{2} = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix},
\]

\[
N_{a1} = N_{a2} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},
\]

\[
N_{b1} = N_{b2} = 0.1.
\]

The minimum value of \( \gamma \) is \( \gamma_{\text{min}} = 0.5487 \) by solving the corresponding convex optimisation problem (36) with the help of Matlab LMI Toolbox. By solving (31)-(32), the parameters of the \( H_{\infty} \) filter can be obtained below:

\[
G_{1} = \begin{bmatrix} 0.1171 & -0.3809 \\ 0.5549 & 0.4219 \end{bmatrix},
\]

\[
K_{1} = \begin{bmatrix} 0.0711 \\ -0.6840 \end{bmatrix},
\]

\[
H_{1} = \begin{bmatrix} -0.1000 & -0.2000 \end{bmatrix};
\]

\[
G_{2} = \begin{bmatrix} 0.1660 & -0.1318 \\ 0.2093 & 0.8124 \end{bmatrix},
\]

\[
K_{2} = \begin{bmatrix} 0.2376 \\ -1.5140 \end{bmatrix},
\]

\[
H_{2} = \begin{bmatrix} -0.0996 & -0.1999 \end{bmatrix}.
\]

Figure 1 shows the measurements without and with missing influence after running 200 steps each simulation. The corresponding realization of the jumping mode is plotted in Figure 2.

By setting the initial condition \( x_{0} = [0.4 \ 0.6]^{T} \) and choosing the energy bounded noise \( u_{k} = 0.5\exp(-0.1k)\sin(0.01\pi k) \), the error response of the resulting error system is given in Figure 3. It is clearly observed that the filtering error is stable against the parametric uncertainties and missing measurements effects, which implies the feasibility and effectiveness of the proposed filter.

Figure 4 illustrates the missing measurements effects on the optimal \( H_{\infty} \) performance index, by comparing the minimum \( H_{\infty} \) performance index \( y_{\text{min}} \) under different level of probability \( \beta \), indicating that the \( H_{\infty} \) performance deteriorates with the increasing of the missing probability.
6. Conclusion

In this paper, the analysis and design of robust $H_{\infty}$ filtering for DMJLS with parametric uncertainties and missing measurements are investigated. The missing measurements are assumed to follow a Bernoulli distributed sequence. With a mode-dependent Lyapunov function, the $H_{\infty}$ filter is designed and sufficient conditions are established to ensure the robustly stochastic stability and $H_{\infty}$ attenuation level of the filtering error system. The numerical example implies the effectiveness and feasibility of the proposed approach.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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