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Research Article

Maximum Principle for Forward-Backward Stochastic Control System Driven by Lévy Process

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We study a stochastic optimal control problem where the controlled system is described by a forward-backward stochastic differential equation driven by Lévy process. In order to get our main result of this paper, the maximum principle, we prove the continuity result depending on parameters about fully coupled forward-backward stochastic differential equations driven by Lévy process. Under some additional convexity conditions, the maximum principle is also proved to be sufficient. Finally, the result is applied to the linear quadratic problem.

1. Introduction

The stochastic optimal control problem is one of the central themes of modern control science. Forward-backward stochastic control systems which the controlled systems described by forward-backward stochastic differential equations (FBSDEs) are widely used in mathematics and finance. Peng and Wu [1] firstly used a probabilistic method to get the existence and uniqueness results of fully coupled FBSDEs; then Peng [2] considered one kind of forward-backward stochastic control systems with economic background when the control domain is convex and obtained the maximum principle; since then a number of developments in this direction were reported in Wu [3] and Shi and Wu [4]. Wu [5] firstly proved the existence and uniqueness results of the solutions to fully coupled FBSDEs with Brownian motion and Poisson process; then Shi and Wu [6] got the stochastic maximum principle for fully coupled FBSDEs with random jumps. More conclusions about stochastic maximum principle about forward-backward stochastic control systems driven by Brownian motion and Poisson process can be seen in [7–9].

It is natural to extend the stochastic differential equations (SDEs) with Brownian motion and Poisson process to the case of Lévy process with independent and stationary increments. Baghery et al. [10] firstly considered the following fully coupled forward-backward stochastic differential equation driven by Lévy process (FBSDEL):

$$\begin{aligned} dx_t &= b(t, x_t, y_t, z_t, r_t)\,dt + \sum_{i=1}^{d} \sigma^i(t, x_t, y_t, z_t, r_t)\,dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g^i(t, x_{t-}, y_{t-}, z_t, r_t)\,dH_t^i, \\ -dy_t &= f(t, x_t, y_t, z_t, r_t)\,dt - \sum_{i=1}^{d} z_t^i dB_t^i - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\ x_0 &= a, \\ y_T &= \Phi(x_T), \end{aligned} \tag{1}$$

and, under some monotonicity assumptions, they got the existence and uniqueness of solutions for this equation. Zhu [11] had proposed the asymptotic stability in the Pth moment for SDE with Lévy noise. Nualart and Schoutens [12] constructed a set of pairwise strongly orthonormal martingales called Teugels martingale and they also proved a martingale representation theorem for Lévy processes satisfying some

exponential moment condition. Using the martingale representation theorem they [13] had proved the existence and uniqueness of a solution for backward stochastic differential equations (BSDEs) driven by Teugels martingale. Bahlali et al. [14] extended this conclusion to the BSDEs driven by Teugels martingale and an independent Brownian motion; they got the existence, uniqueness, and comparison of solutions for these equations under Lipschitz and locally Lipschitz conditions on the coefficient. Based on these consequences, Mitsui and Tabata [15] established the closeness property of the solution of the multidimensional backward stochastic Riccati differential equation with Lévy process; then they used this solution to study a linear quadratic regulation problem with Lévy process. After the foundation of the existence and uniqueness of the solutions of SDEs and multidimensional BSDEs driven by Lévy process, Tang and Wu [16] proceed to study a stochastic linear quadratic optimal control problem with a Lévy process, where the cost weighting matrices of the state and control were allowed to be indefinite.

These consequences are important for the researching of maximum principle for forward stochastic control system driven by Lévy process, as the adjoint equation for forward stochastic control system is a BSDE. Meng and Tang [17] firstly were concerned with optimal control for forward stochastic control system driven by Teugels martingale; they got the maximum principle and verification theorem for this system. In 2012, Tang and Zhang [18] were concerned with optimal control of BSDE driven by Teugels martingale and an independent multidimensional Brownian motion; they derived the necessary and sufficient conditions for the existence of the optimal control by means of convex variation methods and duality techniques. When the control domain was nonconcave and the control variable was allowed to enter the coefficients of the Teugels martingales, Lin [19] got the necessary maximum principle for optimal control of stochastic system driven by multidimensional Teugels martingales. Zhang et al. [20] firstly studied the forward-backward stochastic control system where the system was driven by Teugels martingale and an independent multidimensional Brownian motion as follows:

$$\begin{aligned} dx_t &= b(t,x_t,u_t)\,dt + \sum_{i=1}^{d}\sigma^i(t,x_t,u_t)\,dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g^i(t,x_{t-},u_t)\,dH_t^i, \\ -dy_t &= f(t,x_t,y_t,z_t,r_t,u_t)\,dt - \sum_{i=1}^{d} z_t^i dB_t^i - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\ x_0 &= a, \\ y_T &= \Phi(x_T), \end{aligned} \tag{2}$$

and they had got the maximum principle and verification theorem in the condition of the SDE part did not contain the backward state variables; the forward-backward stochastic control system they studied was not fully coupled.

In this paper, we extend the result of Zhang et al. [20] to the fully coupled forward-backward stochastic control system. Here the state variables are described by fully coupled FBSDEs driven by Brownian motion and an independent Teugels martingale. Before applying the convex variation and duality technique to obtain the stochastic maximum principle, we use the same method in [5] to get the continuity result depending on parameters, as the continuity result is not only important for us to get the stochastic maximum principle but also important property of FBSDEL especially in practice. Different from the Wu [3] and Shi and Wu [6] about maximum principles to Brownian motions and Poisson process, we also need more general *Ito's* formula about *càdlàg* semimartingale.

This paper is organized as follows. In Section 2, we will give some preliminaries used in this paper. Section 3 presents the continuity result depending on parameters about fully coupled FBSDEs driven by Lévy process. In Section 4, we obtain the main result of this paper, the maximum principle. We also prove that, under some additional convexity conditions, the maximum principle can be a sufficient condition for optimal control. And, in Section 5, an application of our stochastic maximum principle to the linear quadratic problem which the linear control system described by fully coupled FBSDEL is proved.

2. Preliminaries and Notations

Let $(\Omega, F, P, \mathscr{F}_t, B_t, L_t)$ $(t \in [0,T])$ be a complete space driving by Brownian motion and Lévy process in $R^m \times R \setminus \{0\}$, with Lévy measure ν; that is, $\{B_t\}_{0\le t\le T}$ is a standard Brownian motion. $\{L_t\}_{0\le t\le T}$ is R-valued Lévy process of the form $L_t = bt + \ell_t$ independent of $\{B_t\}_{0\le t\le T}$, corresponding to a standard Lévy measure ν satisfying the following conditions:

(i) $\int_R (1\wedge x^2)\nu(dx) < \infty$,

(ii) $\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|x|}\nu(dx) < \infty$, for every $\varepsilon > 0$ and for some $\lambda > 0$, and

$$\mathscr{F}_t = \sigma(L_s,\ s\le t) \vee \sigma(B_s,\ s\le t) \vee \aleph. \tag{3}$$

Here $\aleph$ is the totality of P-null sets and $g_1 \vee g_2$ denotes the σ-field generated by $g_1 \cup g_2$.

Let x be a Lévy process and denote the left limit process by $x_{t-} = \lim_{s\to t, s\le t} x_s$ and the jump size at time t by $\Delta x_t = x_t - x_{t-}$. Set

$$x_t^i = \begin{cases} \sum_{0<s\le t} (\Delta x_s)^i, & i \ge 2 \\ x_t, & i = 1 \end{cases} \tag{4}$$

and we denote the compensated power jump process of order i by $Y_t^i = x_t^i - E[x_t^i]$; then Teugels martingale $(H_t^i)_{0\le t\le T}$ can be defined as follows:

$$H_t^i = c_{i,i}Y_t^i + c_{i,i-1}Y_t^{i-1} + c_{i,i-2}Y_t^{i-2} + \cdots + c_{i,1}Y_t^1. \tag{5}$$

Here the coefficients $c_{i,k}$ correspond to orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = \upsilon(dx) + \sigma^2\delta_0(dx)$.

Now we introduce some notations adopted in this paper as follows:

(1) H: Hilbert space,

(2) $\langle \alpha, \beta \rangle$: the inner product in R^n, $\forall \alpha, \beta \in R^n$,

(3) $|\alpha| = \sqrt{\langle \alpha, \alpha \rangle}$: the norm in R^n, $\forall \alpha \in R^n$,

(4) $\langle A, B \rangle = \mathrm{tr}(AB^T)$: the inner product in $R^{n\times m}$, $\forall A, B \in R^{n\times m}$,

(5) $|A| = \sqrt{\mathrm{tr}(AA^T)}$: the inner product in $R^{n\times m}$, $\forall A \in R^{n\times m}$,

(6) l^2: the space of real valued sequences $X = (x_n)_{n\geq 0}$ such that

$$\|x\|_{l^2}^2 = \sum_{i=1}^{\infty} x_i^2 < \infty, \tag{6}$$

(7) $l^2(H)$: the space of H-valued sequences $\phi = \varphi^i{}_{i\geq 1}$ such that

$$\|\phi\|_{l^2(H)}^2 = \sum_{i=1}^{\infty} \left\|\phi^i\right\|^2 < \infty, \tag{7}$$

(8) $l^2(0,T;H)$: the corresponding spaces of $l^2(H)$ valued $\mathscr{F}_t$-measurable processes equipped with the norm

$$\|\phi\|_{l^2(0,T;H)}^2 = E\int_0^T \sum_{i=1}^{\infty} \left\|\phi_t^i\right\|_H^2 dt < \infty, \tag{8}$$

(9) $L^2(\Omega, \mathscr{F}_t, P; H)$: the space of H-valued random variable ξ with the norm

$$\|\xi\|^2 = E\|\xi\|_{L^2(\Omega,\mathscr{F}_t,P;H)}^2 < \infty, \tag{9}$$

(10) $M^2(0,T;H)$: the space of H-valued $\mathscr{F}_t$-measurable process $\phi(\cdot) = \{\phi(t,\omega) : (t,\omega) \in [0,T]\times\Omega\}$ with the norm

$$\|\phi(\cdot)\|_{M^2(0,T;H)}^2 = E\int_0^T \|\phi_t\|_{M^2(0,T;H)}^2 dt < \infty, \tag{10}$$

(11) $S^2(0,T;H)$: the space of H-valued $\mathscr{F}_t$-measurable *càdlàg* process $f(\cdot) = \{f(t,\omega) : (t,\omega) \in [0,T]\times\Omega\}$ with the norm

$$\|f(\cdot)\|_{S^2(0,T;H)}^2 = E\sup_{0\leq t\leq T} \|f_t\|_H^2 dt < \infty, \tag{11}$$

(12) for notational brevity:

$$\begin{aligned} M^2(0,T) &= M^2(0,T;R^n)\times M^2(0,T;R^m) \\ &\quad \times M^2\left(0,T;R^{m\times d}\right)\times l^2(0,T;R^m). \end{aligned} \tag{12}$$

Let us recall more general *Ito's* formula about *càdlàg* semimartingales which is important for us to get the maximum principle. Let $X = \{X_t : t \in [0,T]\}$ be *càdlàg* semimartingales, and we denoted the quadratic variation by $[X] = \{[X]_t : t \in [0,T]\}$; F is a $\mathscr{C}^2$ real valued function; then $F(X)$ is also semimartingales and following *Ito's* formula holds:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-})\, dX_s \\ &\quad + \frac{1}{2}\int_0^t F''(X_s)\, d[X]_s^{\mathscr{C}} \\ &\quad + \sum_{0<s\leq t} \left\{F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\right\}, \end{aligned} \tag{13}$$

where $[X]^{\mathscr{C}}$ is the continuous part of the quadratic variation $[X]$.

When $F(X) = X^2$ and $F(X) = X_tY_t$, where X, Y are two *càdlàg* semimartingales, we get

$$\begin{aligned} X_t^2 &= X_0^2 + \int_0^t 2X_{s-}dX_s + \int_0^t d[X]_s, \\ X_tY_t &= X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s \\ &\quad + \int_0^t d[X,Y]_s. \end{aligned} \tag{14}$$

Here $[X,Y]$ is the quadratic covariation of X, Y. We can refer to Protter [21] for a complete survey in this topic.

Next, we introduce the existence and uniqueness results for fully coupled FBSDEL (1):

$$\begin{aligned} dx_t &= b(t,x_t,y_t,z_t,r_t)\, dt + \sum_{i=1}^{d} \sigma^i(t,x_t,y_t,z_t,r_t)\, dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g^i(t,x_{t-},y_{t-},z_t,r_t)\, dH_t^i, \\ -dy_t &= f(t,x_t,y_t,z_t,r_t)\, dt - \sum_{i=1}^{d} z_t^i dB_t^i - \sum_{i=1}^{\infty} r_t^i dH_t^i, \\ x_0 &= a, \\ y_T &= \Phi(x_T), \end{aligned} \tag{15}$$

where

$$\begin{aligned} b &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \longrightarrow R^n, \\ \sigma &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \longrightarrow R^{n\times d}, \\ g &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \\ &\quad \longrightarrow l^2(R^n), \\ f &: \Omega\times[0,T]\times R^n\times R^m\times R^{m\times d}\times l^2(R^m) \longrightarrow R^m. \end{aligned} \tag{16}$$

For a given $m \times n$ full rank matrix G, we set

$$\lambda = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A(t,\lambda,r) = \begin{pmatrix} -G^T f(t,\lambda,r) \\ Gb(t,\lambda,r) \\ G\sigma(t,\lambda,r) \end{pmatrix}. \tag{17}$$

Assumption 1. Assume the following.

(i) b, σ, g, and f are uniformly Lipschitz continuous with respect to (x, y, z, r).

(ii) For each $(\omega, t) \in \Omega \times [0,T]$, $l(\omega, t, 0, 0, 0, 0) \in M^2(0,T)$ and $g(\omega, t, 0, 0, 0, 0) \in H^2(l^2)$, where $l = b, \sigma, f$, respectively.

(iii) $\Phi(\cdot)$ is uniformly Lipschitz continuous with respect to x and $\forall x, \Phi(x) \in L^2(\Omega, F_T, P)$.

Assumption 2. We also assume that

$$\begin{aligned} &\langle A(t,\lambda_1,r_1) - A(t,\lambda_2,r_2), \lambda_1 - \lambda_2 \rangle + \sum_{i=1}^{\infty} \left\langle G\widehat{g^i}, \widehat{r^i} \right\rangle \\ &\quad \le -\beta_1 |G\widehat{x}|^2 \\ &\qquad - \beta_2 \left(\left|G^T \widehat{y}\right|^2 + \left|G^T \widehat{z}\right|^2 + \sum_{i=1}^{\infty} \left\|G^T \widehat{r^i}\right\|^2 \right); \\ &\langle \Phi(x_1) - \Phi(x_2), G(x_1 - x_2) \rangle \ge \mu_1 |G\widehat{x}|^2, \end{aligned} \tag{18}$$

where $\lambda_1 = (x_1, y_1, z_1)$, $\lambda_2 = (x_2, y_2, z_2)$, $\widehat{x} = x_1 - x_2$, $\widehat{y} = y_1 - y_2$, $\widehat{z} = z_1 - z_2$, $\widehat{g^i} = g^i(t,\lambda_1,r_1) - g^i(t,\lambda_2,r_2)$, $\widehat{r^i} = r_1^i - r_2^i$, and β_1, β_2 and μ_1 are nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$. Moreover, we have $\beta_1 > 0$, $\mu_1 > 0$ (resp., $\beta_2 > 0$) when $m > n$ (resp., $n > m$). Under Assumptions 1 and 2, in [10], they have got the following lemma.

Lemma 3 (existence and uniqueness theorem of FBSDEL [10]). *Under Assumptions 1 and 2, FBSDEL (15) has a unique solution.*

3. Continuity Result Depending on Parameters about FBSDEL

Next, we are going to get the continuity result depending on parameters about FBSDEL.

Let $(b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$, $\alpha \in R$ be a family of FBSDEL:

$$\begin{aligned} dx_t^\alpha &= b_\alpha(t, x_t^\alpha, y_t^\alpha, z_t^\alpha, r_t^\alpha)\, dt \\ &\quad + \sum_{i=1}^{d} \sigma_\alpha^i (t, x_t^\alpha, y_t^\alpha, z_t^\alpha, r_t^\alpha)\, dB_t^i \\ &\quad + \sum_{i=1}^{\infty} g_\alpha^i (t, x_{t-}^\alpha, y_{t-}^\alpha, z_t^\alpha, r_t^\alpha)\, dH_t^i, \\ -dy_t &= f_\alpha(t, x_t^\alpha, y_t^\alpha, z_t^\alpha, r_t^\alpha)\, dt - \sum_{i=1}^{d} z_t^{i,\alpha} dB_t^i \\ &\quad - \sum_{i=1}^{\infty} r_t^{i,\alpha} dH_t^i, \\ x_0^\alpha &= a, \\ y_T^\alpha &= \Phi_\alpha(x_T^\alpha). \end{aligned} \tag{19}$$

Assumption 4. (i) The family $(b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$, $\alpha \in R$ are equi-Lipschitz with respect to (x, y, z, r) and x separately.

(ii) The function $\alpha \to (b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$ is continuous in their existing space norm sense, respectively.

Then we can get the following continuity result depending on parameters of forward-backward stochastic differential equation driven by Lévy processes.

Theorem 5. *Let $(b_\alpha, \sigma_\alpha, g_\alpha, f_\alpha, \Phi_\alpha)$, $\alpha \in R$ be a family of FBSDEL satisfying Assumptions 1, 2, and 4 with solutions denoted by $(x^\alpha, y^\alpha, z^\alpha, r^\alpha)$. Thus, the function*

$$\begin{aligned} &\alpha \longrightarrow (x^\alpha, y^\alpha, z^\alpha, r^\alpha, x_T^\alpha): \\ &R \longrightarrow M^2(0,T) \times L^2(\Omega, \mathcal{F}_t, P, R^n) \end{aligned} \tag{20}$$

is continuous.

Proof. For notational brevity, we only prove the continuity of FBSDEL (19) at $\alpha = 0$. Set $\widehat{\lambda}_t = \lambda_t^\alpha - \lambda_t^0 = (x_t^\alpha - x_t^0, y_t^\alpha - y_t^0, z_t^\alpha - z_t^0)$ and $\widehat{r}_t = r_t^\alpha - r_t^0$. From Assumptions 1, 2, and 4, applying the usual technique to $\widehat{x}_t$ of *Itô's* SDE with Lévy process, we can get

$$\begin{aligned} \sup_{0 \le t \le T} E\left|\widehat{x}_t\right|^2 &\le K_1 E \int_0^T \left(\left|\widehat{y}_t\right|^2 + \left|\widehat{z}_t\right|^2 + \left\|\widehat{r}_t^i\right\|^2 \right) dt \\ &\quad + K_1 E \int_0^T \left(\left|\widehat{b}_t\right|^2 + \left|\widehat{\sigma}_t\right|^2 + \left\|\widehat{g}_t\right\|^2 \right) dt. \end{aligned} \tag{21}$$

Applying the same technique to $\widehat{y}_t$ of BSDE with Lévy process, then

$$\begin{aligned} &E \int_0^T \left(\left|\widehat{y}_t\right|^2 + \left|\widehat{z}_t\right|^2 + \left\|\widehat{r}_t^i\right\|^2 \right) dt \\ &\quad \le K_1 \left(E \int_0^T \left|\widehat{x}_t\right|^2 dt + E\left|\widehat{x}_T\right|^2 + E \int_0^T \left|\widehat{f}_t\right|^2 dt \right. \\ &\quad \left. + E\left|\Phi_\alpha(x_T^0) - \Phi_0(x_T^0)\right|^2 \right). \end{aligned} \tag{22}$$

Here K_1, K_2 depend on the Lipschitz constants and T, and

$$\begin{aligned}\widehat{f}_t &= f_\alpha\left(t,\lambda_t^0,r_t^0\right)-f_0\left(t,\lambda_t^0,r_t^0\right),\\ \widehat{b}_t &= b_\alpha\left(t,\lambda_t^0,r_t^0\right)-b_0\left(t,\lambda_t^0,r_t^0\right),\\ \widehat{\sigma}_t &= \sigma_\alpha\left(t,\lambda_t^0,r_t^0\right)-\sigma_0\left(t,\lambda_t^0,r_t^0\right),\\ \widehat{g}_t &= g_\alpha\left(t,\lambda_t^0,r_t^0\right)-g_0\left(t,\lambda_t^0,r_t^0\right).\end{aligned} \tag{23}$$

Set

$$A_\alpha(t,\lambda,r)=\begin{pmatrix}-G^T f_\alpha(t,\lambda,r)\\ Gb_\alpha(t,\lambda,r)\\ G\sigma_\alpha(t,\lambda,r)\end{pmatrix}, \tag{24}$$

and applying *Itô's* formula to $\langle G\widehat{x}_t,\widehat{y}_t\rangle$ yields

$$\begin{aligned}&E\left\langle\Phi_\alpha\left(x_T^\alpha\right)-\Phi_\alpha\left(x_T^0\right),G\widehat{x}_T\right\rangle+E\left\langle\Phi_0\left(x_T^\alpha\right)\right.\\ &\quad\left.-\Phi_0\left(x_T^0\right),G\widehat{x}_T\right\rangle\\ &\quad=E\int_0^T\left[\left\langle A_\alpha\left(t,\lambda_t^\alpha,r_t^\alpha\right)-A_\alpha\left(t,\lambda_t^0,r_t^0\right),\widehat{\lambda}_t\right\rangle\right.\\ &\quad\left.+\sum_{i=1}^\infty\left\langle G\left(g_\alpha^i\left(t,\lambda_t^\alpha,r_t^\alpha\right)-g_\alpha^i\left(t,\lambda_t^0,r_t^0\right)\right),\widehat{r}_t^i\right\rangle\right]dt\\ &\quad+E\int_0^T\left[\left\langle\widehat{x}_t,-G^T\widehat{f}_t\right\rangle+\left\langle G^T\widehat{y}_t,\widehat{b}_t\right\rangle+\sum_{i=1}^d\left\langle\widehat{z}_t^i,G\widehat{\sigma}_t^i\right\rangle\right.\\ &\quad\left.+\sum_{i=1}^\infty\left\langle G^T\widehat{r}_t^i,\widehat{g}_t^i\right\rangle\right]dt.\end{aligned} \tag{25}$$

From the above three estimates, we get

$$\begin{aligned}&\beta_1E\int_0^T\left|G\widehat{x}_t\right|^2dt+\mu_1E\left|G\widehat{x}_T\right|^2\\ &\quad+\beta_2E\int_0^T\left[\left|G^T\widehat{y}_t\right|^2+\left|G^T\widehat{z}_t\right|^2+\left\|G^T\widehat{r}_t\right\|^2\right]dt\\ &\quad\le C_1\left[E\left|\Phi_\alpha\left(x_T^0\right)-\Phi_0\left(x_T^0\right)\right|^2\right.\\ &\quad\left.+E\int_0^T\left(\left|\widehat{f}_t\right|^2+\left|\widehat{b}_t\right|^2+\left|\widehat{\sigma}_t\right|^2+\left\|\widehat{g}_t\right\|^2\right)dt\right]\\ &\quad+\delta E\left|\widehat{x}_T\right|^2\\ &\quad+\delta E\int_0^T\left(\left|\widehat{x}_t\right|^2+\left|\widehat{y}_t\right|^2+\left|\widehat{z}_t\right|^2+\left\|\widehat{r}_t\right\|^2\right)dt,\end{aligned} \tag{26}$$

where the constant C_1 depends on the Lipschitz constants T and δ. When $m\ge n$, $\beta_1>0$, and $\mu_1>0$, then

$$\delta=\min\left(\frac{1}{3},\frac{1}{3K_1},\frac{\beta_1\left|G^TG\right|}{3},\frac{\mu_1\left|G^TG\right|}{3}\right). \tag{27}$$

If $m\le n$, $\beta_2>0$, and $\mu_1\ge 0$, then

$$\delta=\min\left(\frac{1}{3},\frac{1}{3K_1},\frac{1}{3K_1T},\frac{\beta_2\left|GG^T\right|}{3}\right). \tag{28}$$

Thus, it is clear whatever $\beta_1>0$, $\beta_2\ge 0$, and $\mu_1>0$ or $\beta_1\ge 0$, $\beta_2>0$, and $\mu_1\ge 0$ we always have

$$\begin{aligned}&E\left|\widehat{x}_T\right|^2+E\int_0^T\left(\left|\widehat{\lambda}_t\right|^2+\left\|\widehat{r}_t\right\|^2\right)dt\\ &\quad\le C\left[E\left|\Phi_\alpha\left(x_T^0\right)-\Phi_0\left(x_T^0\right)\right|^2\right.\\ &\quad\left.+E\int_0^T\left(\left|\widehat{f}_t\right|^2+\left|\widehat{b}_t\right|^2+\left|\widehat{\sigma}_t\right|^2+\left\|\widehat{g}_t\right\|^2\right)\right].\end{aligned} \tag{29}$$

The proof is completed. □

4. Maximum Principle

Let us consider the following full coupled forward-backward stochastic control system:

$$\begin{aligned}dx_t &= b\left(t,x_t,y_t,z_t,r_t,u_t\right)dt\\ &\quad+\sum_{i=1}^d\sigma^i\left(t,x_t,y_t,z_t,u_t\right)dB_t^i\\ &\quad+\sum_{i=1}^\infty g^i\left(t,x_{t-},y_{t-},r_t,u_t\right)dH_t^i,\\ -dy_t &= f\left(t,x_t,y_t,z_t,r_t,u_t\right)dt-\sum_{i=1}^d z_t^idB_t^i-\sum_{i=1}^\infty r_t^idH_t^i,\\ x_0 &= a,\\ y_T &= \Phi\left(x_T\right),\end{aligned} \tag{30}$$

where (x_t,y_t,z_t,r_t) take values in $R^n\times R^m\times R^{m\times d}\times l^2(R^m)$; $a\in R^n$ is given.

Let U be a nonempty convex subset of R^k. We define the admissible control set $U_{ad}=\{u(\cdot)\in M^2(0,T;R^k);u_t\in U,0\le t\le T,\text{a.e., a.s.}\}$ and the cost functional:

$$J(u)=E\int_0^T L\left(x_t,y_t,z_t,r_t,u_t\right)dt+h\left(x_T\right)+\gamma\left(y_0\right). \tag{31}$$

The optimal control problem is to find $\overline{u}\in U_{ad}$, such that

$$J(\overline{u}(\cdot))=\inf_{u(\cdot)\in U[0,T]}J(u(\cdot)). \tag{32}$$

Assumption 6. Now we introduce the basic assumptions of this section as follows.

(i) b, f, and L are continuously differentiable with respect to (x,y,z,r,u); σ is continuously differentiable with respect to (x,y,z,u); g is continuously differentiable with respect to (x,y,r,u); Φ and h are continuously differentiable with respect to x; γ is continuously differentiable with respect to y. And the derivatives of each function are all bounded.

(ii) For each $(x, y, z, r, u) \in R^n \times R^m \times R^{m\times d} \times l^2(R^m) \times U_{ad}$, there exists a constant $C > 0$, such that

$$\begin{aligned}
&|L| \le C\left(1+|x|^2+|y|^2+|z|^2+\|r\|^2+|u|^2\right),\\
&|h| \le C\left(1+|x|^2\right),\\
&|\gamma| \le \left(1+|y|^2\right),\\
&|L_x|+|L_y|+|L_z|+|L_r|+|L_u|\\
&\quad \le C\left(1+|x|+|y|+|z|+\|r\|+|u|\right),\\
&|h_x| \le C\left(1+|x|\right),\\
&|\gamma_y| \le C\left(1+|y|\right).
\end{aligned} \tag{33}$$

(iii) For any given admissible control $u(\cdot)$, (30) satisfies Assumptions 1 and 2.

Then, for a given admissible control, from Lemma 3, there exists a unique solution satisfying control system (30).

Let $\overline{u}_t$ be an optimal control and let $(\overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t)$ be the corresponding trajectory. For any given admissible control u_t and $0 \le \varepsilon \le 1$, we define

$$u_t^\varepsilon = \overline{u}_t + \varepsilon\left(u_t - \overline{u}_t\right). \tag{34}$$

Since U_{ad} is convex, then u_t^ε is in U_{ad}; that is, u_t^ε is an admissible control and $(x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon)$ is the corresponding trajectory.

We introduce the following variational equation:

$$\begin{aligned}
dX_t &= \left[b_x(t)X_t + b_y(t)Y_t + b_z(t)Z_t + b_r(t)R_t\right.\\
&\quad \left.+ b_u(t)\left(u_t-\overline{u}_t\right)\right]dt + \sum_{i=1}^{d}\left[\sigma_x^i(t)X_t + \sigma_y^i(t)Y_t\right.\\
&\quad \left.+ \sigma_z^i(t)Z_t + \sigma_u^i(t)\left(u_t-\overline{u}_t\right)\right]dB_t^i + \sum_{i=1}^{\infty}\left[g_x^i(t)X_{t-}\right.\\
&\quad \left.+ g_y^i(t)Y_{t-} + g_r^i(t)R_t + g_u^i(t)\left(u_t-\overline{u}_t\right)\right]dH_t^i,\\
-dy_t &= \left[f_x(t)X_t + f_y(t)Y_t + f_z(t)Z_t + f_r(t)R_t\right.\\
&\quad \left.+ f_u^i(t)\left(u_t-\overline{u}_t\right)\right]dt - \sum_{i=1}^{d}Z_t^i dB_t^i - \sum_{i=1}^{\infty}R_t^i dH_t^i,\\
X_0 &= 0,\\
Y_T &= \Phi_x\left(\overline{x}_t\right)X_T.
\end{aligned} \tag{35}$$

From Assumption 6, we can verify that variational equation (35) satisfies Lemma 3. Thus, there exists a unique solution (X_t, Y_t, Z_t, R_t) satisfying variational equation. In order to get the maximum principle, we also need the following lemma.

Lemma 7. *Assume that Assumption 6 holds. We have*

$$\begin{aligned}
&E\sup_{0\le t\le T}\left|x_t^\varepsilon - \overline{x}_t - \varepsilon X_t\right|^2 = o\left(\varepsilon^2\right),\\
&E\sup_{0\le t\le T}\left|y_t^\varepsilon - \overline{y}_t - \varepsilon Y_t\right|^2 + E\int_0^T\left|z_t^\varepsilon - \overline{z}_t - \varepsilon Z_t\right|^2 dt\\
&\quad + E\int_0^T\left\|r_t^\varepsilon - \overline{r}_t - \varepsilon R_t\right\|^2 dt = o\left(\varepsilon^2\right),
\end{aligned} \tag{36}$$

where (X_t, Y_t, Z_t, R_t) is the solution of variational equation (35).

Proof. Set

$$\begin{aligned}
\Delta x_t &= \varepsilon^{-1}\left(x_t^\varepsilon - \overline{x}_t\right),\\
\Delta y_t &= \varepsilon^{-1}\left(y_t^\varepsilon - \overline{y}_t\right),\\
\Delta z_t &= \varepsilon^{-1}\left(z_t^\varepsilon - \overline{z}_t\right),\\
\Delta r_t &= \varepsilon^{-1}\left(r_t^\varepsilon - \overline{r}_t\right),
\end{aligned} \tag{37}$$

and then

$$\begin{aligned}
d\Delta x_t &= \varepsilon^{-1}\left[b\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- b\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dt\\
&\quad + \varepsilon^{-1}\left[\sum_{i=1}^{d}\sigma^i\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- \sum_{i=1}^{d}\sigma^i\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dB_t^i\\
&\quad + \varepsilon^{-1}\left[\sum_{i=1}^{\infty}g^i\left(t, x_{t-}^\varepsilon, y_{t-}^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- \sum_{i=1}^{\infty}g^i\left(t, \overline{x}_{t-}, \overline{y}_{t-}, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dH_t^i,\\
-d\Delta y_t &= \varepsilon^{-1}\left[f\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right.\\
&\quad \left.- f\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right]dt - \sum_{i=1}^{d}\Delta z_t^i dB_t^i\\
&\quad - \sum_{i=1}^{\infty}\Delta r_t^i dH_t^i,\\
\Delta x_0 &= 0,\\
\Delta y_T &= \varepsilon^{-1}\left[\Phi\left(x_T^\varepsilon\right) - \Phi\left(\overline{x}_T\right)\right].
\end{aligned} \tag{38}$$

We can transform (38) into

$$\begin{aligned}
&d\Delta x_t \\
&\quad = \widehat{b}\left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dt \\
&\qquad + \sum_{i=1}^{d} \widehat{\sigma}^i \left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dB_t^i \\
&\qquad + \sum_{i=1}^{\infty} \widehat{g}^i \left(t, \Delta x_{t-}, \Delta y_{t-}, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dH_t^i, \\
&-d\Delta y_t \\
&\quad = \widehat{f}\left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) dt - \sum_{i=1}^{d} \Delta z_t^i dB_t^i \\
&\qquad - \sum_{i=1}^{\infty} \Delta r_t^i dH_t^i, \\
&\Delta x_0 = 0, \\
&\Delta y_T = \varepsilon^{-1}\left[\Phi\left(x_T^\varepsilon\right) - \Phi\left(\overline{x}_T\right)\right],
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
&\widehat{l}\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, u_t - \overline{u}_t\right) \\
&\quad = A^l(t)\,\overline{x}_t + B^l(t)\,\overline{y}_t + C^l(t)\,\overline{z}_t + D^l(t)\,\overline{r}_t \\
&\qquad + E^l(t)\left(u_t - \overline{u}_t\right),
\end{aligned} \tag{40}$$

for $l = b, \sigma, g, f$, respectively, and

$$\begin{aligned}
A^l(t) &= \begin{cases} A(t), & x_t^\varepsilon - \overline{x}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
B^l(t) &= \begin{cases} B(t), & y_t^\varepsilon - \overline{y}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
C^l(t) &= \begin{cases} C(t), & z_t^\varepsilon - \overline{z}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
D^l(t) &= \begin{cases} D(t), & r_t^\varepsilon - \overline{r}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases} \\
E^l(t) &= \begin{cases} E(t), & u_t - \overline{u}_t \neq 0 \\ 0, & \text{otherwise}, \end{cases}
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
A(t) &= \left(x_t^\varepsilon - \overline{x}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right], \\
B(t) &= \left(y_t^\varepsilon - \overline{y}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right)\right], \\
C(t) &= \left(z_t^\varepsilon - \overline{z}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, \overline{y}_t, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, r_t^\varepsilon, u_t^\varepsilon\right)\right], \\
D(t) &= \left(r_t^\varepsilon - \overline{r}_t\right)^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, r_t^\varepsilon, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, u_t^\varepsilon\right)\right], \\
E(t) &= \left[\varepsilon\left(u_t - \overline{u}_t\right)\right]^{-1} \\
&\quad \cdot \left[l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, u_t^\varepsilon\right) - l\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\right].
\end{aligned} \tag{42}$$

From the continuity result depending on parameters we have got in Section 3, we know that

$$\begin{aligned}
&E \sup_{0 \le t \le T} \left|x_t^\varepsilon - \overline{x}_t\right|^2 + E \sup_{0 \le t \le T} \left|y_t^\varepsilon - \overline{y}_t\right|^2 \\
&\qquad + E \int_0^t \left|z_t^\varepsilon - \overline{z}_t\right|^2 dt + E \int_0^t \left\|r_t^\varepsilon - \overline{r}_t\right\|^2 dt \le K\varepsilon^2;
\end{aligned} \tag{43}$$

that is, $(x_t^\varepsilon - \overline{x}_t, y_t^\varepsilon - \overline{y}_t, z_t^\varepsilon - \overline{z}_t, r_t^\varepsilon - \overline{r}_t) \to 0$ in $M^2(0, T)$, as $\varepsilon \to 0$. Together with Assumption 6, we can get

$$\begin{aligned}
&\lim_{\varepsilon \to 0} A^l(t) = l_x\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} B^l(t) = l_y\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} C^l(t) = l_z\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} D^l(t) = l_r\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} E^l(t) = l_u\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right), \\
&\lim_{\varepsilon \to 0} \widehat{l}\left(t, \Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t, u_t - \overline{u}_t\right) \\
&\quad = l_x\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta x_t \\
&\qquad + l_y\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta y_t \\
&\qquad + l_z\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta z_t \\
&\qquad + l_r\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right) \Delta r_t \\
&\qquad + l_u\left(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t\right)\left(u_t - \overline{u}_t\right).
\end{aligned} \tag{44}$$

As FBSDEL (35) has a unique solution (X_t, Y_t, Z_t, R_t) under Assumption 6, from the continuity and uniqueness result by Lemma 3, we know that $(\Delta x_t, \Delta y_t, \Delta z_t, \Delta r_t)$ converges to (X_t, Y_t, Z_t, R_t) in $M^2([0, T])$ as $\varepsilon \to 0$.

The proof is complete. □

Notice that $\lim_{\varepsilon\to 0^+}(J(u_t^\varepsilon - J(\overline{u}_t))/\varepsilon) \geq 0$ and we can get the following variational inequality.

Lemma 8. *Assume that Assumption 6 holds; then*

$$\begin{aligned} &E\int_0^T L_x(t)X_t dt + E\int_0^T L_y(t)Y_t dt \\ &\quad + E\int_0^T L_z(t)Z_t dt + E\int_0^T L_r(t)R_t dt \\ &\quad + E\int_0^T L_u(t)(u_t - \overline{u}_t)dt + Eh_x(x_T)X_T \\ &\quad + E\gamma_y(y_0)Y_0 \geq 0. \end{aligned} \tag{45}$$

Proof. First we have

$$\begin{aligned} \frac{J(u_t^\varepsilon - J(\overline{u}_t))}{\varepsilon} &= \frac{1}{\varepsilon}\Big\{E\int_0^T [L(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon) \\ &\quad - L(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)]dt + E[h(x_T^\varepsilon) - h(\overline{x}_T)] \\ &\quad + E[\gamma(y_0^\varepsilon) - \gamma(\overline{y}_0)]\Big\}. \end{aligned} \tag{46}$$

Under Assumption 6, from Lemma 7, we can get

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E\int_0^T [L(t, x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, u_t^\varepsilon) \\ &\quad - L(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)]dt \\ &\quad \longrightarrow E\int_0^T \big[L_x(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)X_t \\ &\quad + L_y(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)Y_t \\ &\quad + L_z(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)Z_t \\ &\quad + L_r(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)R_t \\ &\quad + L_u(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)(u_t - \overline{u}_t)\big]dt, \\ &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E[h(x_T^\varepsilon) - h(\overline{x}_T)] \longrightarrow E[h_x(\overline{x}_T)X_T], \\ &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E[\gamma(y_0^\varepsilon) - \gamma(\overline{y}_0)] \longrightarrow E[\gamma_y(\overline{y}_0)Y_0]. \end{aligned} \tag{47}$$

Then (45) is proved.

The proof is complete. □

We define the Hamiltonian function H as follows:

$$\begin{aligned} &H(t, x, y, z, r, u, p, q, k, \rho) \\ &\quad = \langle p, -f(t, x, y, z, r, u)\rangle + \langle q, b(t, x, y, z, r, u)\rangle \\ &\quad + \langle k, \sigma(t, x, y, z, u)\rangle + \langle \rho, g(t, x, y, r, u)\rangle \\ &\quad + L(t, x, y, z, r, u), \end{aligned} \tag{48}$$

and the following adjoint forward-backward equation to variational equation (35):

$$\begin{aligned} &dp_t \\ &\quad = -H_y(t, x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t)dt \\ &\quad - \sum_{i=1}^d H_z^i(t, x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t)dB_t^i \\ &\quad - \sum_{i=1}^\infty H_r^i(t, x_{t-}, y_{t-}, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t)dH_t^i, \\ &-dq_t \\ &\quad = H_x(t, x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t)dt - \sum_{i=1}^d k_t^i dB_t^i \\ &\quad - \sum_{i=1}^\infty \rho_t^i dH_t^i, \\ &p_0 = -\gamma_y(y_0), \\ &q_T = h_x(x_T) - \Phi_x(x_T)p_T. \end{aligned} \tag{49}$$

It is easy to verify that (49) satisfies Assumptions 1 and 2; then there exists a unique quarter (p_t, q_t, k_t, ρ_t) satisfying (49).

Then we have the main result of this paper which is the following theorem.

Theorem 9. *Supposing that Assumptions 1, 2, 4, and 6 hold, $(\overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t)$ is an optimal pair for our optimal control problem and (p_t, q_t, k_t, ρ_t) is the solution to corresponding adjoint equation (49). Then for each admissible control $u_t \in U(0, T)$ we have*

$$\langle H_u(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t, p_t, q_t, k_t, \rho_t), (u_t - \overline{u}_t)\rangle \geq 0 \quad a.e., a.s., \tag{50}$$

where H is defined by (48).

Proof. Applying *Ito's* formula to $\langle X_t, q_t\rangle + \langle Y_t, p_t\rangle$, we can obtain

$$\begin{aligned} &E[h_x(\overline{x}_T)X_T] + E[\gamma_y(\overline{y}_0)Y_0] + E\int_0^T \big[L_x(t)X_t \\ &\quad + L_y(t)Y_t + L_z(t)Z_t + L_r(t)R_t + L_u(t) \\ &\quad \cdot (u_t - \overline{u}_t)\big]dt \\ &\quad = E\int_0^T \langle H_u(t, \overline{x}_t, \overline{y}_t, \overline{z}_t, \overline{r}_t, \overline{u}_t, p_t, q_t, k_t, \rho_t), \\ &\quad (u_t - \overline{u}_t)\rangle dt. \end{aligned} \tag{51}$$

The variational inequality implies for each $u_t \in U[0,T]$ that

$$E\int_0^T \langle H_u(t,\overline{x}_t,\overline{y}_t,\overline{z}_t,\overline{r}_t,\overline{u}_t,p_t,q_t,k_t,\rho_t),(u_t-\overline{u}_t)\rangle\,dt \geq 0. \tag{52}$$

The proof is completed. □

Next, under some additional convexity conditions, we prove that the maximum principle can be a sufficient condition for optimal control.

Theorem 10. *For stochastic control system (30) and the cost functional $J(u)$, if Assumptions 1, 2, 4, and 6 hold, and $y_T = Mx_T$, $M \in L^2(\Omega,\mathcal{F}_T,P;R^{m\times n})$, h is convex in x, and γ is convex in y. Let u_t be an admissible control and let (x_t,y_t,z_t,r_t) be the corresponding trajectory. Let (p_t,q_t,k_t,ρ_t) be the solution of corresponding adjoint equation (49). Suppose that the Hamiltonian function H is convex in (x,y,z,r,u) and inequality (50) holds; then u_t is an optimal control.*

Proof. Let v_t be an arbitrary admissible control and the corresponding trajectory is $(x_t^v,y_t^v,z_t^v,r_t^v)$; then

$$\begin{aligned} &J(v(\cdot)) - J(u(\cdot)) \\ &\quad = E\int_0^T [L(x_t^v,y_t^v,z_t^v,r_t^v,v_t) - L(x_t,y_t,z_t,r_t,u_t)]\,dt \\ &\qquad + E[h(x_T^v) - h(x_T)] + E[\gamma(y_0^v) - \gamma(y_0)] \\ &\quad = I_1 + I_2, \end{aligned} \tag{53}$$

where

$$\begin{aligned} &I_1 \\ &\quad = E\int_0^T [L(x_t^v,y_t^v,z_t^v,r_t^v,v_t) - L(x_t,y_t,z_t,r_t,u_t)]\,dt, \\ &I_2 = E[h(x_T^v) - h(x_T)] + E[\gamma(y_0^v) - \gamma(y_0)]. \end{aligned} \tag{54}$$

From the definition of Hamiltonian function H, we get

$$\begin{aligned} I_1 &= E\int_0^T [H(x_t^v,y_t^v,z_t^v,r_t^v,v_t,p_t^v,q_t^v,k_t^v,\rho_t^v) \\ &\quad - H(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t)]\,dt \\ &\quad + E\int_0^T \langle f(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - f(x_t,y_t,z_t,r_t,u_t),p_t\rangle\,dt \\ &\quad - E\int_0^T \langle b(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - b(x_t,y_t,z_t,r_t,u_t),q_t\rangle\,dt - \sum_{i=1}^d E \\ &\quad \cdot \int_0^T \langle \sigma^i(x_t^v,y_t^v,z_t^v,v_t) - \sigma^i(x_t,y_t,z_t,u_t),k_t^i\rangle\,dt \\ &\quad - \sum_{i=1}^\infty E\int_0^T \langle g^i(x_t^v,y_t^v,r_t^v,v_t) \\ &\quad - g^i(x_t,y_t,r_t,u_t),\rho_t^i\rangle\,dt. \end{aligned} \tag{55}$$

By convexity of h,γ and using *Itô's* formula to $\langle q_t, x_t^v - x_t\rangle + \langle p_t, y_t^v - y_t\rangle$ we can get

$$\begin{aligned} I_2 &\geq E\langle h_x(x_T), x_T^v - x_T\rangle + E\langle \gamma_y(y_0), y_0^v - y_0\rangle \\ &= -E\int_0^T \langle H_x(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), x_t^v \\ &\quad - x_t\rangle\,dt \\ &\quad - E\int_0^T \langle H_y(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), y_t^v \\ &\quad - y_t\rangle\,dt - \sum_{i=1}^d E \\ &\quad \cdot \int_0^T \langle H_z^i(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), z_t^{iv} - z_t^i\rangle\,dt \\ &\quad - \sum_{i=1}^\infty E\int_0^T \langle H_r^i(x_t,y_t,z_t,r_t,u_t,p_t,q_t,k_t,\rho_t), r_t^{iv} \\ &\quad - r_t^i\rangle\,dt - E\int_0^T \langle f(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - f(x_t,y_t,z_t,r_t,u_t),p_t\rangle\,dt \\ &\quad + E\int_0^T \langle b(x_t^v,y_t^v,z_t^v,r_t^v,v_t) \\ &\quad - b(x_t,y_t,z_t,r_t,u_t),q_t\rangle\,dt + \sum_{i=1}^d E \\ &\quad \cdot \int_0^T \langle \sigma^i(x_t^v,y_t^v,z_t^v,v_t) - \sigma^i(x_t,y_t,z_t,u_t),k_t^i\rangle\,dt \\ &\quad + \sum_{i=1}^\infty E\int_0^T \langle g^i(x_t^v,y_t^v,r_t^v,v_t) \\ &\quad - g^i(x_t,y_t,r_t,u_t),\rho_t^i\rangle\,dt, \end{aligned} \tag{56}$$

and from (53) to (56) we have

$$\begin{aligned}
&J(v(\cdot)) - J(u(\cdot)) \\
&\quad = E\int_0^T [H(x_t^v, y_t^v, z_t^v, r_t^v, v_t, p_t^v, q_t^v, k_t^v, \rho_t^v) \\
&\quad - H(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t)]\, dt \\
&\quad - E\int_0^T \langle H_x(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), x_t^v - x_t\rangle\, dt \\
&\quad - E\int_0^T \langle H_y(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), y_t^v \\
&\quad - y_t\rangle\, dt - \sum_{i=1}^d E \\
&\quad \cdot \int_0^T \langle H_z^i(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), z_t^{iv} - z_t^i\rangle\, dt \\
&\quad - \sum_{i=1}^\infty E\int_0^T \langle H_r^i(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), r_t^{iv} \\
&\quad - r_t^i\rangle\, dt.
\end{aligned} \tag{57}$$

Moreover, as the Hamiltonian function H is convex in (x, y, z, r, u), the following inequality holds:

$$\begin{aligned}
&H(x_t^v, y_t^v, z_t^v, r_t^v, v_t, p_t^v, q_t^v, k_t^v, \rho_t^v) \\
&\quad - H(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t) \\
&\quad \geq \langle H_x(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), x_t^v - x_t\rangle \\
&\qquad + \langle H_y(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), y_t^v - y_t\rangle \\
&\qquad + \sum_{i=1}^d \langle H_z^i(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), z_t^{iv} - z_t^i\rangle \\
&\qquad + \sum_{i=1}^\infty \langle H_r^i(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), r_t^{iv} - r_t^i\rangle \\
&\qquad + \langle H_u(x_t, y_t, z_t, r_t, u_t, p_t, q_t, k_t, \rho_t), v_t - u_t\rangle,
\end{aligned} \tag{58}$$

and from (57) to (58) and together with (50) for arbitrary admissible control v_t we have

$$J(v(\cdot)) - J(u(\cdot)) \geq 0, \tag{59}$$

then admissible control u_t is an optimal control.

The proof is completed. □

5. Applications in Linear Quadratic Problem

In this section, we will apply our stochastic maximum principle to the linear quadratic problem which the linear control system described by fully coupled forward-backward stochastic differential equation driven by Lévy processes:

$$\begin{aligned}
dx_t &= \left(A_t x_t + B_t y_t + \sum_{i=1}^d C_t^i z_t^i + \sum_{i=1}^\infty D_t^i r_t^i + E_t u_t\right) dt \\
&\quad + \sum_{i=1}^d \left(F_t^i x_t + G_t^i y_t + V_t^i z_t^i + W_t^i u_t\right) dB_t^i \\
&\quad + \sum_{i=1}^\infty \left(J_t^i x_{t_-} + K_t^i y_{t_-} + L_t^i r_t^i + M_t^i u_t\right) dH_t^i, \\
-dy_t &= \left(N_t x_t + P_t y_t + \sum_{i=1}^d Q_t^i z_t^i + \sum_{i=1}^\infty R_t^i r_t^i + S_t u_t\right) dt \\
&\quad - \sum_{i=1}^d z_t^i dB_t - \sum_{i=1}^\infty r_t^i dH_t^i, \\
x_0 &= a, \\
y_T &= h x_T + \xi
\end{aligned} \tag{60}$$

and the cost functional:

$$\begin{aligned}
&J(u) \\
&\quad = E\langle \widetilde{Q} x_T, x_T\rangle + E\langle \widetilde{P} y_0, y_0\rangle \\
&\qquad + E\int_0^T \left(\langle \widetilde{R}_t x_t, x_t\rangle + \langle \widetilde{N}_t u_t, u_t\rangle + \langle \widetilde{L}_t y_t, y_t\rangle\right) dt \\
&\qquad + \sum_{i=1}^d E\int_0^T \langle \widetilde{M}_t^i z_t^i, z_t^i\rangle\, dt + \sum_{i=1}^\infty E\int_0^T \langle \widetilde{K}_t^i r_t^i, r_t^i\rangle\, dt,
\end{aligned} \tag{61}$$

where the $\mathcal{F}_t$-predictable matrix processes

$$\begin{aligned}
A, F^i, J^j, \widetilde{R} &: [0,T]\times\Omega \longrightarrow R^{n\times n}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
B, C^i, D^j, G^i, V^i, K^j, L^j &: [0,T]\times\Omega \longrightarrow R^{n\times m}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
E, W^i, M^j &: [0,T]\times\Omega \longrightarrow R^{n\times k}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
P, Q^i, R^j, \widetilde{L}, \widetilde{M}^i, \widetilde{K}^j &: [0,T]\times\Omega \longrightarrow R^{m\times m}, \\
&\quad i = 1,2,\ldots,d,\ \ j = 1,2,\ldots \\
N &: [0,T]\times\Omega \longrightarrow R^{m\times n}, \\
S &: [0,T]\times\Omega \longrightarrow R^{m\times k}, \\
\widetilde{N} &: [0,T]\times\Omega \longrightarrow R^{k\times k};
\end{aligned} \tag{62}$$

the $\mathcal{F}_T$-predictable random matrix $\widetilde{Q} : \Omega \to R^{n\times n}$ and the $\mathcal{F}_0$-predictable stochastic matrix $\widetilde{P} : \Omega \to R^{m\times m}$ are all

uniformly bounded. And $a \in \mathbb{R}^{n\times n}$, $h \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^{m\times n})$, and $\xi \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R}^m)$.

To study this problem, we need the assumptions on the coefficients as follows.

Assumption 11. The state weighting matrix processes $\widetilde{R}, \widetilde{L}, \widetilde{M}^i$, and $\widetilde{K}^j, i = 1, 2, \ldots, d,\ j = 1, 2, \ldots$, and the control weighting matrix process $\widetilde{N}$ and random matrixes $\widetilde{Q}$ and $\widetilde{P}$ are almost everywhere almost surely symmetric and nonnegative. Furthermore, $\widetilde{N}$ is almost everywhere almost surely uniformly positive; that is, $N \geq \delta I$, for some positive constant δ almost everywhere almost surely.

Assumption 12. For the control processes there is no further constraint:

$$\begin{aligned} U_{ad} = \Big\{ & u(\cdot) : u(\cdot) \\ & \text{is } \mathscr{F}_t\text{-predictable with values in } \mathbb{R}^k, \\ & E\int_0^T |u(t)|\, dt < \infty \Big\}. \end{aligned} \tag{63}$$

If we denote the norm of U_{ad} by

$$\|u(\cdot)\|_{U_{ad}} = E\sqrt{\int_0^T |u(t)|^2\, dt}; \tag{64}$$

then U_{ad} is a Hilbert space. And by Lemma 3 we also know that the linear FBSDEL (60) has a unique solution; that is, the linear quadratic problem is well defined.

Theorem 13. *Under Assumptions 11 and 12, LQ problems (60) and (61) have a unique optimal control, and the optimal control is*

$$\begin{aligned} u_t = -\frac{1}{2}\widetilde{N}_t^{-1}\Big[& E_t^T q_t - S_t^T p_t + \sum_{i=1}^{d}\left(W_t^i\right)^T k_t^i \\ & + \sum_{i=1}^{\infty}\left(M_t^i\right)^T \rho_t^i \Big]; \end{aligned} \tag{65}$$

here p, q, k, and ρ are the solution of the following adjoint FBSDEL:

$$\begin{aligned} dp_t = \Big[& P_t p_t - B_t^T q_t - \sum_{i=1}^{d}\left(G_t^i\right)^T k_t^i - \sum_{i=1}^{\infty}\left(K_t^i\right)^T \rho_t^i \\ & - 2\widetilde{L}_t y_t \Big] dt \\ & - \sum_{i=1}^{d}\Big[Q_t^i p_t - \left(C_t^i\right)^T q_t - \left(V_t^i\right)^T k_t^i - 2\left(\widetilde{M}_t^i\right) z_t^i \Big] dB_t^i \\ & - \sum_{i=1}^{\infty}\Big[R_t^i p_t^i - \left(D_t^i\right)^T q_t - \left(L_t^i\right)^T \rho_t - 2\widetilde{K}_t^i r_t^i \Big] dH_t^i, \\ -dq_t = \Big[& N_t^T p_t - A_t q_t - \sum_{i=1}^{d} F_t^i k_t^i - \sum_{i=1}^{\infty} J_t^i \rho_t^i - 2\widetilde{R}_t x_t \Big] dt \\ & - \sum_{i=1}^{d} k_t^i dB_t - \sum_{i=1}^{\infty} \rho_t^i dH_t^i, \\ p_0 = & -2\widetilde{P} y_0, \\ q_T = & 2\widetilde{Q} x_T - h p_T. \end{aligned} \tag{66}$$

Proof. From Assumptions 11 and 12 and inequality (29), we can verify that the cost function $J(u(\cdot))$ (61) is strictly convex and continuous over U_{ad} and

$$\lim_{\|u(\cdot)\|_{U_{ad}} \to \infty} J(u(\cdot)) = \infty; \tag{67}$$

then, from Lemma 5.3 in [18], the cost function has a unique minimal value and, together with Lemma 3 (existence and uniqueness theorem of FBSDEL), the LQ problem has a unique optimal control. Next, we will prove that the optimal control u_t has an expression as (65).

Let (x_t, y_t, z_t, r_t) be the optimal state process corresponding to the optimal control u_t and let (p_t, q_t, k_t, ρ_t) be the unique solution to adjoint equation (66) corresponding to the optimal pair $(u_t; x_t, y_t, z_t, r_t)$; then the Hamiltonian function

$$\begin{aligned} H(t, x, y, z, r, u, p, q, k, \rho) = & -\Big\langle p, \\ & \Big(N_t x_t + P_t y_t + \sum_{i=1}^{d} Q_t^i z_t^i + \sum_{i=1}^{\infty} R_t^i r_t^i + S_t u_t \Big) \Big\rangle + \Big\langle q, \\ & \Big(A_t x_t + B_t y_t + \sum_{i=1}^{d} C_t^i z_t^i + \sum_{i=1}^{\infty} D_t^i r_t^i + E_t u_t \Big) \Big\rangle + \Big\langle k, \\ & \sum_{i=1}^{d}\left(F_t^i x_t + G_t^i y_t + V_t^i z_t^i + W_t^i u_t \right) \Big\rangle + \Big\langle \rho, \\ & \sum_{i=1}^{\infty}\left(J_t^i x_{t_-} + K_t^i y_{t_-} + L_t^i r_t^i + M_t^i u_t \right) \Big\rangle + \left\langle \widetilde{R}_t x_t, x_t \right\rangle \\ & + \left\langle \widetilde{N}_t u_t, u_t \right\rangle + \left\langle \widetilde{L}_t y_t, y_t \right\rangle + \sum_{i=1}^{d}\left\langle \widetilde{M}_t^i z_t^i, z_t^i \right\rangle \\ & + \sum_{i=1}^{\infty}\left\langle \widetilde{K}_t^i r_t^i, r_t^i \right\rangle. \end{aligned} \tag{68}$$

From Theorem 9 and Assumption 12 we have

$$H_u = -S_t^T p_t + E_t^T q_t + \sum_{i=1}^{d} \left(W_t^i\right)^T k_t^i + \sum_{i=1}^{\infty} \left(M_t^i\right)^T \rho_t^i + 2N_t u_t = 0; \tag{69}$$

that is

$$u_t = -\frac{1}{2}\widetilde{N}_t^{-1}\left[E_t^T q_t - S_t^T p_t + \sum_{i=1}^{d} \left(W_t^i\right)^T k_t^i + \sum_{i=1}^{\infty} \left(M_t^i\right)^T \rho_t^i\right]; \tag{70}$$

then (65) holds.

The proof is completed. □

6. Conclusion

In this paper, the continuity result depending on parameters of forward-backward stochastic differential equation driven by Lévy process is proved. Based on this result, we get the stochastic maximum principle for fully coupled forward-backward stochastic control system driven by Lévy process. And then we use the stochastic maximum principle to solve LQ problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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