Research Article

Global Asymptotic Stability of Switched Neural Networks with Delays

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This paper investigates the global asymptotic stability of a class of switched neural networks with delays. Several new criteria ensuring global asymptotic stability in terms of linear matrix inequalities (LMIs) are obtained via Lyapunov-Krasovskii functional. And here, we adopt the quadratic convex approach, which is different from the linear and reciprocal convex combinations that are extensively used in recent literature. In addition, the proposed results here are very easy to be verified and complemented. Finally, a numerical example is provided to illustrate the effectiveness of the results.

1. Introduction

In the past thirty years, neural networks have found extensive applications in associative memory, pattern recognition, and image processing [1–3]. It is true that most applications of neural networks are heavily dependent on the dynamic behaviors of neural networks, especially on global asymptotic stability of neural networks. On the other hand, time delays are inevitably encountered in the hardware implementation due to the finite switching speed of amplifier, which may destroy the system performance and become a source of oscillation or instability in neural networks. Therefore, stability of neural networks with delays has attracted increasing attention and lots of stability criteria have been reported in the literature [4, 5].

As a special class of hybrid systems, switched systems are organized by a switching rule that orchestrates the switching. In reality, neural networks sometimes have finite modes that switch from one to another at different times according to a switching law. In [6, 7], the authors studied the stability problem of different kinds of switched neural networks with time delays. Different from the model in these works, in

this paper, we consider a class of neural networks with state-dependent switchings. Our switched neural networks model is general and it generalizes the conventional neural networks.

Recently, convex analysis has been significantly employed in the stability analysis of time-delay systems [8–17]. According to the feature of different convex functions, different convex combination approaches are adopted in the literature, such as the linear convex combination [8–10], reciprocal convex combination [11–13], and quadratic convex combination [15–17]. In [8, 9, 12, 14, 16, 17], convex combination technology was successfully used to derive some stability criteria for neural networks with time delays. It should be pointed out that the lower bound of the time delay in [8, 9, 16] is zero, which means the information on the lower bound of the time delay cannot be sufficiently used. Namely, the conditions obtained in [9, 14, 16] fail to take effect on the stability of neural networks when the lower bound of the time delay is strictly greater than zero.

In this paper, some delay-dependent stability criteria in terms of LMIs are derived. The advantages are as follows. Firstly, differential inclusions and set-valued maps are

employed to deal with the switched neural networks with discontinuous right-hand sides. Secondly, our results employ the quadratic convex approach, which is different from the linear and reciprocal convex combinations that are extensively used in recent literature on stability. Thirdly, the lower bound τ_1 of the time-varying delays is not zero and its information is adequately used to construct the Lyapunov-Krasovskii functional. Fourthly, we resort to neither Jensen's inequality with delay-dividing approach nor the free-weighting matrix method compared with previous results.

The organization of this paper is as follows. Some preliminaries are introduced in Section 2. In Section 3, based on the quadratic convex approach, delay-dependent stability criteria in terms of LMIs are established for switched neural networks with time-varying delays. Then, an example is given to demonstrate the effectiveness of the obtained results in Section 4. Finally, conclusions are given in Section 5.

Notations. Throughout this paper, R^n denotes the n -dimensional Euclidean space. A^T and A^{-1} denote the transpose and the inverse of the matrix A , respectively. $A > 0$ ($A \geq 0$) means that the matrix A is symmetric and positive definite (semipositive definite). $*$ represents the elements below the main diagonal of a symmetric matrix. The identity and zero matrices of appropriate dimensions are denoted by I and 0 , respectively. $\text{SYM}(A)$ is defined as $\text{SYM}(A) = A + A^T$. $\text{diag}\{\dots\}$ denotes a block-diagonal matrix.

2. System Description and Preliminaries

In this paper, we consider a class of switched neural networks with delays as follows:

$$\begin{aligned} \dot{x}_i(t) = & -d_i(x_i(t))x_i(t) + \sum_{j=1}^n a_{ij}(x_i(t))f_j(x_j(t)) \\ & + \sum_{j=1}^n b_{ij}(x_i(t))f_j(x_j(t - \tau_j(t))), \end{aligned} \quad (1)$$

$$t \geq 0, \quad i = 1, 2, \dots, n,$$

where

$$\begin{aligned} d_i(x_i(t)) = & \begin{cases} d_i^*, & x_i(t) \leq 0, \\ d_i^{**}, & x_i(t) > 0, \end{cases} \\ a_{ij}(x_i(t)) = & \begin{cases} a_{ij}^*, & x_i(t) \leq 0, \\ a_{ij}^{**}, & x_i(t) > 0, \end{cases} \\ b_{ij}(x_i(t)) = & \begin{cases} b_{ij}^*, & x_i(t) \leq 0, \\ b_{ij}^{**}, & x_i(t) > 0, \end{cases} \end{aligned} \quad (2)$$

$x_i(t)$ is the state variable of the i th neuron, and $a_{ij}(x_i(t))$ and $b_{ij}(x_i(t))$ denote the feedback connection weight and the

delayed feedback connection weight, respectively. $f_j : R \rightarrow R$ is bounded continuous function; $\tau_j(t)$ corresponds to the transmission delay and satisfies $i, j = 1, 2, \dots, n$. $d_i^* > 0$, $d_i^{**} > 0$, a_{ij}^* , a_{ij}^{**} , b_{ij}^* , and b_{ij}^{**} , $i, j = 1, 2, \dots, n$, are all constant numbers. The initial condition of system (1) is $x(s) = \phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T \in \mathcal{C}([-\tau_2, 0], R^n)$.

The following assumptions are given for system (1):

(H1) For $j \in 1, 2, \dots, n$, f_j is bounded and there exist constants h_j^-, h_j^+ such that

$$h_j^- \leq \frac{f_j(s_1) - f_j(s_2)}{s_1 - s_2} \leq h_j^+, \quad f_j(0) = 0, \quad (3)$$

for all $s_1, s_2 \in R, s_1 \neq s_2$.

(H2) The transmission delay $\tau_j(t)$ is a differential function and there exist constants $0 \leq \tau_1 < \tau_2, \mu$ such that

$$\begin{aligned} 0 \leq \tau_1 \leq \tau_j(t) \leq \tau_2, \\ \dot{\tau}_j(t) \leq \mu, \end{aligned} \quad (4)$$

for all $t \geq 0, j = 1, 2, \dots, n$.

Obviously, system (1) is a discontinuous system; then its solution is different from the classic solution and cannot be defined in the conventional sense. In order to obtain the solution of system (1), some definitions and lemmas are given.

Definition 1. For a system with discontinuous right-hand sides,

$$\frac{dx}{dt} = F(t, x), \quad x(0) = x_0, \quad x \in R^n, \quad t \geq 0. \quad (5)$$

A set-valued map is defined as

$$\Phi(t, x) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{\text{co}}[F(t, B(x, \delta) \setminus N)], \quad (6)$$

where $\overline{\text{co}}[E]$ is the closure of the convex hull of set E , $E \subset R^n$, $B(x, \delta) = \{y : \|y - x\| < \delta, x, y \in R^n, \delta \in R^+\}$, and $N \subset R^n$, $\mu(N)$ is Lebesgue measure of set N .

A solution in Filippov's sense of system (5) with initial condition $x(0) = x_0 \in R^n$ is an absolutely continuous function $x(t)$, $t \in [0, T]$, $T > 0$, which satisfy $x(0) = x_0$ and differential inclusion:

$$\frac{dx}{dt} \in \Phi(t, x), \quad \text{for a.a. } t \in [0, T]. \quad (7)$$

If $F(t, x)$ is bounded, then the set-valued function $\Phi(t, x)$ is upper semicontinuous with nonempty, convex, and compact values [18]. Then, the solution $x(t)$ of system (5) with

initial condition exists and it can be extended to the interval $[0, +\infty)$ in the sense of Filippov.

By applying the theories of set-valued maps and differential inclusions [18–20], system (1) can be rewritten as the following differential inclusion:

$$\begin{aligned} \dot{x}_i(t) \in & -[\underline{d}_i, \bar{d}_i] x_i(t) + \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] f_j(x_j(t)) \\ & + \sum_{j=1}^n [\underline{b}_{ij}, \bar{b}_{ij}] f_j(x_j(t - \tau_i(t))), \end{aligned} \quad (8)$$

for a.a. $t \geq 0$, $i = 1, 2, \dots, n$,

where $[\underline{\xi}_i, \bar{\xi}_i]$ is the convex hull of $[\underline{\xi}_i, \bar{\xi}_i]$, $\underline{\xi}_i, \bar{\xi}_i \in R$. $\underline{d}_i = \min\{d_i^*, d_i^{**}\}$, $\bar{d}_i = \max\{d_i^*, d_i^{**}\}$, $\underline{a}_{ij} = \min\{a_{ij}^*, a_{ij}^{**}\}$, $\bar{a}_{ij} = \max\{a_{ij}^*, a_{ij}^{**}\}$, $\underline{b}_{ij} = \min\{b_{ij}^*, b_{ij}^{**}\}$, and $\bar{b}_{ij} = \max\{b_{ij}^*, b_{ij}^{**}\}$. The other parameters are the same as in system (1).

Definition 2. A constant vector $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ is called an equilibrium point of system (1), if, for $i = 1, 2, \dots, n$,

$$\begin{aligned} 0 \in & -[\underline{d}_i, \bar{d}_i] x_i^* + \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] f_j(x_j^*) \\ & + \sum_{j=1}^n [\underline{b}_{ij}, \bar{b}_{ij}] f_j(x_j^*). \end{aligned} \quad (9)$$

It is easy to find that the origin $(0, 0, \dots, 0)^T$ is an equilibrium point of system (1).

Definition 3 (see [18]). A function $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is a solution of (1), with the initial condition $x(s) = \phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T \in \mathcal{C}([- \tau_2, 0], R^n)$, if $x(t)$ is an absolutely continuous function and satisfies differential inclusion (8).

Lemma 4 (see [18]). Suppose that assumption (H1) is satisfied; then solution $x(t)$ with initial condition $\phi(s) = (\phi_1(s), \phi_2(s), \dots, \phi_n(s))^T \in \mathcal{C}([- \tau_2, 0], R^n)$ of (1) exists and it can be extended to the interval $[0, +\infty)$.

Before giving our main results, we present the following important lemmas that will be used in the proof to derive the stability conditions of the switched neural networks.

Lemma 5 (see [21]). Given constant matrices Σ_1, Σ_2 , and Σ_3 , where $\Sigma_1^T = \Sigma_1$, $\Sigma_2^T = \Sigma_2$,

$$\begin{pmatrix} \Sigma_1 & \Sigma_3 \\ \Sigma_3^T & -\Sigma_2 \end{pmatrix} < 0 \quad (10)$$

is equivalent to the following conditions:

$$\begin{aligned} \Sigma_2 & > 0, \\ \Sigma_1 + \Sigma_3 \Sigma_2^{-1} \Sigma_3^T & < 0. \end{aligned} \quad (11)$$

Lemma 6 (see [17]). For real symmetric matrices $W_1, W_2, W_3 \in R^{m \times m}$ and a scalar continuous function τ satisfy $\tau_1 \leq \tau \leq \tau_2$, where τ_1 and τ_2 are constants satisfying $0 \leq \tau_1 \leq \tau_2$. If $W_1 \geq 0$, then

$$\begin{aligned} \tau^2 W_1 + \tau W_2 + W_3 & < 0, \quad \forall \tau \in [\tau_1, \tau_2] \\ \iff \tau_i^2 W_1 + \tau_i W_2 + W_3 & < 0, \quad (i = 1, 2). \end{aligned} \quad (12)$$

Lemma 7 (see [16]). Let $W > 0$, and let $y(s)$ be an appropriate dimensional vector. Then, one has the following facts for any scalar function $\eta(s) \geq 0$, $\forall s \in [t_1, t_2]$:

$$\begin{aligned} (i) \quad & - \int_{t_1}^{t_2} y^T(s) W y(s) ds \leq (t_2 - t_1) \xi_t^T F_1^T W^{-1} F_1 \xi_t \\ & + 2 \xi_t^T F_1^T \int_{t_1}^{t_2} y(s) ds, \end{aligned} \quad (13)$$

$$\begin{aligned} (ii) \quad & - \int_{t_1}^{t_2} \eta(s) y^T(s) W y(s) ds \\ & \leq \int_{t_1}^{t_2} \eta(s) ds \xi_t^T F_2^T W^{-1} F_2 \xi_t \\ & + 2 \xi_t^T F_2^T \int_{t_1}^{t_2} \eta(s) y(s) ds, \end{aligned} \quad (14)$$

$$\begin{aligned} (iii) \quad & - \int_{t_1}^{t_2} \eta^2(s) y^T(s) W y(s) ds \\ & \leq (t_2 - t_1) \xi_t^T F_3^T W^{-1} F_3 \xi_t \\ & + 2 \xi_t^T F_3^T \int_{t_1}^{t_2} \eta(s) y(s) ds, \end{aligned} \quad (15)$$

where matrices F_i ($i = 1, 2, 3$) and vector ξ_t independent of the integral variable are appropriate dimensional arbitrary ones.

3. Main Results

For presentation convenience, in the following we denote $\tau_{21} = \tau_2 - \tau_1$, $H^- = \text{diag}\{h_1^-, h_2^-, \dots, h_n^-\}$, $H^+ = \text{diag}\{h_1^+, h_2^+, \dots, h_n^+\}$, $E_i = [0_{n \times (i-1)n}, I_n, 0_{n \times (10-i)n}]^T$, $i = 1, 2, \dots, 10$. $A_c = [-D, 0, 0, 0, 0, 0, 0, A, B, 0]^T$, $D = \text{diag}(D_i)_{n \times n}$, $D_i = \min\{\underline{d}_i, \bar{d}_i\}$, $A = (A_{ij})_{n \times n}$, $A_{ij} = \max\{|\underline{a}_{ij}|, |\bar{a}_{ij}|\}$, $B = (B_{ij})_{n \times n}$, $B_{ij} = \max\{|\underline{b}_{ij}|, |\bar{b}_{ij}|\}$.

Theorem 8. Suppose assumptions (H1) and (H2) hold; the origin of system (1) is globally asymptotically stable if there exist matrices $P > 0$, $S > 0$, $Q_i > 0$ ($i = 1, 2, 3, 4, 5$), $R_j > 0$ ($j = 1, 2, 3, 4$), and $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) > 0$, $K_1 = \text{diag}(k_{11}, k_{12}, \dots, k_{1n}) > 0$, $K_2 = \text{diag}(k_{21}, k_{22}, \dots, k_{2n}) > 0$,

and F_i ($i = 1, 2, \dots, 12$), such that the following two LMIs hold:

$$\Xi = \begin{bmatrix} \Xi_{11} & \tau_1 F_4^T & \sqrt{3}\tau_1 F_6^T & \tau_1 F_5^T & \tau_{21} F_1^T & \sqrt{3}\tau_{21} F_3^T & \tau_{21} F_2^T & \tau_{21} F_7^T & \sqrt{3}\tau_{21} F_9^T & \tau_{21} F_8^T \\ * & -\tau_1 Q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\tau_1 R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\tau_{21} Q_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -\tau_{21} R_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -R_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\tau_{21} Q_5 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\tau_{21} R_4 & 0 \\ * & * & * & * & * & * & * & * & * & -R_3 \end{bmatrix} < 0, \quad (16)$$

$$\Pi = \begin{bmatrix} \Pi_{11} & \tau_2 F_4^T & \sqrt{3}\tau_2 F_6^T & \tau_2 F_5^T & \tau_{21} F_{10}^T & \sqrt{3}\tau_{21} F_{12}^T & \tau_{21} F_{11}^T \\ * & -\tau_2 Q_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\tau_2 R_2 & 0 & 0 & 0 & 0 \\ * & * & * & -R_1 & 0 & 0 & 0 \\ * & * & * & * & -\tau_{21} Q_5 & 0 & 0 \\ * & * & * & * & * & -\tau_{21} R_4 & 0 \\ * & * & * & * & * & * & -R_3 \end{bmatrix} < 0,$$

where $\Xi_{11} = \Delta_0 + \text{SYM}((\tau_1 E_2 - E_7)(2F_2 + 3F_3 + 2F_8 + 3F_9)) + \text{SYM}((\tau_1 E_1 - (E_1 + E_5))(2F_5 + 3F_6))$, $\Pi_{11} = \Delta_0 + \text{SYM}([A_c, 0]Q_1[\tau_{21} E_1, E_6]^T) - \text{SYM}(E_7(2F_2 + 3F_3 + 2F_8 + 3F_9)) + \text{SYM}((\tau_2 E_1 - (E_5 + E_6))(2F_5 + 3F_6))$ with

$$\Delta_0 = \text{SYM}([E_1, E_5, E_6 + E_7]$$

$$\begin{aligned} & \cdot P[A_c, E_1 - E_3, E_3 - E_4]^T + E_8 \Gamma A_c^T) + E_8 S E_8^T \\ & - (1 - \mu) E_9 S E_9^T + [E_1, E_3](Q_1 + Q_4)[E_1, E_3]^T \\ & - (1 - \mu)[E_1, E_2]Q_1[E_1, E_2]^T + \text{SYM}([A_c, 0] \\ & \cdot Q_2[\tau_2 E_1, E_5 + E_6 + E_7]^T + [A_c, 0] \\ & \cdot Q_4[\tau_{21} E_1, E_6 + E_7]^T) + [E_1, E_1]Q_2[E_1, E_1]^T \\ & - [E_1, E_4](Q_2 + Q_4)[E_1, E_4]^T + \tau_2[E_1, A_c] \\ & \cdot Q_3[E_1, A_c]^T + \tau_{21}[E_3, E_{10}]Q_5[E_3, E_{10}]^T \\ & + A_c(\tau_2^2 R_1 + \tau_2^3 R_2)A_c^T + E_{10}(\tau_{21}^2 R_3 + \tau_{21}^3 R_4)E_{10}^T \\ & + \text{SYM}([E_5 + E_6, E_1 - E_2]F_4 + [E_7, E_2 - E_4]F_7 \\ & + [E_6, E_3 - E_2]F_{10}) + \text{SYM}(E_8 K_1 H^+ E_1^T \end{aligned}$$

$$\begin{aligned} & + E_1 H^- K_1 E_8^T + E_9 K_2 H^+ E_2^T + E_2 H^- K_2 E_9^T) \\ & - \text{SYM}(E_8 K_1 E_8^T + E_1 H^- K_1 H^+ E_1^T + E_9 K_2 E_9^T \\ & + E_2 H^- K_2 H^+ E_2^T). \end{aligned} \quad (17)$$

Proof. Define a vector $\xi_t \in R^{10n}$ as

$$\begin{aligned} \xi_t^T = & \left[x^T(t), x^T(t - \tau(t)), x^T(t - \tau_1), x^T(t - \tau_2), \right. \\ & \int_{t-\tau_1}^t x^T(s) ds, \int_{t-\tau(t)}^{t-\tau_1} x^T(s) ds, \int_{t-\tau_2}^{t-\tau(t)} x^T(s) ds, \\ & \left. f^T(x(t)), f^T(x(t - \tau(t))), \dot{x}^T(t - \tau_1) \right]. \end{aligned} \quad (18)$$

Consider the following Lyapunov-Krasovskii functional candidate as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \quad (19)$$

where

$$V_1(t) = \zeta^T(t) P \zeta(t),$$

$$V_2(t) = 2 \sum_{i=1}^n \gamma_i \int_0^{x_i(t)} f_i(s) ds,$$

$$\begin{aligned}
V_3(t) &= \int_{t-\tau(t)}^t f^T(x(s)) Sf(x(s)) ds \\
&+ \int_{t-\tau(t)}^{t-\tau_1} [x^T(t) \ x^T(s)] Q_1 [x^T(t) \ x^T(s)]^T ds, \\
V_4(t) &= \int_{t-\tau_2}^t \left\{ [x^T(t) \ x^T(s)] Q_2 [x^T(t) \ x^T(s)]^T \right. \\
&+ (\tau_2 - t + s) [x^T(s) \ \dot{x}^T(s)] Q_3 [x^T(s) \ \dot{x}^T(s)]^T \\
&+ (\tau_2 - t + s)^2 \dot{x}^T(s) R_1 \dot{x}(s) + (\tau_2 - t + s)^3 \dot{x}^T(s) \\
&\cdot R_2 \dot{x}(s) \left. \right\} ds \\
&+ \int_{t-\tau_2}^{t-\tau_1} \left\{ [x^T(t) \ x^T(s)] Q_4 [x^T(t) \ x^T(s)]^T \right. \\
&+ (\tau_2 - t + s) [x^T(s) \ \dot{x}^T(s)] Q_5 [x^T(s) \ \dot{x}^T(s)]^T \\
&+ (\tau_2 - t + s)^2 \dot{x}^T(s) R_3 \dot{x}(s) + (\tau_2 - t + s)^3 \dot{x}^T(s) \\
&\cdot R_4 \dot{x}(s) \left. \right\} ds,
\end{aligned} \tag{20}$$

where $\zeta^T(t) = [x^T(t) \ \int_{t-\tau_1}^t x^T(s) ds \ \int_{t-\tau_2}^{t-\tau_1} x^T(s) ds]$.

Calculating the time derivatives of $V_i(t)$ ($i = 1, 2, 3, 4$) along the trajectories of system (8), we obtain

$$\begin{aligned}
\dot{V}_1(t) &= 2\zeta^T(t) P \dot{\zeta}(t) \leq 2x^T(t) \\
&\cdot P \begin{bmatrix} -Dx(t) + Af(x(t)) + Bf(x(t-\tau(t))) \\ x(t) - x(t-\tau_1) \\ x(t-\tau_1) - x(t-\tau_2) \end{bmatrix} \\
&\leq 2\xi_t^T [E_1, E_5, E_6 + E_7] P [A_c, E_1 - E_3, E_3 - E_4]^T \xi_t \\
&= \xi_t^T \text{SYM}([E_1, E_5, E_6 + E_7] \\
&\cdot P [A_c, E_1 - E_3, E_3 - E_4]^T) \xi_t, \\
\dot{V}_2(t) &= 2 \sum_{i=1}^n \gamma_i f_i(x_i(t)) \dot{x}(t) = 2f^T(x(t)) \Gamma \dot{x}(t) \\
&\leq 2\xi_t^T E_8 \Gamma A_c^T \xi_t, \\
\dot{V}_3(t) &= f^T(x(t)) Sf(x(t)) - (1 - \dot{\tau}(t)) \\
&\cdot f^T(x(t-\tau(t))) Sf(x(t-\tau(t))) \\
&+ [x^T(t) \ x^T(t-\tau_1)] Q_1 [x^T(t) \ x^T(t-\tau_1)]^T \\
&- (1 - \dot{\tau}(t)) [x^T(t) \ x^T(t-\tau(t))] Q_1 [x^T(t) \\
&\cdot x^T(t-\tau(t))]^T
\end{aligned}$$

$$\begin{aligned}
&+ 2 \int_{t-\tau(t)}^{t-\tau_1} \frac{\partial [x^T(t) \ x^T(s)]}{\partial t} Q_1 [x^T(t) \ x^T(s)]^T ds \\
&\leq f^T(x(t)) Sf(x(t)) - (1 - \mu) f^T(x(t-\tau(t))) \\
&\cdot Sf(x(t-\tau(t))) \\
&+ [x^T(t) \ x^T(t-\tau_1)] Q_1 [x^T(t) \ x^T(t-\tau_1)]^T \\
&- (1 - \mu) [x^T(t) \ x^T(t-\tau(t))] Q_1 [x^T(t) \\
&\cdot x^T(t-\tau(t))]^T + 2 [x^T(t), 0] \\
&\cdot Q_1 \int_{t-\tau(t)}^{t-\tau_1} [x^T(t) \ x^T(s)]^T ds \leq \xi_t^T \{E_8 S E_8^T \\
&- (1 - \mu) E_9 S E_9^T + [E_1, E_3] Q_1 [E_1, E_3]^T - (1 - \mu) \\
&\cdot [E_1, E_2] Q_1 [E_1, E_2]^T + 2 [A_c, 0] \\
&\cdot Q_1 [(\tau(t) - \tau_1) E_1, E_6]^T\} \xi_t, \\
\dot{V}_4(t) &= [x^T(t) \ x^T(t)] Q_2 [x^T(t) \ x^T(t)]^T \\
&+ [x^T(t) \ x^T(t-\tau_1)] Q_4 [x^T(t) \ x^T(t-\tau_1)]^T \\
&- [x^T(t) \ x^T(t-\tau_2)] Q_2 [x^T(t) \ x^T(t-\tau_2)]^T \\
&- [x^T(t) \ x^T(t-\tau_2)] Q_4 [x^T(t) \ x^T(t-\tau_2)]^T \\
&+ 2 [\dot{x}^T(t), 0] Q_2 \int_{t-\tau_2}^t [x^T(t) \ x^T(s)]^T ds \\
&+ 2 [\dot{x}^T(t), 0] Q_4 \int_{t-\tau_2}^{t-\tau_1} [x^T(t) \ x^T(s)]^T ds \\
&+ \tau_2 [x^T(t) \ \dot{x}^T(t)] Q_3 [x^T(t) \ \dot{x}^T(t)]^T \\
&+ \tau_{21} [x^T(t-\tau_1) \ \dot{x}^T(t-\tau_1)] Q_5 [x^T(t-\tau_1) \\
&\cdot \dot{x}^T(t-\tau_1)]^T + \dot{x}^T(t) (\tau_2^2 R_1 + \tau_2^3 R_2) \dot{x}(t) + \dot{x}^T(t \\
&- \tau_1) (\tau_{21}^2 R_3 + \tau_{21}^3 R_4) \dot{x}(t-\tau_1) + V_x + V_y \\
&= \xi_t^T \{[E_1, E_1] Q_2 [E_1, E_1]^T + [E_1, E_3] Q_4 [E_1, E_3]^T \\
&- [E_1, E_4] (Q_2 + q_4) [E_1, E_4]^T + 2 [A_c, 0] \\
&\cdot Q_2 [\tau_2 E_1, E_5 + E_6 + E_7]^T + 2 [A_c, 0] \\
&\cdot Q_4 [(\tau_2 - \tau_1) E_1, E_6 + E_7]^T + \tau_2 [E_1, A_c] \\
&\cdot Q_3 [E_1, A_c]^T + \tau_{21} [E_3, E_{10}] Q_5 [E_3, E_{10}]^T \\
&+ A_c (\tau_2^2 R_1 + \tau_2^3 R_2) A_c^T + E_{10} (\tau_{21}^2 R_3 + \tau_{21}^3 R_4) E_{10}^T \} \\
&\cdot \xi_t + V_x + V_y,
\end{aligned}$$

where

$$\begin{aligned}
 V_x = & - \int_{t-\tau_2}^t \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds, \\
 V_y = & - \int_{t-\tau_2}^{t-\tau_1} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_5 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_3 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_4 \dot{x}(s) \right\} ds.
 \end{aligned} \tag{22}$$

It is easy to obtain the following identities:

$$\begin{aligned}
 \tau_2 - t + s &= (\tau(t) - t + s) + (\tau_2 - \tau(t)), \\
 (\tau_2 - t + s)^2 &= (\tau(t) - t + s)^2 + (\tau_2^2 - \tau^2(t)) \\
 &\quad + 2(\tau_2 - \tau(t))(s - t).
 \end{aligned} \tag{23}$$

Thus, we have

$$\begin{aligned}
 V_x = & - \int_{t-\tau_2}^{t-\tau(t)} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 & - \int_{t-\tau(t)}^t \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 = & - \int_{t-\tau_2}^{t-\tau(t)} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 & - \int_{t-\tau(t)}^t \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau(t) - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau(t) - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 & - \int_{t-\tau(t)}^t \dot{x}^T(s) \left[2(\tau_2 - \tau(t)) R_1 \right. \\
 & \left. + 3(\tau_2^2 - \tau^2(t)) R_2 + 6(\tau_2 - \tau(t))(s - t) R_2 \right] \\
 & \cdot \dot{x}(s) ds
 \end{aligned}$$

$$\begin{aligned}
 \leq & - \int_{t-\tau_2}^{t-\tau(t)} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 & - \int_{t-\tau(t)}^t \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau(t) - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau(t) - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 & - 2 \int_{t-\tau(t)}^t \dot{x}^T(s) (\tau_2 - \tau(t)) R_1 \dot{x}(s) ds \\
 & - 3 \int_{t-\tau(t)}^t \dot{x}^T(s) (\tau_2 - \tau(t))^2 R_2 \dot{x}(s) ds \\
 \leq & - \int_{t-\tau_2}^{t-\tau(t)} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds \\
 & - \int_{t-\tau(t)}^t \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_3 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau(t) - t + s) \dot{x}^T(s) R_1 \dot{x}(s) \\
 & \left. + 3(\tau(t) - t + s)^2 \dot{x}^T(s) R_2 \dot{x}(s) \right\} ds.
 \end{aligned} \tag{24}$$

Similarly, we have

$$\begin{aligned}
 V_y \leq & - \int_{t-\tau_2}^{t-\tau(t)} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_5 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau_2 - t + s) \dot{x}^T(s) R_3 \dot{x}(s) \\
 & \left. + 3(\tau_2 - t + s)^2 \dot{x}^T(s) R_4 \dot{x}(s) \right\} ds \\
 & - \int_{t-\tau(t)}^{t-\tau_1} \left\{ \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix} Q_5 \begin{bmatrix} x^T(s) & \dot{x}^T(s) \end{bmatrix}^T \right. \\
 & + 2(\tau(t) - t + s) \dot{x}^T(s) R_3 \dot{x}(s) \\
 & \left. + 3(\tau(t) - t + s)^2 \dot{x}^T(s) R_4 \dot{x}(s) \right\} ds.
 \end{aligned} \tag{25}$$

Applying Lemma 7 to V_x and V_y , we get

$$\begin{aligned}
 V_x \leq & \xi_t^T \left\{ (\tau_2 - \tau(t)) F_1^T Q_3^{-1} F_1 + 2F_1^T [E_7, E_2 - E_4]^T \right. \\
 & + (\tau_2 - \tau(t))^2 F_2^T R_1^{-1} F_2 \\
 & \left. + 4F_2^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \right\}
 \end{aligned}$$

$$\begin{aligned}
& + 3 (\tau_2 - \tau(t)) F_3^T R_2^{-1} F_3 \\
& + 6 F_3^T [(\tau_2 - \tau(t)) E_2 - E_7]^T + \tau(t) F_4^T Q_3^{-1} F_4 \\
& + 2 F_4^T [E_5 + E_6, E_1 - E_2]^T + \tau^2(t) F_5^T R_1^{-1} F_5 \\
& + 4 F_5^T [\tau(t) E_1 - (E_5 + E_6)]^T + 3 \tau(t) F_6^T R_2^{-1} F_6 \\
& + 6 F_6^T [\tau(t) E_1 - (E_5 + E_6)]^T \xi_t, \\
V_y \leq & \xi_t^T \{ (\tau_2 - \tau(t)) F_7^T Q_5^{-1} F_7 + 2 F_7^T [E_7, E_2 - E_4]^T \\
& + (\tau_2 - \tau(t))^2 F_8^T R_3^{-1} F_8 \\
& + 4 F_8^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \\
& + 3 (\tau_2 - \tau(t)) F_9^T R_4^{-1} F_9 \\
& + 6 F_9^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \\
& + (\tau(t) - \tau_1) F_{10}^T Q_5^{-1} F_{10} + 2 F_{10}^T [E_6, E_3 - E_2]^T \\
& + (\tau(t) - \tau_1)^2 F_{11}^T R_3^{-1} F_{11} \\
& + 4 F_{11}^T [(\tau(t) - \tau_1) E_3 - E_6]^T \\
& + 3 (\tau(t) - \tau_1) F_{12}^T R_4^{-1} F_{12} \\
& + 6 F_{12}^T [(\tau(t) - \tau_1) E_3 - E_6]^T \xi_t \}.
\end{aligned} \tag{26}$$

On the other hand, from assumption (H1), we have

$$\begin{aligned}
& - 2 (f_i(x_i(t)) - h_i^- x_i(t)) K_1 (f_i(x_i(t)) - h_i^+ x_i(t)) \\
& \geq 0, \\
& - 2 (f_i(x_i(t - \tau(t))) - h_i^- x_i(t - \tau(t))) \\
& \cdot K_1 (f_i(x_i(t - \tau(t))) - h_i^+ x_i(t - \tau(t))) \geq 0,
\end{aligned} \tag{27}$$

where $K_1 = \text{diag}(k_{11}, k_{12}, \dots, k_{1n}) > 0$, $K_2 = \text{diag}(k_{21}, k_{22}, \dots, k_{2n}) > 0$, $i = 1, 2, \dots, n$.

From (27), it is easy to get that

$$\begin{aligned}
& 2 \xi_t^T \{ -E_8 K_1 E_8^T + E_8 K_1 H^+ E_1^T + E_1 H^- K_1 E_8^T \\
& - E_1 H^- K_1 H^+ E_1^T - E_9 K_2 E_9^T + E_9 K_2 H^+ E_2^T \\
& + E_2 H^- K_2 E_9^T - E_2 H^- K_2 H^+ E_2^T \} \geq 0.
\end{aligned} \tag{28}$$

It follows from (19)–(26) and (28) that

$$\begin{aligned}
\dot{V}(t) \leq & \xi_t^T \{ 2 [E_1, E_5, E_6 + E_7] \\
& \cdot P [A_c, E_1 - E_3, E_3 - E_4]^T + 2 E_8 \Gamma A_c^T + E_8 S E_8^T \\
& - (1 - \mu) E_9 S E_9^T + [E_1, E_3] Q_1 [E_1, E_3]^T - (1 - \mu) \\
& \cdot [E_1, E_2] Q_1 [E_1, E_2]^T + 2 [A_c, 0]
\end{aligned}$$

$$\begin{aligned}
& \cdot Q_1 [(\tau(t) - \tau_1) E_1, E_6]^T + [E_1, E_1] Q_2 [E_1, E_1]^T \\
& + [E_1, E_3] Q_4 [E_1, E_3]^T - [E_1, E_4] (Q_2 + q_4) \\
& \cdot [E_1, E_4]^T + 2 [A_c, 0] Q_2 [\tau_2 E_1, E_5 + E_6 + E_7]^T \\
& + 2 [A_c, 0] Q_4 [\tau_{21} E_1, E_6 + E_7]^T + \tau_2 [E_1, A_c] \\
& \cdot Q_3 [E_1, A_c]^T + \tau_{21} [E_3, E_{10}] Q_5 [E_3, E_{10}]^T \\
& + A_c (\tau_2^2 R_1 + \tau_2^3 R_2) A_c^T + E_{10} (\tau_{21}^2 R_3 + \tau_{21}^3 R_4) E_{10}^T \\
& + (\tau_2 - \tau(t)) F_1^T Q_3^{-1} F_1 + 2 F_1^T [E_7, E_2 - E_4]^T \\
& + (\tau_2 - \tau(t))^2 F_2^T R_1^{-1} F_2 \\
& + 4 F_2^T [(\tau_2 - \tau(t)) E_2 - E_7]^T + 3 (\tau_2 - \tau(t)) \\
& \cdot F_3^T R_2^{-1} F_3 + 6 F_3^T [(\tau_2 - \tau(t)) E_2 - E_7]^T + \tau(t) \\
& \cdot F_4^T Q_3^{-1} F_4 + 2 F_4^T [E_5 + E_6, E_1 - E_2]^T + \tau^2(t) \\
& \cdot F_5^T R_1^{-1} F_5 + 4 F_5^T [\tau(t) E_1 - (E_5 + E_6)]^T + 3 \tau(t) \\
& \cdot F_6^T R_2^{-1} F_6 + 6 F_6^T [\tau(t) E_1 - (E_5 + E_6)]^T \\
& + (\tau_2 - \tau(t)) F_7^T Q_5^{-1} F_7 + 2 F_7^T [E_7, E_2 - E_4]^T \\
& + (\tau_2 - \tau(t))^2 F_8^T R_3^{-1} F_8 \\
& + 4 F_8^T [(\tau_2 - \tau(t)) E_2 - E_7]^T + 3 (\tau_2 - \tau(t)) \\
& \cdot F_9^T R_4^{-1} F_9 + 6 F_9^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \\
& + (\tau(t) - \tau_1) F_{10}^T Q_5^{-1} F_{10} + 2 F_{10}^T [E_6, E_3 - E_2]^T \\
& + (\tau(t) - \tau_1)^2 F_{11}^T R_3^{-1} F_{11} \\
& + 4 F_{11}^T [(\tau(t) - \tau_1) E_3 - E_6]^T + 3 (\tau(t) - \tau_1) \\
& \cdot F_{12}^T R_4^{-1} F_{12} + 6 F_{12}^T [(\tau(t) - \tau_1) E_3 - E_6]^T \\
& - 2 E_8 K_1 E_8^T + 2 E_8 K_1 H^+ E_1^T + 2 E_1 H^- K_1 E_8^T \\
& - 2 E_1 H^- K_1 H^+ E_1^T - 2 E_9 K_2 E_9^T + 2 E_9 K_2 H^+ E_2^T \\
& + 2 E_2 H^- K_2 E_9^T - 2 E_2 H^- K_2 H^+ E_2^T \} \xi_t = \xi_t^T (\Delta_0 \\
& + \Delta_1) \xi_t,
\end{aligned} \tag{29}$$

where Δ_0 is defined in the theorem context, and

$$\begin{aligned}
\Delta_1 = & 2 [A_c, 0] Q_1 [(\tau(t) - \tau_1) E_1, E_6]^T \\
& + (\tau_2 - \tau(t)) F_1^T Q_3^{-1} F_1 + (\tau_2 - \tau(t))^2 F_2^T R_1^{-1} F_2 \\
& + 4 F_2^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \\
& + 3 (\tau_2 - \tau(t)) F_3^T R_2^{-1} F_3
\end{aligned}$$

$$\begin{aligned}
& + 6F_3^T [(\tau_2 - \tau(t)) E_2 - E_7]^T + \tau(t) F_4^T Q_3^{-1} F_4 \\
& + \tau^2(t) F_5^T R_1^{-1} F_5 + 4F_5^T [\tau(t) E_1 - (E_5 + E_6)]^T \\
& + 3\tau(t) F_6^T R_2^{-1} F_6 + 6F_6^T [\tau(t) E_1 - (E_5 + E_6)]^T \\
& + (\tau_2 - \tau(t)) F_7^T Q_5^{-1} F_7 + (\tau_2 - \tau(t))^2 F_8^T R_3^{-1} F_8 \\
& + 4F_8^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \\
& + 3(\tau_2 - \tau(t)) F_9^T R_4^{-1} F_9 \\
& + 6F_9^T [(\tau_2 - \tau(t)) E_2 - E_7]^T \\
& + (\tau(t) - \tau_1) F_{10}^T Q_5^{-1} F_{10} \\
& + (\tau(t) - \tau_1)^2 F_{11}^T R_3^{-1} F_{11} \\
& + 4F_{11}^T [(\tau(t) - \tau_1) E_3 - E_6]^T \\
& + 3(\tau(t) - \tau_1) F_{12}^T R_4^{-1} F_{12} \\
& + 6F_{12}^T [(\tau(t) - \tau_1) E_3 - E_6]^T.
\end{aligned} \tag{30}$$

It is easy to see that $\Delta_0 + \Delta_1$ is a quadratic convex combination of matrices on $\tau(t) \in [\tau_1, \tau_2]$.

Applying Lemma 5 to (16), we have

$$\begin{aligned}
(\Delta_0 + \Delta_1)|_{\tau(t)=\tau_1} &= \Delta_0 + \Delta_1|_{\tau(t)=\tau_1} = \Delta_0 + 2(\tau_{21} E_2 \\
&- E_7)(2F_2 + 3F_3 + 2F_8 + 3F_9) + 2(\tau_1 E_1 \\
&- (E_1 + E_5))(2F_5 + 3F_6) + \tau_1(F_4^T Q_3^{-1} F_4 \\
&+ 3F_6^T R_2^{-1} F_6) + \tau_1^2 F_5^T R_1^{-1} F_5 + \tau_{21}(F_1^T Q_3^{-1} F_1 \\
&+ F_7^T Q_5^{-1} F_7 + 3F_3^T R_2^{-1} F_3 + 3F_9^T R_4^{-1} F_9) \\
&+ \tau_{21}^2(F_2^T R_1^{-1} F_2 + F_8^T R_3^{-1} F_8) < 0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
(\Delta_0 + \Delta_1)|_{\tau(t)=\tau_2} &= \Delta_0 + \Delta_1|_{\tau(t)=\tau_2} = \Delta_0 + 2[A_c, 0] \\
&\cdot Q_1[\tau_{21} E_1, E_6]^T - 2E_7(2F_2 + 3F_3 + 2F_8 + 3F_9) \\
&+ 2(\tau_2 E_1 - (E_5 + E_6))(2F_5 + 3F_6) + \tau_2(F_4^T Q_3^{-1} F_4 \\
&+ 3F_6^T R_2^{-1} F_6) + \tau_2^2 F_5^T R_1^{-1} F_5 < 0.
\end{aligned}$$

Since $\xi_t^T \{F_2^T R_1^{-1} F_2 + F_5^T R_1^{-1} F_5 + F_8^T R_3^{-1} F_8 + F_{11}^T R_3^{-1} F_{11}\} \xi_t \geq 0$, from Lemma 6, if LMIs (16) are true, then $\Delta_0 + \Delta_1 < 0, \forall \tau(t) \in [\tau_1, \tau_2]$. Then, we can see that the origin of system (1) is asymptotically stable.

The proof is completed. $\square$

Remark 9. In [16], stability analysis for neural networks with time-varying delay is studied by using quadratic convex combination. Our results have two advantages compared with the results of that paper. On the one hand, the information on the lower bound τ_1 of the time-varying delays is considered.

On the other hand, the augmented vector $\zeta(t)$ includes the distributed delay terms.

Remark 10. We use three inequalities in Lemma 7 combined with the quadratic convex combination implied by Lemma 6, rather than Jensen's inequality and the linear convex combination. In addition, our theoretical proof is not concerned with free-weighting matrix method.

Remark 11. To use the quadratic convex approach, we construct the Lyapunov-Krasovskii functional with the following term: $\int_{t-\tau_2}^t \sum_{j=1}^3 (\tau_2 + t - s)^j g(s) ds$. The degree increase of $\tau_2 - t + s$ by 1 means the number increase of the integral by 1 due to the fact that $\int_{\tau_2}^t \int_{t+\theta}^t g(s) ds d\theta = \int_{t-\tau_2}^t (\tau_2 - t + s) g(s) ds$.

In the case $\tau_1 = 0$, we have the following result.

Corollary 12. Suppose assumptions (H1) and (H2) hold with $\tau_1 = 0$; the origin of system (1) is globally asymptotically stable if there exist matrices $P > 0, S > 0, Q_i > 0$ ($i = 1, 2, 3$), $R_j > 0$ ($j = 1, 2$), and $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) > 0, K_1 = \text{diag}(k_{11}, k_{12}, \dots, k_{1n}) > 0, K_2 = \text{diag}(k_{21}, k_{22}, \dots, k_{2n}) > 0$, and F_i ($i = 1, 2, \dots, 6$), such that the following two LMIs hold:

$$\begin{aligned}
\Xi &= \begin{bmatrix} \Delta_0 & \tau_2 F_1^T & \sqrt{3} \tau_2 F_3^T & \tau_2 F_2^T \\ * & -\tau_2 Q_3 & 0 & 0 \\ * & * & -\tau_2 R_2 & 0 \\ * & * & * & -R_1 \end{bmatrix} < 0, \\
\Pi &= \begin{bmatrix} \Pi_{11} & \tau_2 F_4^T & \sqrt{3} \tau_2 F_6^T & \tau_2 F_5^T \\ * & -\tau_2 Q_3 & 0 & 0 \\ * & * & -\tau_2 R_2 & 0 \\ * & * & * & -R_1 \end{bmatrix} < 0,
\end{aligned} \tag{32}$$

where $\Pi_{11} = \Delta_0 + \tau_2 \text{SYM}([A_c, 0] Q_1 [E_1, 0]^T) - \tau_2 \text{SYM}(E_2(2F_2 + 3F_3)) + \tau_2 \text{SYM}(E_1(2F_5 + 3F_6))$ with

$$\begin{aligned}
\Delta_0 &= \text{SYM}([E_1, E_4 + E_5] P [A_c, E_1 - E_3]^T + E_6 \Gamma A_c^T) \\
&+ E_6 S E_6^T - (1 - \mu) E_7 S E_7^T + [E_1, E_1] (Q_1 + Q_2) \\
&\cdot [E_1, E_1]^T - (1 - \mu) [E_1, E_2] Q_1 [E_1, E_2]^T \\
&- [E_1, E_3] Q_2 [E_1, E_3]^T + \text{SYM}([A_c, 0] Q_1 [0, E_5]^T \\
&+ [A_c, 0] Q_2 [\tau_2 E_1, E_4 + E_5]^T) + \tau_2 [E_1, A_c] \\
&\cdot Q_3 [E_1, A_c]^T + A_c (\tau_2^2 R_1 + \tau_2^3 R_2) A_c^T \\
&+ \text{SYM}([E_5, E_1 - E_2] F_1 \\
&+ (\tau_2 E_2 - E_5)(2F_2 + 3F_3) + [E_4, E_1 - E_2] F_4) \\
&- \text{SYM}(E_4(2F_5 + 3F_6)) + \text{SYM}(E_8 K_1 H^+ E_1^T \\
&+ E_1 H^- K_1 E_8^T + E_9 K_2 H^+ E_2^T + E_2 H^- K_2 E_9^T)
\end{aligned}$$

$$\begin{aligned}
& - \text{SYM} \left(E_8 K_1 E_8^T + E_1 H^- K_1 H^+ E_1^T + E_9 K_2 E_9^T \right. \\
& \left. + E_2 H^- K_2 H^+ E_2^T \right).
\end{aligned} \tag{33}$$

In addition, when the information of the time derivative of delays is unknown or the derivative of the delays does not exist, then we have the following result.

Corollary 13. Suppose assumption (H1) holds with $\tau_1 = 0$; the origin of system (1) is globally asymptotically stable if there exist matrices $P > 0$, $Q_i > 0$ ($i = 2, 3$), $R_j > 0$ ($j = 1, 2$), and $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n) > 0$, $K_1 = \text{diag}(k_{11}, k_{12}, \dots, k_{1n}) > 0$, $K_2 = \text{diag}(k_{21}, k_{22}, \dots, k_{2n}) > 0$, and F_i ($i = 1, 2, \dots, 6$), such that the following two LMIs hold:

$$\begin{aligned}
\Xi &= \begin{bmatrix} \Delta_0 & \tau_2 F_1^T & \sqrt{3}\tau_2 F_3^T & \tau_2 F_2^T \\ * & -\tau_2 Q_3 & 0 & 0 \\ * & * & -\tau_2 R_2 & 0 \\ * & * & * & -R_1 \end{bmatrix} < 0, \\
\Pi &= \begin{bmatrix} \Pi_{11} & \tau_2 F_4^T & \sqrt{3}\tau_2 F_6^T & \tau_2 F_5^T \\ * & -\tau_2 Q_3 & 0 & 0 \\ * & * & -\tau_2 R_2 & 0 \\ * & * & * & -R_1 \end{bmatrix} < 0,
\end{aligned} \tag{34}$$

where $\Pi_{11} = \Delta_0 + \tau_2 \text{SYM}([A_c, 0]Q_1[E_1, 0]^T) - \tau_2 \text{SYM}(E_2(2F_2 + 3F_3)) + \tau_2 \text{SYM}(E_1(2F_5 + 3F_6))$ with

$$\begin{aligned}
\Delta_0 &= \text{SYM}([E_1, E_4 + E_5]P[A_c, E_1 - E_3]^T + E_6 \Gamma A_c^T) \\
&+ [E_1, E_1]Q_2[E_1, E_1]^T - [E_1, E_3]Q_2[E_1, E_3]^T \\
&+ \text{SYM}([A_c, 0]Q_1[0, E_5]^T \\
&+ [A_c, 0]Q_2[\tau_2 E_1, E_4 + E_5]^T) + \tau_2 [E_1, A_c] \\
&\cdot Q_3[E_1, A_c]^T + A_c(\tau_2^2 R_1 + \tau_2^3 R_2)A_c^T \\
&+ \text{SYM}([E_5, E_1 - E_2]F_1 \\
&+ (\tau_2 E_2 - E_5)(2F_2 + 3F_3) + [E_4, E_1 - E_2]F_4) \\
&- \text{SYM}(E_4(2F_5 + 3F_6)) + \text{SYM}(E_8 K_1 H^+ E_1^T \\
&+ E_1 H^- K_1 E_8^T + E_9 K_2 H^+ E_2^T + E_2 H^- K_2 E_9^T) \\
&- \text{SYM}(E_8 K_1 E_8^T + E_1 H^- K_1 H^+ E_1^T + E_9 K_2 E_9^T \\
&+ E_2 H^- K_2 H^+ E_2^T).
\end{aligned} \tag{35}$$

Remark 14. It is worth noting that when we consider system (1) without switching, that is, $d_i^* = d_i^{**}$, $a_{ij}^* = a_{ij}^{**}$, and $b_{ij}^* = b_{ij}^{**}$, then Corollary 12 is the main Theorem 1 of [16].

Remark 15. Compared with the results on stability of neural networks with continuous right-hand side [6], our results

on stability of neural networks are with discontinuous right-hand sides. So the results of this paper are less conservative and more general.

4. Numerical Example

In this section, an example is provided to verify the effectiveness of the results obtained in the previous section.

Example 1. Consider two-dimensional switched neural networks with time-varying delays

$$\begin{aligned}
\dot{x}_i(t) &= -d_i(x_i(t))x_i(t) + \sum_{j=1}^2 a_{ij}(x_i(t))f_j(x_j(t)) \\
&+ \sum_{j=1}^2 b_{ij}(x_i(t))f_j(x_j(t - \tau_j(t))),
\end{aligned} \tag{36}$$

$t \geq 0, i = 1, 2,$

where

$$\begin{aligned}
d_1(x_1(t)) &= \begin{cases} 2, & x_1(t) \leq 0, \\ 2.1, & x_1(t) > 0, \end{cases} \\
d_2(x_2(t)) &= \begin{cases} 2.1, & x_2(t) \leq 0, \\ 2, & x_2(t) > 0, \end{cases} \\
a_{11}(x_1(t)) &= \begin{cases} 1, & x_1(t) \leq 0, \\ 0.9, & x_1(t) > 0, \end{cases} \\
a_{12}(x_1(t)) &= \begin{cases} 1, & x_1(t) \leq 0, \\ 1.2, & x_1(t) > 0, \end{cases} \\
a_{21}(x_2(t)) &= \begin{cases} -0.9, & x_2(t) \leq 0, \\ -1, & x_2(t) > 0, \end{cases} \\
a_{22}(x_2(t)) &= \begin{cases} -1.2, & x_2(t) \leq 0, \\ -1, & x_2(t) > 0, \end{cases} \\
b_{11}(x_1(t)) &= \begin{cases} -0.7, & x_1(t) \leq 0, \\ -0.8, & x_1(t) > 0, \end{cases} \\
b_{12}(x_1(t)) &= \begin{cases} 1, & x_1(t) \leq 0, \\ 1.2, & x_1(t) > 0, \end{cases} \\
b_{21}(x_2(t)) &= \begin{cases} 1, & x_2(t) \leq 0, \\ 0.9, & x_2(t) > 0, \end{cases} \\
b_{22}(x_2(t)) &= \begin{cases} -1, & x_2(t) \leq 0, \\ -1.2, & x_2(t) > 0, \end{cases}
\end{aligned} \tag{37}$$

with $\tau_j(t) = 0.5 \cos(1.6t) + 3$, $j = 1, 2$, and take the activation function as $f(x) = (\sin(x_1), \sin(x_2))^T$. We can obtain that $\tau_1 = 2.5$, $\tau_2 = 3.5$, $\tau_{21} = 1$, $\mu = 0.8$,

$$\begin{aligned} D &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ A &= \begin{bmatrix} 1 & 1.2 \\ 1 & 1.2 \end{bmatrix}, \\ B &= \begin{bmatrix} 0.8 & 1.2 \\ 1 & 1.2 \end{bmatrix} \\ H^- &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ H^+ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned} \quad (38)$$

For written simplification, we use the Matlab LMI Control Toolbox and then a solution to LMIs (32) is obtained (F_i , $i = 1, \dots, 6$, is omitted since the dimension is too big) as follows:

$$\begin{aligned} P &= \begin{bmatrix} 28.5171 & -20.9241 & 0.5484 & -0.4280 \\ -20.9241 & 27.7701 & -0.4064 & 0.4984 \\ 0.5484 & -0.4064 & 10.8451 & -3.9493 \\ -0.4280 & 0.4984 & -3.9493 & 10.6288 \end{bmatrix}, \\ Q_1 &= \begin{bmatrix} 7.3594 & -1.0832 & 0.0588 & -0.0060 \\ -1.0832 & 7.4606 & -0.0074 & 0.0723 \\ 0.0588 & -0.0074 & 3.5541 & -0.4045 \\ -0.0060 & 0.0723 & -0.4045 & 3.6751 \end{bmatrix}, \\ Q_2 &= \begin{bmatrix} 67.4363 & -64.0296 & 3.2893 & -3.0414 \\ -64.0296 & 65.2276 & -3.0862 & 0.0723 \\ 3.2893 & -3.0862 & 35.1371 & -5.2829 \\ -3.0414 & 0.0723 & -5.2829 & 34.8395 \end{bmatrix}, \\ Q_3 &= \begin{bmatrix} 15.7736 & -2.2877 & 0.8863 & -0.8539 \\ -2.2877 & 15.6801 & -0.8085 & 0.4984 \\ 0.8863 & -0.8085 & 7.6796 & -6.7268 \\ -0.8539 & 0.4984 & -6.7268 & 7.3682 \end{bmatrix}, \\ S &= \begin{bmatrix} 2.4075 & 1.0722 \\ 1.0722 & 2.5766 \end{bmatrix}, \\ G &= \begin{bmatrix} 3.5681 & 0 \\ 0 & 3.7071 \end{bmatrix}, \end{aligned}$$

FIGURE 1: The state curves of system (36).

$$\begin{aligned} K_1 &= \begin{bmatrix} 57.4100 & 0 \\ 0 & 65.2979 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} 26.5401 & 0 \\ 0 & 29.9701 \end{bmatrix}. \end{aligned} \quad (39)$$

Therefore, according to Corollary 12, we can see that the origin of system (36) is globally asymptotically stable. The state trajectories of variables $x_1(t)$ and $x_2(t)$ are shown in Figure 1.

Remark 16. Because the parameters of system (1) are discontinuous, the results obtained in [6] about neural networks with continuous right-hand sides cannot be used here. In addition, the lower bounds of the delays of system (36) are not zero, so the results obtained in [8, 9, 16] cannot be used here.

5. Conclusions

In this paper, the delay-dependent stability for a class of switched neural networks with time-varying delays has been studied by using the quadratic convex combination. Some delay-dependent criteria in terms of LMIs have been obtained. The lower bound τ_1 of the time-varying delays is considered to be nonzero so that the information of τ_1 can be used adequately. It is worth noting that we resort to neither Jensen's inequality with delay-dividing approach nor the free-weighting matrix method compared with previous results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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