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Research Article

On a Time-Fractional Integrodifferential Equation via Three-Point Boundary Value Conditions

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The existence and the uniqueness theorems play a crucial role prior to finding the numerical solutions of the fractional differential equations describing the models corresponding to the real world applications. In this paper, we study the existence of solutions for a time-fractional integrodifferential equation via three-point boundary value conditions.

1. Introduction

There is no doubt that the fractional calculus has various important applications in many fields as mathematics (see the following monographs [1, 2] and the references therein), physics [3], and economics [4], as well as in many other branches of science and engineering [3, 5, 6]. Recently the numerical methods were applied intensively to solve complicated fractional differential equations (FDE) which describe the real world applications. As an example, we mention that several physical processes exhibit fractional order behavior varying with time and/or space [7]. Thus, a class of phenomena can be obtained by analyzing this most general model. A fundamental step is to prove the existence and uniqueness of the proposed fractional nonlinear differential equations for these models. There are many works on fractional differential equations and inclusions. The existence of solutions for integrodifferential equations of fractional order with nonlocal three-point fractional boundary conditions was developed in [8] while the importance of antiperiodic type integral boundary conditions was discussed in [9]. The existence of solutions for fractional differential inclusions in the presence of the separated boundary conditions in Banach space was developed in [10]. We recall that the existence and multiplicity of positive solutions for singular fractional boundary value problems were analyzed in [11]. The existence and multiplicity of positive solutions for singular fractional boundary value problems were the topic debated in [12]. Making use of the fixed point results on cones, the existence and uniqueness of positive solutions for some nonlinear fractional differential equations were developed in [13]. Further results on positive solutions of a boundary value problem for nonlinear fractional differential equations were reported in [14] and new results on the existence results on nonlinear fractional differential equations were reported in [15]. For other related results we suggest for the readers [16, 17] as well as the references therein. We recall that the existence of solutions for some fractional partial differential equations was investigated in [18, 19] and the references therein.

Let n be a natural number, $n-1 < \alpha < n$, $a, b \in \mathbb{R}$, and $u \in C([a,b] \times [a,b], \mathbb{R})$. The Riemann-Liouville time-fractional order integral of the function u is defined by $I_{*t}^{\alpha} u(x,t) = (1/\Gamma(\alpha)) \int_0^t (u(x,\tau)/(t-\tau)^{1-\alpha}) d\tau$ for all $x, t \in [a,b]$ whenever the integral exists (more details regarding the basic definitions of the fractional calculus can be seen in [1]). Also, the Caputo derivative of time-fractional of order

α for the function u is defined by ${}^cD^{\alpha}_{*t}u(x,t) = (1/\Gamma(n-\alpha))\int_0^t (t-\tau)^{n-\alpha-1}(\partial^n u(x,\tau)/\partial\tau^n)d\tau$ (see for more details [1]). It has been proved that the general solution of the time-fractional differential equation ${}^cD^{\alpha}_{*t}u(x,t) = 0$ is given by $u(x,t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_0,\ldots,c_{n-1}$ are real constants and $n = [\alpha]+1$ (see [20]). Also, for each $T > 0$ and $u \in C([0,T]\times[0,T])$ we have $I^{\alpha}_{*t}\,{}^cD^{\alpha}_{*t}u(x,t) = u(x,t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$, where $c_0,\ldots,c_{n-1}$ are real constants and $n = [\alpha]+1$ ([20]). Let M be a bounded set in a metric space (X,d). Then the Kuratowski measure of noncompactness is defined by (see [21])

$$\alpha(M) = \inf\{\epsilon : M \text{ covered by finitely many sets in which the diameter of each set is less than or equal to } \epsilon\}. \tag{1}$$

Let $\Phi : D(\Phi) \subseteq X \to X$ be a bounded and continuous operator on a Banach space X. Then Φ is called a condensing map whenever $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset D(\Phi)$, where α denotes the Kuratowski measure of noncompactness (see [21]).

In this paper, we study the existence of solutions for a time-fractional integrodifferential equation via three-point boundary value conditions under certain conditions. We mention that the investigated equation generalized a huge class of classical ordinary differential equations which can be found in applications in engineering and physics. The boundary conditions used in this paper are as general as possible and for particular cases of the parameters we recover several cases of fractional nonlinear differential equations. In this way, we use the next Lemma and Sadovskii's fixed point theorem for condensing maps (see [22]).

Lemma 1 (see [23]). *Let X be a Banach space, $D \subseteq X$, $k \in [0,1)$, $K : D \to X$ a k-contraction, and $C : D \to X$ a compact operator. Then $K + C$ is a k-set contraction and so a condensing map.*

Theorem 2 (see [22]). *Let B be a convex, bounded, and closed subset of a Banach space X and $\Phi : B \to B$ a condensing map. Then Φ has a fixed point.*

2. Main Results

Suppose that $J = [0,1]\times[0,1]$, $1 \le q < 2$, and $X = \{u : u, (\partial^q/\partial t^q)u \in C(J,\mathbb{R})\}$ endowed via the norm

$$\|u\| = \sup_{(x,t)\in J}|u(x,t)| + \sup_{(x,t)\in J}\left|\frac{\partial^q}{\partial t^q}u(x,t)\right|, \tag{2}$$

where $(\partial^q/\partial t^q)u(x,t)$ denotes the standard Caputo time-fractional derivative. Let $2 \le \alpha < 3$, $0 < \beta < 1$, $1 \le \delta < 2$, $0 \le \eta \le 1$, and $\lambda,\mu \in \mathbb{R}$, and $f : J\times X^2 \to X$ and $g : J\times X \to X$ are continuous functions. We investigate the nonlinear time-fractional integrodifferential equation as follows:

$$\begin{aligned} -\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t) &= \lambda f\left(x,t,u(x,t),\frac{\partial^q}{\partial t^q}u(x,t)\right) \\ &\quad + \mu I^{\beta}_{*t}g(x,t,u(x,t)) \end{aligned} \tag{$*$}$$

via the three-point boundary value conditions $(\partial^{\delta+1}/\partial t^{\delta+1})u(x,0) = (\partial^{\delta+1}/\partial t^{\delta+1})u(x,1)$, $(\partial^{\delta}/\partial t^{\delta})u(x,1) - (\partial^{\delta}/\partial t^{\delta})u(x,\eta) = 0$, and $u(x,0) = 0$. In this way, we give first next result.

Lemma 3. *An element u_0 in X is a solution for the problem $(*)$ via the three-point boundary value conditions if and only if u_0 is a solution for the time-fractional integral equation*

$$\begin{aligned} u(x,t) = \int_0^1 G_x(t,\tau)\Big[&\lambda f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial\tau^q}u(x,\tau)\right) \\ &+ \mu I^{\beta}_{*t}g(x,\tau,u(x,\tau))\Big]d\tau, \end{aligned} \tag{3}$$

where

$$\begin{aligned} G_x(t,\tau) &= -\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \Big(\Big[2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right) \\ &\qquad + t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)\Big](1-\tau)^{\alpha-\delta-2}\Big) \\ &\quad \times\left(2\Gamma(\alpha-\delta-1)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]\right)^{-1} \\ &\quad + \frac{\left[t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-1}}{2\Gamma(\alpha-\delta)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad + \frac{\left[2t\Gamma(3-\delta)-t^2(1-\delta)\Gamma(3-\delta)\right](\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]}, \end{aligned} \tag{4}$$

whenever $0 \le \tau \le \eta \le t \le 1$,

$$\begin{aligned} G_x(t,\tau) &= -\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \Big(\Big[2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right) \\ &\qquad + t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)\Big](1-\tau)^{\alpha-\delta-2}\Big) \\ &\quad \times\left(2\Gamma(\alpha-\delta-1)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]\right)^{-1} \\ &\quad + \frac{\left[t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-1}}{2\Gamma(\alpha-\delta)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]}, \end{aligned} \tag{5}$$

whenever $0 \le \eta \le \tau \le t \le 1$,

$$\begin{aligned} G_x(t,\tau) &= \left(\left[2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right) + t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-2}\right) \\ &\quad \times \left(2\Gamma(\alpha-\delta-1)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]\right)^{-1} \\ &\quad + \frac{\left[t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-1}}{2\Gamma(\alpha-\delta)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]}, \end{aligned} \tag{6}$$

whenever $0 \le \eta \le t \le \tau \le 1$,

$$\begin{aligned} G_x(t,\tau) &= \left(\left[2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right) + t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-2}\right) \\ &\quad \times \left(2\Gamma(\alpha-\delta-1)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]\right)^{-1} \\ &\quad + \frac{\left[t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-1}}{2\Gamma(\alpha-\delta)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]}, \end{aligned} \tag{7}$$

whenever $0 \le t \le \eta \le \tau \le 1$,

$$\begin{aligned} G_x(t,\tau) &= \left(\left[2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right) + t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-2}\right) \\ &\quad \times \left(2\Gamma(\alpha-\delta-1)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]\right)^{-1} \\ &\quad + \frac{\left[t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-1}}{2\Gamma(\alpha-\delta)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad + \frac{\left[2t\Gamma(3-\delta)-t^2(1-\delta)\Gamma(3-\delta)\right](\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)\,2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]}, \end{aligned} \tag{8}$$

whenever $0 \le t \le \tau \le \eta \le 1$, *and*

$$\begin{aligned} G_x(t,\tau) &= -\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ &\quad + \left(\left[2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right) + t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-2}\right) \\ &\quad \times \left(2\Gamma(\alpha-\delta-1)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]\right)^{-1} \\ &\quad + \frac{\left[t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)\right](1-\tau)^{\alpha-\delta-1}}{2\Gamma(\alpha-\delta)\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad + \frac{\left[2t\Gamma(3-\delta)-t^2(1-\delta)\Gamma(3-\delta)\right](\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)\,2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]}, \end{aligned} \tag{9}$$

whenever $0 \le \tau \le t \le \eta \le 1$.

Proof. Let u_0 be a solution for the time-fractional integrodifferential equation $(*)$ via the boundary value conditions. Put $y_0(x,t) = \lambda f(x,t,u_0(x,t),(\partial^q/\partial t^q)u_0(x,t)) + \mu I_{*t}^{\beta} g(x,t,u_0(x,t))$. Choose $c_0, c_1, c_2 \in \mathbb{R}$ such that $u_0(x,t) = -\int_0^t ((t-\tau)^{\alpha-1}/\Gamma(\alpha)) y_0(x,\tau)d\tau + c_0 + c_1 t + c_2 t^2$. Thus, we get

$$\begin{aligned} \frac{\partial^\delta}{\partial t^\delta} u_0(x,t) &= -\int_0^t \frac{(t-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau \\ &\quad + c_1 \frac{t^{1-\delta}}{\Gamma(2-\delta)} + c_2 \frac{2t^{2-\delta}}{\Gamma(3-\delta)} \end{aligned} \tag{10}$$

and $(\partial^{\delta+1}/\partial t^{\delta+1})u_0(x,t) = -\int_0^t ((t-\tau)^{\alpha-\delta-2}/\Gamma(\alpha-\delta-1)) y_0(x,\tau)d\tau + c_1(t^{-\delta}/\Gamma(1-\delta)) + c_2(2t^{1-\delta}/\Gamma(2-\delta))$. Now by using the boundary value conditions, we obtain $c_0 = 0$,

$$\begin{aligned} c_1 &= \frac{\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right)}{\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad \times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} y_0(x,\tau)\,d\tau \\ &\quad - \frac{\Gamma(3-\delta)}{\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad \times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau \\ &\quad + \frac{\Gamma(3-\delta)}{\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad \times \int_0^\eta \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau, \\ c_2 &= \frac{(1-\delta)\Gamma(3-\delta)}{2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\ &\quad \times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&- \frac{(1-\delta)\Gamma(3-\delta)}{2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\
&\times \int_0^{\eta} \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau \\
&+ \frac{\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)}{2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\
&\times \int_0^{1} \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} y_0(x,\tau)\,d\tau.
\end{aligned} \tag{11}
$$

Hence,

$$
\begin{aligned}
u_0(x,t) = &- \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_0(x,\tau)\,d\tau \\
&+ \frac{2t\Gamma(2-\delta)\left(1-\eta^{2-\delta}\right)+t^2\left(\eta^{1-\delta}-1\right)\Gamma(3-\delta)}{2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\
&\times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} y_0(x,\tau)\,d\tau \\
&+ \frac{t^2(1-\delta)\Gamma(3-\delta)-2t\Gamma(3-\delta)}{2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\
&\times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau \\
&+ \frac{2t\Gamma(3-\delta)-t^2(1-\delta)\Gamma(3-\delta)}{2\left[\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1\right]} \\
&\times \int_0^{\eta} \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} y_0(x,\tau)\,d\tau \\
= &\int_0^1 G_x(t,\tau)\, y_0(x,\tau)\,d\tau.
\end{aligned} \tag{12}
$$

Thus, u_0 is a solution for the time-fractional integral equation. It is obvious that u_0 is a solution for the time-fractional integrodifferential equation $(*)$ whenever u_0 is a solution for the time-fractional integral equation. This completes the proof. □

By considering the proof of last result, one can get that the solution of the time-fractional integrodifferential equation $(*)$ is in the form

$$
\begin{aligned}
u(x,t) = &-\lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)d\tau \\
&- \mu \int_0^t \frac{(t-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(x,\tau,u(x,\tau))\,d\tau \\
&+ \frac{tQ+t^2R}{2N}\lambda \\
&\times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)d\tau \\
&+ \frac{tQ+t^2R}{2N}\mu \\
&\times \int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)} g(x,\tau,u(x,\tau))\,d\tau \\
&+ \frac{t^2S-tM}{2N}\lambda \\
&\times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)d\tau \\
&+ \frac{t^2S-tM}{2N}\mu \int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau \\
&+ \frac{tM-t^2S}{2N}\lambda \\
&\times \int_0^{\eta} \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)d\tau \\
&+ \frac{tM-t^2S}{2N}\mu \int_0^{\eta} \frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau,
\end{aligned} \tag{13}
$$

where $Q = 2\Gamma(2-\delta)(1-\eta^{2-\delta})$, $R = (\eta^{1-\delta}-1)\Gamma(3-\delta)$, $S = (1-\delta)\Gamma(3-\delta)$, $M = 2\Gamma(3-\delta)$, and $N = [\eta^{1-\delta}(2-\delta)+\eta^{2-\delta}(\delta-1)-1]$ with $\eta^{1-\delta}, \eta^{2-\delta} \neq 1$.

Theorem 4. *Let $p_1 \in (0,1)$, $p_2 \in (0,\alpha-\delta)$, $p = \min\{p_1,p_2\}$, $L, m \in L_{1/p}([0,1],\mathbb{R}^+)$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing bounded function. Suppose that $f : J\times X^2 \to X$ and $g : J\times X \to X$ are continuous functions such that $|f(x,t,u,w) - f(x,t,v,z)| \le L(t)(|u-v|+|w-z|)$ and $|g(x,t,u(x,t)| \le m(t)\psi(\|u\|)$ for all $(x,t) \in J$ and $u,v,w,z \in X$. Put*

$$
\begin{aligned}
\gamma := |\lambda| \Bigg[&\frac{1}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p} + \frac{1}{\Gamma(\alpha-q)}\left(\frac{1-p}{\alpha-q-p}\right)^{1-p} \\
&+ \left|\frac{Q+R}{2N}\right| \frac{1}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&+ \left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right| \frac{1}{\Gamma(\alpha-\delta-1)} \\
&\times \left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&+ \left|\frac{S}{2N}\right| \frac{\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p} \\
&+ \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \frac{\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)} \\
&\times \left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p} \Bigg] \|L\|_{1/p}.
\end{aligned} \tag{14}
$$

If $\gamma < 1$, then the time-fractional integrodifferential equation $(*)$ has at least one solution.

Proof. First, put

$$\begin{aligned} w = {} & |\lambda|\,\sigma + |\mu|\,b \\ & \times \Bigg[\frac{1}{\Gamma(\alpha+\beta)} \left(\frac{1-p}{\alpha+\beta-p} \right)^{1-p} + \frac{1}{\Gamma(\alpha+\beta-q)} \\ & \times \left(\frac{1-p}{\alpha+\beta-q-p} \right)^{1-p} + \left| \frac{Q+R}{2N} \right| \frac{1}{\Gamma(\alpha+\beta-\delta-1)} \\ & \times \left(\frac{1-p}{\alpha+\beta-\delta-p-1} \right)^{1-p} + \left| \frac{(2-q)Q+2R}{2N\Gamma(3-q)} \right| \\ & \times \frac{1}{\Gamma(\alpha+\beta-\delta-1)} \left(\frac{1-p}{\alpha+\beta-\delta-p-1} \right)^{1-p} \\ & + \left| \frac{S}{2N} \right| \frac{\left(1+\eta^{\alpha+\beta-\delta-p}\right)}{\Gamma(\alpha+\beta-\delta)} \left(\frac{1-p}{\alpha+\beta-\delta-p} \right)^{1-p} \\ & + \left| \frac{2S-(2-q)M}{2N\Gamma(3-q)} \right| \frac{\left(1+\eta^{\alpha+\beta-\delta-p}\right)}{\Gamma(\alpha+\beta-\delta)} \\ & \times \left(\frac{1-p}{\alpha+\beta-\delta-p} \right)^{1-p} \Bigg] \|m\|_{1/p}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \sigma = {} & K \Bigg(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-q+1)} \\ & + \left| \frac{Q+R}{2N} \right| \frac{1}{\Gamma(\alpha-\delta)} + \left| \frac{(2-q)Q+2R}{2N\Gamma(3-q)} \right| \frac{1}{\Gamma(\alpha-\delta)} \\ & + \left| \frac{S}{2N} \right| \left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right) \\ & + \left| \frac{2S-(2-q)M}{2N\Gamma(3-q)} \right| \left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right) \Bigg). \end{aligned} \tag{16}$$

Now, choose $r \geq w/(1-\gamma)$ and consider the closed, convex, and bounded subset $B_r = \{u \in X : \|u\| \leq r\}$ of X. Define the operator $\Phi : B_r \to X$ by

$$\begin{aligned} & (\Phi u)(x,t) \\ & = -\lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad - \mu \int_0^t \frac{(t-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(x,\tau,u(x,\tau))\,d\tau + \frac{tQ+t^2R}{2N} \\ & \quad \times \Bigg[\lambda \int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad + \mu \int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)} g(x,\tau,u(x,\tau))\,d\tau \Bigg] \\ & \quad + \frac{t^2S-tM}{2N} \\ & \quad \times \Bigg[\lambda \int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad + \mu \int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau \\ & \quad - \lambda \int_0^\eta \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad - \mu \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau \Bigg], \end{aligned} \tag{17}$$

for all $(x,t) \in J$. Now, decompose Φ by $\Phi = \Phi_1 + \Phi_2$, where

$$\begin{aligned} & (\Phi_1 u)(x,t) \\ & = -\lambda \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad + \frac{tQ+t^2R}{2N}\lambda \\ & \quad \times \int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad + \frac{t^2S-tM}{2N}\lambda \\ & \quad \times \Bigg[\int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \\ & \quad - \int_0^\eta \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) d\tau \Bigg], \end{aligned}$$

$$\begin{aligned} & (\Phi_2 u)(x,t) \\ & = -\mu \int_0^t \frac{(t-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(x,\tau,u(x,\tau))\,d\tau \\ & \quad + \frac{tQ+t^2R}{2N}\mu \int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)} g(x,\tau,u(x,\tau))\,d\tau \\ & \quad + \frac{t^2S-tM}{2N} \Bigg[\mu \int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau \\ & \quad - \mu \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau \Bigg], \end{aligned} \tag{18}$$

for all $(x,t) \in J$. Now, we show that $\Phi(B_r) \subset B_r$. Let $\sup_{x,t\in[0,1]}|f(x,t,0,0)| \le K$, $(x,t) \in J$, and $u \in B_r$. Then, we have

$$
\begin{aligned}
&\left|\Phi_1 u(x,t)\right| \\
&\quad \le |\lambda| \left| \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \times \left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) - f(x,\tau,0,0) + f(x,\tau,0,0)\right] d\tau \right| \\
&\qquad + \left|\frac{Q+R}{2N}\right| |\lambda| \times \left| \int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} \left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) - f(x,\tau,0,0) + f(x,\tau,0,0)\right] d\tau \right| \\
&\qquad + \left|\frac{S}{2N}\right| |\lambda| \left| \int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \times \left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) - f(x,\tau,0,0) + f(x,\tau,0,0)\right] d\tau \right| \\
&\qquad + \left|\frac{S}{2N}\right| |\lambda| \left| \int_0^\eta \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \times \left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) - f(x,\tau,0,0) + f(x,\tau,0,0)\right] d\tau \right| \\
&\quad \le |\lambda| r \left[\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} L(\tau)\, d\tau \right] + \frac{|\lambda| K}{\Gamma(\alpha+1)} \\
&\qquad + \left|\frac{Q+R}{2N}\right| |\lambda| r \left[\int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)} L(\tau)\, d\tau \right] \\
&\qquad + \left|\frac{Q+R}{2N}\right| |\lambda| \frac{K}{\Gamma(\alpha-\delta)} \\
&\qquad + \left|\frac{S}{2N}\right| |\lambda| r \left[\int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} L(\tau)\, d\tau \right] \\
&\qquad + \left|\frac{S}{2N}\right| |\lambda| \frac{K}{\Gamma(\alpha-\delta+1)} \\
&\qquad + \left|\frac{S}{2N}\right| |\lambda| r \left[\int_0^\eta \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} L(\tau)\, d\tau \right] \\
&\qquad + \left|\frac{S}{2N}\right| |\lambda| \frac{K\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \\
&\quad \le \frac{|\lambda| r}{\Gamma(\alpha)} \left(\int_0^t (t-\tau)^{(\alpha-1)/(1-p)}\, d\tau \right)^{1-p} \left(\int_0^1 L^{1/p}(\tau)\, d\tau \right)^p \\
&\qquad + \left|\frac{Q+R}{2N}\right| \frac{|\lambda| r}{\Gamma(\alpha-\delta-1)} \left(\int_0^1 (1-\tau)^{(\alpha-\delta-2)/(1-p)}\, d\tau \right)^{1-p} \times \left(\int_0^1 L^{1/p}(\tau)\, d\tau \right)^p \\
&\qquad + \left|\frac{S}{2N}\right| \frac{|\lambda| r}{\Gamma(\alpha-\delta)} \left(\int_0^1 (1-\tau)^{(\alpha-\delta-1)/(1-p)}\, d\tau \right)^{1-p} \times \left(\int_0^1 L^{1/p}(\tau)\, d\tau \right)^p \\
&\qquad + \left|\frac{S}{2N}\right| \frac{|\lambda| r}{\Gamma(\alpha-\delta)} \left(\int_0^\eta (\eta-\tau)^{(\alpha-\delta-1)/(1-p)}\, d\tau \right)^{1-p} \times \left(\int_0^1 L^{1/p}(\tau)\, d\tau \right)^p \\
&\qquad + |\lambda| K \left(\frac{1}{\Gamma(\alpha+1)} + \left|\frac{Q+R}{2N}\right| \frac{1}{\Gamma(\alpha-\delta)} + \left|\frac{S}{2N}\right| \left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right) \right) \\
&\quad \le |\lambda| r \left[\frac{1}{\Gamma(\alpha)} \left(\frac{1-p}{\alpha-p} \right)^{1-p} + \left|\frac{Q+R}{2N}\right| \frac{1}{\Gamma(\alpha-\delta-1)} \left(\frac{1-p}{\alpha-\delta-p-1} \right)^{1-p} \right. \\
&\qquad \left. + \left|\frac{S}{2N}\right| \frac{1}{\Gamma(\alpha-\delta)} \left(\frac{1-p}{\alpha-\delta-p} \right)^{1-p} + \left|\frac{S}{2N}\right| \frac{\eta^{\alpha-\delta-p}}{\Gamma(\alpha-\delta)} \left(\frac{1-p}{\alpha-\delta-p} \right)^{1-p} \right] \|L\|_{1/p} \\
&\qquad + |\lambda| K \left(\frac{1}{\Gamma(\alpha+1)} + \left|\frac{Q+R}{2N}\right| \frac{1}{\Gamma(\alpha-\delta)} + \left|\frac{S}{2N}\right| \left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} \right) \right)
\end{aligned} \tag{19}
$$

and similarly

$$
\begin{aligned}
&\left|\frac{\partial^q}{\partial t^q}\Phi_1 u(x,t)\right| \\
&\quad \le |\lambda| \left| \int_0^t \frac{(t-\tau)^{\alpha-q-1}}{\Gamma(\alpha-q)} \left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right) - f(x,\tau,0,0) + f(x,\tau,0,0)\right] d\tau \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right||\lambda| \\
&\times\left|\int_0^1 \frac{(1-\tau)^{\alpha-\delta-2}}{\Gamma(\alpha-\delta-1)}\left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)\right.\right. \\
&\qquad\left.\left.-f(x,\tau,0,0)+f(x,\tau,0,0)\right]d\tau\right| \\
&+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right||\lambda| \\
&\times\left|\int_0^1 \frac{(1-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)\right.\right. \\
&\qquad\left.\left.-f(x,\tau,0,0)+f(x,\tau,0,0)\right]d\tau\right| \\
&+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right||\lambda| \\
&\times\left|\int_0^\eta \frac{(\eta-\tau)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[f\left(x,\tau,u(x,\tau),\frac{\partial^q}{\partial t^q}u(x,\tau)\right)\right.\right. \\
&\qquad\left.\left.-f(x,\tau,0,0)+f(x,\tau,0,0)\right]d\tau\right| \\
\le{}&\frac{|\lambda|r}{\Gamma(\alpha-q)}\left(\int_0^t (t-\tau)^{(\alpha-q-1)/(1-p)}d\tau\right)^{1-p} \\
&\times\left(\int_0^1 L^{1/p}(\tau)\,d\tau\right)^p \\
&+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{|\lambda|r}{\Gamma(\alpha-\delta-1)} \\
&\times\left(\int_0^1 (1-\tau)^{(\alpha-\delta-2)/(1-p)}d\tau\right)^{1-p}\left(\int_0^1 L^{1/p}(\tau)\,d\tau\right)^p \\
&+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{|\lambda|r}{\Gamma(\alpha-\delta)} \\
&\times\left(\int_0^1 (1-\tau)^{(\alpha-\delta-1)/(1-p)}d\tau\right)^{1-p}\left(\int_0^1 L^{1/p}(\tau)\,d\tau\right)^p \\
&+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{|\lambda|r}{\Gamma(\alpha-\delta)} \\
&\times\left(\int_0^\eta (\eta-\tau)^{(\alpha-\delta-1)/(1-p)}d\tau\right)^{1-p}\left(\int_0^1 L^{1/p}(\tau)\,d\tau\right)^p \\
&+|\lambda|K\left(\frac{1}{\Gamma(\alpha-q+1)}+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{1}{\Gamma(\alpha-\delta)}\right. \\
&\qquad\left.+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\right) \\
\le{}&|\lambda|r\left[\frac{1}{\Gamma(\alpha-q)}\left(\frac{1-p}{\alpha-q-p}\right)^{1-p}+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\right. \\
&\qquad\times\frac{1}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&\qquad+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{1}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p} \\
&\qquad\left.+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{\eta^{\alpha-\delta-p}}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p}\right] \\
&\times\|L\|_{1/p} \\
&+|\lambda|K\left(\frac{1}{\Gamma(\alpha-q+1)}+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{1}{\Gamma(\alpha-\delta)}\right. \\
&\qquad\left.+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\right).
\end{aligned}
\tag{20}
$$

Thus, we get

$$
\begin{aligned}
&\|\Phi_1 u(x,t)\| \\
&\le|\lambda|r\left[\frac{1}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}+\frac{1}{\Gamma(\alpha-q)}\left(\frac{1-p}{\alpha-q-p}\right)^{1-p}\right. \\
&\qquad+\left|\frac{Q+R}{2N}\right|\frac{1}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&\qquad+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{1}{\Gamma(\alpha-\delta-1)} \\
&\qquad\times\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&\qquad+\left|\frac{S}{2N}\right|\frac{(1+\eta^{\alpha-\delta-p})}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p} \\
&\qquad+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{(1+\eta^{\alpha-\delta-p})}{\Gamma(\alpha-\delta)} \\
&\qquad\left.\times\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p}\right]\|L\|_{1/p} \\
&+|\lambda|K\left(\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-q+1)}+\left|\frac{Q+R}{2N}\right|\frac{1}{\Gamma(\alpha-\delta)}\right. \\
&\qquad+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{1}{\Gamma(\alpha-\delta)} \\
&\qquad+\left|\frac{S}{2N}\right|\left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right) \\
&\qquad\left.+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\right).
\end{aligned}
\tag{21}
$$

If $\sup_{u\in X}\psi(\|u\|)\le b$, then we have

$$\begin{aligned}
&\left|\Phi_2 u(x,t)\right| \\
&\quad\le |\mu|\left|\int_0^t \frac{(t-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\qquad+\left|\frac{Q+R}{2N}\right||\mu|\left|\int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\qquad+\left|\frac{S}{2N}\right||\mu|\left|\int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\qquad+\left|\frac{S}{2N}\right||\mu|\left|\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\quad\le \frac{|\mu|b}{\Gamma(\alpha+\beta)}\left(\int_0^t (t-\tau)^{(\alpha+\beta-1)/(1-p)}\,d\tau\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p+\left|\frac{Q+R}{2N}\right|\frac{|\mu|b}{\Gamma(\alpha+\beta-\delta-1)} \\
&\qquad\times\left(\int_0^1 (1-\tau)^{(\alpha+\beta-\delta-2)/(1-p)}\,d\tau\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p \\
&\qquad+\left|\frac{S}{2N}\right|\frac{|\mu|b}{\Gamma(\alpha+\beta-\delta)}\left(\int_0^1 (1-\tau)^{(\alpha+\beta-\delta-1)/(1-p)}\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p \\
&\qquad+\left|\frac{S}{2N}\right|\frac{|\mu|b}{\Gamma(\alpha+\beta-\delta)}\left(\int_0^\eta (\eta-\tau)^{(\alpha+\beta-\delta-1)/(1-p)}\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p \\
&\quad\le |\mu|b\left[\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p}+\left|\frac{Q+R}{2N}\right|\right. \\
&\qquad\times\frac{1}{\Gamma(\alpha+\beta-\delta-1)}\left(\frac{1-p}{\alpha+\beta-\delta-p-1}\right)^{1-p} \\
&\qquad+\left|\frac{S}{2N}\right|\frac{1}{\Gamma(\alpha+\beta-\delta)}\left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p} \\
&\qquad\left.+\left|\frac{S}{2N}\right|\frac{\eta^{\alpha+\beta-\delta-p}}{\Gamma(\alpha+\beta-\delta)}\left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p}\right] \\
&\qquad\times\|m\|_{1/p}
\end{aligned}\tag{22}$$

and similarly

$$\begin{aligned}
&\left|\frac{\partial^q}{\partial t^q}\Phi_2 u(x,t)\right| \\
&\quad\le |\mu|\left|\int_0^t \frac{(t-\tau)^{\alpha+\beta-q-1}}{\Gamma(\alpha+\beta-q)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\qquad+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right||\mu| \\
&\qquad\times\left|\int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\qquad+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right||\mu| \\
&\qquad\times\left|\int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\qquad+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right||\mu| \\
&\qquad\times\left|\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} g(x,\tau,u(x,\tau))\,d\tau\right| \\
&\quad\le \frac{|\mu|b}{\Gamma(\alpha+\beta-q)}\left(\int_0^t (t-\tau)^{(\alpha+\beta-q-1)/(1-p)}\,d\tau\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p \\
&\qquad+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{|\mu|b}{\Gamma(\alpha+\beta-\delta-1)} \\
&\qquad\times\left(\int_0^1 (1-\tau)^{(\alpha+\beta-\delta-2)/(1-p)}\,d\tau\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \\
&\qquad\times\frac{|\mu|b}{\Gamma(\alpha+\beta-\delta)}\left(\int_0^1 (1-\tau)^{(\alpha+\beta-\delta-1)/(1-p)}\,d\tau\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \\
&\qquad\times\frac{|\mu|b}{\Gamma(\alpha+\beta-\delta)}\left(\int_0^\eta (\eta-\tau)^{(\alpha+\beta-\delta-1)/(1-p)}\,d\tau\right)^{1-p} \\
&\qquad\times\left(\int_0^1 m^{1/p}(\tau)\,d\tau\right)^p \\
&\quad\le |\mu|b\left[\frac{1}{\Gamma(\alpha+\beta-q)}\left(\frac{1-p}{\alpha+\beta-q-p}\right)^{1-p}\right. \\
&\qquad+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|
\end{aligned}$$

$$\times \frac{1}{\Gamma(\alpha+\beta-\delta-1)}\left(\frac{1-p}{\alpha+\beta-\delta-p-1}\right)^{1-p} + \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \frac{1}{\Gamma(\alpha+\beta-\delta)} \times \left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p} + \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \times \frac{\eta^{\alpha+\beta-\delta-p}}{\Gamma(\alpha+\beta-\delta)}\left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p}\Bigg] \|m\|_{1/p}. \tag{23}$$

Hence,

$$\begin{aligned}
&\|\Phi_2 u(x,t)\| \\
&\le |\mu| b \Bigg[\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p} + \frac{1}{\Gamma(\alpha+\beta-q)} \times \left(\frac{1-p}{\alpha+\beta-q-p}\right)^{1-p} + \left|\frac{Q+R}{2N}\right| \\
&\quad \times \frac{1}{\Gamma(\alpha+\beta-\delta-1)}\left(\frac{1-p}{\alpha+\beta-\delta-p-1}\right)^{1-p} + \left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right| \frac{1}{\Gamma(\alpha+\beta-\delta-1)} \\
&\quad \times \left(\frac{1-p}{\alpha+\beta-\delta-p-1}\right)^{1-p} + \left|\frac{S}{2N}\right| \\
&\quad \times \frac{\left(1+\eta^{\alpha+\beta-\delta-p}\right)}{\Gamma(\alpha+\beta-\delta)}\left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p} \\
&\quad + \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \frac{\left(1+\eta^{\alpha+\beta-\delta-p}\right)}{\Gamma(\alpha+\beta-\delta)} \\
&\quad \times \left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p}\Bigg] \|m\|_{1/p}
\end{aligned} \tag{24}$$

and so

$$\begin{aligned}
&\|\Phi u(x,t)\| \\
&\le \|\Phi_1 u(x,t)\| + \|\Phi_2 u(x,t)\| \\
&\le |\lambda| r \Bigg[\frac{1}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p} + \frac{1}{\Gamma(\alpha-q)}\left(\frac{1-p}{\alpha-q-p}\right)^{1-p} \\
&\quad + \left|\frac{Q+R}{2N}\right| \frac{1}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&\quad + \left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right| \frac{1}{\Gamma(\alpha-\delta-1)} \\
&\quad \times \left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} + \left|\frac{S}{2N}\right| \frac{\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)} \\
&\quad \times \left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p} + \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \\
&\quad \times \frac{\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p}\Bigg] \|L\|_{1/p} \\
&\quad + |\lambda| K \Bigg(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-q+1)} + \left|\frac{Q+R}{2N}\right| \\
&\quad \times \frac{1}{\Gamma(\alpha-\delta)} + \left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right| \frac{1}{\Gamma(\alpha-\delta)} \\
&\quad + \left|\frac{S}{2N}\right| \left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right) \\
&\quad + \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \left(\frac{1+\eta^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)}\right)\Bigg) \\
&\quad + |\mu| b \Bigg[\frac{1}{\Gamma(\alpha+\beta)}\left(\frac{1-p}{\alpha+\beta-p}\right)^{1-p} + \frac{1}{\Gamma(\alpha+\beta-q)} \times \left(\frac{1-p}{\alpha+\beta-q-p}\right)^{1-p} + \left|\frac{Q+R}{2N}\right| \\
&\quad \times \frac{1}{\Gamma(\alpha+\beta-\delta-1)}\left(\frac{1-p}{\alpha+\beta-\delta-p-1}\right)^{1-p} + \left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right| \frac{1}{\Gamma(\alpha+\beta-\delta-1)} \\
&\quad \times \left(\frac{1-p}{\alpha+\beta-\delta-p-1}\right)^{1-p} + \left|\frac{S}{2N}\right| \\
&\quad \times \frac{\left(1+\eta^{\alpha+\beta-\delta-p}\right)}{\Gamma(\alpha+\beta-\delta)}\left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p} \\
&\quad + \left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right| \frac{\left(1+\eta^{\alpha+\beta-\delta-p}\right)}{\Gamma(\alpha+\beta-\delta)} \\
&\quad \times \left(\frac{1-p}{\alpha+\beta-\delta-p}\right)^{1-p}\Bigg] \|m\|_{1/p} \\
&\le \gamma r + w \le r.
\end{aligned} \tag{25}$$

This implies that $\Phi B_r \subset B_r$. Now, we show that Φ_1 is continuous. Let $x, t \in [0,1]$ and $\{u_n\}$ be a sequence with $u_n \to u$. Then, we have

$$\begin{aligned}
&\|\Phi_1 u_n(x,t) - \Phi_1 u(x,t)\| \\
&\le \Bigg[\frac{|\lambda| \|L\|_{1/p}}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p} + \frac{|\lambda| \|L\|_{1/p}}{\Gamma(\alpha-q)}\left(\frac{1-p}{\alpha-q-p}\right)^{1-p} \\
&\quad + \left|\frac{Q+R}{2N}\right| \frac{|\lambda| \|L\|_{1/p}}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p}
\end{aligned}$$

$$+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{|\lambda|\,\|L\|_{1/p}}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p}$$
$$+\left|\frac{S}{2N}\right|\frac{|\lambda|\,\|L\|_{1/p}\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p}$$
$$+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{|\lambda|\,\|L\|_{1/p}\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)}$$
$$\times\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p}\Bigg]\|u_n-u\|. \tag{26}$$

Thus, Φ_1 is continuous. On the other hand, Φ_1 is a γ-contraction because

$$\begin{aligned}
&\|\Phi_1 u(x,t)-\Phi_1 v(x,t)\| \\
&\le\Bigg[\frac{|\lambda|\,\|L\|_{1/p}}{\Gamma(\alpha)}\left(\frac{1-p}{\alpha-p}\right)^{1-p}+\frac{|\lambda|\,\|L\|_{1/p}}{\Gamma(\alpha-q)}\left(\frac{1-p}{\alpha-q-p}\right)^{1-p} \\
&\quad+\left|\frac{Q+R}{2N}\right|\frac{|\lambda|\,\|L\|_{1/p}}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&\quad+\left|\frac{(2-q)Q+2R}{2N\Gamma(3-q)}\right|\frac{|\lambda|\,\|L\|_{1/p}}{\Gamma(\alpha-\delta-1)}\left(\frac{1-p}{\alpha-\delta-p-1}\right)^{1-p} \\
&\quad+\left|\frac{S}{2N}\right|\frac{|\lambda|\,\|L\|_{1/p}\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)}\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p} \\
&\quad+\left|\frac{2S-(2-q)M}{2N\Gamma(3-q)}\right|\frac{|\lambda|\,\|L\|_{1/p}\left(1+\eta^{\alpha-\delta-p}\right)}{\Gamma(\alpha-\delta)} \\
&\quad\times\left(\frac{1-p}{\alpha-\delta-p}\right)^{1-p}\Bigg]\|u-v\|,
\end{aligned} \tag{27}$$

for all $u,v\in B_r$. Now, we show that Φ_2 is a compact map. We showed that Φ_2 is uniformly bounded. Now, we show that Φ_2 maps the bounded sets into equicontinuous sets. Let $(x,t_1),(x,t_2)\in J$ such that $t_1<t_2$. Then, we have

$$\begin{aligned}
&|\Phi_2 u(x,t_2)-\Phi_2 u(x,t_1)| \\
&\le\frac{|\mu|}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left[(t_2-\tau)^{\alpha+\beta-1}-(t_1-\tau)^{\alpha+\beta-1}\right] \\
&\qquad\times|g(x,\tau,u(x,\tau))|\,d\tau \\
&\quad+\frac{|\mu|}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha+\beta-1}|g(x,\tau,u(x,\tau))|\,d\tau \\
&\quad+\frac{|t_2-t_1|\,|Q|+|t_2^2-t_1^2|\,|R|}{|2N|}|\mu| \\
&\quad\times\int_0^1\frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)}|g(x,\tau,u(x,\tau))|\,d\tau \\
&\quad+\frac{|t_2^2-t_1^2|\,|S|-|t_2-t_1|\,|M|}{|2N|}|\mu| \\
&\quad\times\left[\int_0^1\frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}|g(x,\tau,u(x,\tau))|\,d\tau\right. \\
&\qquad\left.-\int_0^{\eta}\frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}|g(x,\tau,u(x,\tau))|\,d\tau\right] \\
&\le\frac{|\mu|\,b}{\Gamma(\alpha+\beta)}\int_0^{t_1}\left[(t_2-\tau)^{\alpha+\beta-1}-(t_1-\tau)^{\alpha+\beta-1}\right]m(\tau)\,d\tau \\
&\quad+\frac{|\mu|\,b}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha+\beta-1}m(\tau)\,d\tau \\
&\quad+\left(\frac{|t_2-t_1|\,|Q|+|t_2^2-t_1^2|\,|R|}{|2N|}\right)|\mu|\,b \\
&\quad\times\int_0^1\frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)}m(\tau)\,d\tau \\
&\quad+\left(\frac{|t_2^2-t_1^2|\,|S|-|t_2-t_1|\,|M|}{|2N|}\right)|\mu|\,b \\
&\quad\times\left[\int_0^1\frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}m(\tau)\,d\tau\right. \\
&\qquad\left.-\int_0^{\eta}\frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)}m(\tau)\,d\tau\right],
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\partial^q}{\partial t^q}\Phi_2 u(x,t_2)-\frac{\partial^q}{\partial t^q}\Phi_2 u(x,t_1)\right| \\
&\le\frac{|\mu|\,b}{\Gamma(\alpha+\beta-q)} \\
&\quad\times\int_0^{t_1}\left[(t_2-\tau)^{\alpha+\beta-q-1}-(t_1-\tau)^{\alpha+\beta-q-1}\right]m(\tau)\,d\tau \\
&\quad+\frac{|\mu|\,b}{\Gamma(\alpha+\beta-q)}\int_{t_1}^{t_2}(t_2-\tau)^{\alpha+\beta-q-1}m(\tau)\,d\tau \\
&\quad+\left(\left(\frac{|t_2^{1-q}-t_1^{1-q}|}{\Gamma(2-q)}|Q|\right.\right. \\
&\qquad\left.\left.+\frac{|t_2^{2-q}-t_1^{2-q}|}{\Gamma(3-q)}|R|\right)\times|2N|^{-1}\right)|\mu|\,b \\
&\quad\times\int_0^1\frac{(1-\tau)^{\alpha+\beta-\delta-2}}{\Gamma(\alpha+\beta-\delta-1)}m(\tau)\,d\tau \\
&\quad+\left(\left(\frac{2|t_2^{2-q}-t_1^{2-q}|}{\Gamma(3-q)}|S|\right.\right. \\
&\qquad\left.\left.-\frac{|t_2^{1-q}-t_1^{1-q}|}{\Gamma(2-q)}|M|\right)\times|2N|^{-1}\right)|\mu|\,b
\end{aligned}$$

$$\times \left[\int_0^1 \frac{(1-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} m(\tau)\, d\tau - \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta-\delta-1}}{\Gamma(\alpha+\beta-\delta)} m(\tau)\, d\tau \right], \tag{28}$$

for all $u \in B_r$. Hence, $\|\Phi_2 u(x,t_2) - \Phi_2 u(x,t_1)\| \to 0$ as $t_2 \to t_1$. By using the Arzela-Ascoli theorem, we get that the operator Φ_2 is completely continuous and so Φ_2 is compact on J. Now by using Lemma 1, the operator $\Phi = \Phi_1 + \Phi_2$ is a condensing map on B_r and so Φ has a fixed point by using Theorem 2. Now by using Lemma 3, it is easy to see that the fixed point of Φ is a solution for the time-fractional integrodifferential equation $(*)$. □

Example 5. Let $\alpha = 5/2$, $\beta = 3/4$, $q = 3/2$, $\delta = 5/4$, $\eta = 2/3$, $\lambda = 1/1000$, $\mu = 1$, and $p = 3/4$. Now, consider the time-fractional integrodifferential equation

$$-\frac{\partial^{5/2}}{\partial t^{5/2}} u(x,t) = \frac{1}{1000}\left(\frac{t|u|}{1+|u|} + \frac{t\left|\left(\partial^{3/2}/\partial t^{3/2}\right)u\right|}{1+\left|\left(\partial^{3/2}/\partial t^{3/2}\right)u\right|} \right) + I_{*t}^{3/4}\left(\frac{t^2|u|^3}{1+|u|^3} \right) \tag{29}$$

via the boundary conditions $u(x,0) = 0$, $(\partial^{9/4}/\partial t^{9/4})u(x,0) = (\partial^{9/4}/\partial t^{9/4})u(x,1)$, and $(\partial^{5/4}/\partial t^{5/4})u(x,1) - (\partial^{5/4}/\partial t^{5/4})u(x,2/3) = 0$. Define the maps $f : J \times X^2 \to X$ by $f(x,t,u(x,t),(\partial^{3/2}/\partial t^{3/2})u(x,t)) = t|u|/(1+|u|) + t|(\partial^{3/2}/\partial t^{3/2})u|/(1+|(\partial^{3/2}/\partial t^{3/2})u|)$ and $g : J \times X \to X$ by $g(x,t,u(x,t)) = t^2|u|^3/(1+|u|^3)$. Define $L(t) = t$ and $m(t) = t^2$ for all t. Then, we have $\|L\|_{1/p} = \|t\|_{4/3} = 0.077$. Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(z) = 1$ for all z. It is easy to check that

$$\left| f\left(x,t,u(x,t),\frac{\partial^{3/2}}{\partial t^{3/2}}u(x,t)\right) - f\left(x,t,v(x,t),\frac{\partial^{3/2}}{\partial t^{3/2}}v(x,t)\right)\right| \le L(t)\left(|u-v| + \left|\frac{\partial^{3/2}}{\partial t^{3/2}}u - \frac{\partial^{3/2}}{\partial t^{3/2}}v\right|\right) \tag{30}$$

and $|g(x,t,u(x,t))| \le m(t)\psi(\|u\|)$ for all $u, v \in X$ and t. One can calculate that $\|m\|_{1/p} = \|t^2\|_{4/3} = (3/11)^{3/4}$ and $\gamma \simeq 0.002579$. Now by using Theorem 4, the time-fractional integrodifferential equation has a solution.

3. Conclusion

A time-fractional integrodifferential equation via three-point boundary value conditions was investigated and the existence of the solution was proved for three-point boundary value conditions. One example was studied in detail. For particular forms of the functions f and g and of the investigated equation and for various values of $2 \le \alpha < 3$, $0 < \beta < 1$, $1 \le \delta < 2$, $0 \le \eta \le 1$, λ, and μ we can reobtain the forms of several nonlinear time-fractional differential equations describing the complex phenomena which arise in science and engineering.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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