Research Article

A Compensated Numerical Method for Solving Stochastic Differential Equations with Variable Delays and Random Jump Magnitudes

Ying Du and Changlin Mei

Department of Statistics, School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

Correspondence should be addressed to Ying Du; duyingxjtu@aliyun.com

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Stochastic differential equations with jumps are of a wide application area especially in mathematical finance. In general, it is hard to obtain their analytical solutions and the construction of some numerical solutions with good performance is therefore an important task in practice. In this study, a compensated split-step θ method is proposed to numerically solve the stochastic differential equations with variable delays and random jump magnitudes. It is proved that the numerical solutions converge to the analytical solutions in mean-square with the approximate rate of $1/2$. Furthermore, the mean-square stability of the exact solutions and the numerical solutions are investigated via a linear test equation and the results show that the proposed numerical method shares both the mean-square stability and the so-called A-stability.

1. Introduction

Stochastic differential equations (SDEs) for jump-diffusions arise in a variety of practical areas and have successfully been used to describe unexpected and abrupt changes in the present structure (for an overview, see [1, 2]). Generally, SDEs for jump-diffusions cannot be solved explicitly. Therefore, constructing some forms of numerical solution and studying their properties have received a great deal of attention. When the jump magnitude is deterministic, Higham and Kloeden [3–5] studied the convergence and stability of the numerical solutions of the stochastic differential equations with jumps. Wang et al. [6] proved that the semi-implicit numerical solutions of stochastic delay differential equations with jumps are convergent to their corresponding analytical solutions. Jiang et al. [7] investigated the Taylor approximation for stochastic delay differential equation with jumps (SDDEJs). Bao et al. [8] obtained the convergence rate of the Euler-Maruyama method for SDDEJs under the local Lipschitz condition. Furthermore, some researchers [9–11] extended the constant delay τ in SDDEJs to variable delay $\tau(t)$.

Chalmers and Higham [12] extended the results in [4] to the case where the jump magnitudes are random. Stochastic

differential equations with compound Poisson processes have been commonly used in mathematical finance and covering a wide range of finance models [13–16]. Considering the aftereffect of the past state, Jiang et al. [17] proposed a semi-implicit Euler numerical method for stochastic differential delay equations with Poisson driven jumps of random magnitudes. Furthermore, Mao [18] studied the stochastic differential equations with variable delays and random jump magnitudes (SDEVDRJMs) which are of the form

$$\begin{aligned} dx(t) = & f(x(t), x(t - \tau(t))) dt \\ & + g(x(t), x(t - \tau(t))) dW(t) \\ & + h(x(t), x(t - \tau(t)), \gamma_{N(t)+1}) dN(t), \quad (1) \\ & t \in [0, T]; \end{aligned}$$

$$x(t) = \xi(t), \quad t \in [-r, 0],$$

where $\tau(t)$ is a variable delay, $W(t)$ is an m -dimensional standard Wiener process, $N(t)$ is a scalar Poisson process with intensity λ , γ_i ($i = 1, 2, \dots$) are independent and identically distributed random variables representing jump magnitudes,

the drift coefficient $f(x, y)$ and the jump coefficient $h(x, y, \gamma)$ are $\mathbb{R}^n$ -valued, the diffusion coefficient $g(x, y)$ is $\mathbb{R}^{n \times m}$ -valued for $x, y \in \mathbb{R}^n$, and $\xi(t)$ is starting delay condition function. Moreover, it is generally assumed that, for some $p \geq 2$, there exists a constant B such that $\mathbb{E}[|\gamma_i|^p] \leq B$. Mao [18] proposed a semi-implicit Euler numerical method for SDEVDRJMs and proved that the numerical solutions converge to their analytical solution both in mean-square and in probability.

However, as pointed out by Chalmers and Higham [12], the semi-implicit Euler method for stochastic differential equations with random jump magnitudes is stable in mean-square but lose the A-stable result for constant jump magnitude. Here, A-stable property means that “problem stable $\Rightarrow$ method stable for all time step size Δ .” Furthermore, it seems that there is little literature on the study of the stability of numerical methods for SDEVDRJMs.

Our work is motivated by [3], where compensated split-step backward Euler method is introduced and proved to have a satisfactory stability property. In this paper, a compensated split-step θ method is proposed to construct the numerical solutions of SDEVDRJMs, and the mean-square convergence of the numerical solutions is proved. Furthermore, the mean-square stability of the analytical solutions and the numerical solutions is investigated via a test equation; the results demonstrate that the proposed numerical method possesses not only the mean-square stability but also the A-stability.

2. Compensated Split-Step θ Numerical Solutions for SDEVDRJMs

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ satisfying the usual conditions that $\mathcal{F}_t$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets. Let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$ and let $\langle \cdot \rangle$ be the scalar product of vectors $x, y \in \mathbb{R}^n$. Denote $\|z\|_{L_2} = (\mathbb{E}[|z|^2])^{1/2}$ with z being a n -dimensional real random variable.

We firstly introduce the following assumptions for the establishment of the convergence and stability of the proposed numerical solution of the SDEVDRJMs in (1).

(i) The time delay $\tau : [0, \infty] \rightarrow \mathbb{R}$ satisfies

$$0 \leq \tau(t) \leq t + r. \quad (2)$$

For $t, s \geq 0$, there exists positive constant K_1 such that

$$\mathbb{E}(|\tau(t) - \tau(s)|) \leq K_1 |t - s|. \quad (3)$$

(ii) Global Lipschitz condition: there exists a positive constant K_2 such that, for any $x_k, y_k, z_k \in \mathbb{R}^n$ ($k = 1, 2$), letting $y(t) = x(t - \tau(t))$,

$$\begin{aligned} & |f(x_1, y_1) - f(x_2, y_2)|^2 + |g(x_1, y_1) - g(x_2, y_2)|^2 \\ & \leq K_2 (|x_1 - x_2|^2 + |y_1 - y_2|^2), \\ & |h(x_1, y_1, z_1) - h(x_2, y_2, z_2)|^2 \\ & \leq K_2 (|x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2). \end{aligned} \quad (4)$$

(iii) Quadratic growth condition: for any $x_k \in \mathbb{R}^n$ ($k = 1, 2, 3$), there exists a positive constant K_3 such that

$$\begin{aligned} & |f(x_1, x_2)|^2 + |g(x_1, x_2)|^2 \leq K_3 (1 + |x_1|^2 + |x_2|^2), \\ & |h(x_1, x_2, x_3)|^2 \leq K_3 (1 + |x_1|^2 + |x_2|^2 + |x_3|^2). \end{aligned} \quad (5)$$

(iv) For $-r \leq t, s \leq 0$, there exist positive constants K_4 and K_5 such that $\mathbb{E}|\xi(t) - \xi(s)| \leq K_4 |t - s|$ and $\sup_{-r \leq t \leq 0} |\xi(t)|^2 \leq K_5$.

Theorem 1. Under the assumptions (i) and (iii), there exist positive constants C_1 and C_2 , such that the solution of the SDEVDRJM in (1) satisfies

$$\begin{aligned} & \mathbb{E} \left(\sup_{-r \leq t \leq T} |x(t)|^2 \right) \leq C_1, \\ & \mathbb{E} (|x(t) - x(s)|^2) \leq C_2 (t - s), \end{aligned} \quad (6)$$

for any $0 \leq s \leq t \leq T$.

Proof. From (1) and the linear growth condition, it can be obtained that, for any $t_1 \in [0, T]$,

$$\begin{aligned} & \mathbb{E} \left(\sup_{0 \leq t \leq t_1} |x(t)|^2 \right) \\ & \leq 4\mathbb{E} (|\xi(0)|^2) + 16BK_3^2 T + 4(T + 1 + 2\lambda^2 T + 4\lambda) \\ & \quad \cdot K_3^2 \mathbb{E} \left(\sup_{0 \leq t \leq t_1} \int_0^t (1 + |x(s)| + |x(s - \tau(s))|)^2 ds \right). \end{aligned} \quad (7)$$

Let $J_1 = 12(T + 1 + 2\lambda^2 T + 4\lambda)K_3^2$ and $J_2 = 4 \sup_{-r \leq t \leq 0} \mathbb{E}|\xi(t)|^2 + 16BK_3^2 T + J_1$. We have

$$\begin{aligned} & \mathbb{E} \left(\sup_{-r \leq t \leq t_1} |x(t)|^2 \right) \\ & \leq J_2 + 2J_1 \int_0^{t_1} \mathbb{E} \left(\sup_{-r \leq v \leq s} |x(v)|^2 \right) ds. \end{aligned} \quad (8)$$

Using Gronwall inequality, we obtain

$$\mathbb{E} \left(\sup_{-r \leq t \leq T} |x(t)|^2 \right) \leq J_2 e^{2J_1 T} = C_1. \quad (9)$$

Furthermore,

$$\begin{aligned} & \mathbb{E} (|x(t) - x(s)|^2) \\ & \leq 16BK_3^2 (t - s) \\ & \quad + \frac{J_1}{3} \int_s^t \mathbb{E} [(1 + |x(s)| + |x(s - \tau(s))|)^2] ds \\ & \leq (16BK_3^2 + J_1 + J_1 C_1) (t - s) \leq C_2 (t - s) \end{aligned} \quad (10)$$

with $C_2 = 16BK_3^2 + J_1 + J_1 C_1$. $\square$

Using the compensated Poisson process $\bar{N}(t) = N(t) - \lambda \Delta$ which is a martingale, we can rewrite

$$\begin{aligned} dx(t) &= f_\lambda(x(t), x(t - \tau(t)), \gamma_{N(t)+1}) dt \\ &\quad + g(x(t), x(t - \tau(t))) dW(t) \\ &\quad + h(x(t), x(t - \tau(t)), \gamma_{N(t)+1}) d\bar{N}(t), \end{aligned} \quad (11)$$

where

$$\begin{aligned} f_\lambda(x(t), x(t - \tau(t)), \gamma_{N(t)+1}) \\ = f(x(t), x(t - \tau(t))) \\ + \lambda h(x(t), x(t - \tau(t)), \gamma_{N(t)+1}). \end{aligned} \quad (12)$$

Obviously, $f_\lambda(x(t), x(t - \tau(t)), \gamma_{N(t)+1})$ still satisfies the global Lipschitz condition and the linear growth condition. However, this equation is complicated and its analytical solution is in general hard to be obtained. In what follows, we use the compensated split-step θ method to construct its numerical solution.

For a given constant time step size Δ , let

$$\begin{aligned} m &= \left\lfloor \frac{(n\Delta - \tau(n\Delta))}{\Delta} \right\rfloor, \\ \bar{m} &= \left\lfloor \frac{((n+1)\Delta - \tau((n+1)\Delta))}{\Delta} \right\rfloor. \end{aligned} \quad (13)$$

Then the compensated split-step θ numerical solutions of the SDEVDRJM is of the form

$$\begin{aligned} x_k^* &= x_k + \left[\theta f_\lambda(x_k^*, x_{\bar{m}}, \gamma_{N(t_k)+1}) \right. \\ &\quad \left. + (1 - \theta) f_\lambda(x_k, x_m, \gamma_{N(t_k)+1}) \right] \Delta; \end{aligned} \quad (14a)$$

$$\begin{aligned} x_{k+1} &= x_k^* + g(x_k^*, x_m) \Delta W_k \\ &\quad + h(x_k^*, x_m, \gamma_{N(t_k)+1}) \Delta \bar{N}_k, \end{aligned} \quad (14b)$$

where x_k is the numerical approximation to $x(t_k)$, $\Delta W_k = W_{t_{k+1}} - W_{t_k}$, and $\Delta N_k = N_{t_{k+1}} - N_{t_k}$ for $t_k = k\Delta$ with $k = 0, 1, \dots, N$ and $N = [T/\Delta]$ being the integer part of T/Δ .

3. Convergence of the Numerical Solutions

In this section, we will prove that the above numerical solutions converge in mean-square to the true solution of the SDEVDRJM with the approximate rate of $1/2$.

Replacing the numerical approximations with the exact solution values on the right-hand side of (14a) and (14b), we obtain

$$\begin{aligned} \bar{x}_k^* &= x(t_k) + \left[\theta f_\lambda(\bar{x}_k^*, x(t_{\bar{m}}), \gamma_{N(t_k)+1}) \right. \\ &\quad \left. + (1 - \theta) f_\lambda(x(t_k), x(t_m), \gamma_{N(t_k)+1}) \right] \Delta, \end{aligned} \quad (15a)$$

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k^* + g(\bar{x}_k^*, x(t_m)) \Delta W_k \\ &\quad + h(\bar{x}_k^*, x(t_m), \gamma_{N(t_k)+1}) \Delta \bar{N}_k. \end{aligned} \quad (15b)$$

Then, the local error of the compensated split-step θ method for the approximation of the solution $x(t)$ of (1) is

$$\delta_{n+1} = x(t_{n+1}) - \bar{x}_{n+1} \quad (16)$$

and the global error is

$$\epsilon_n = x(t_n) - x_n. \quad (17)$$

Obviously, ϵ_n is $\mathcal{F}_{t_n}$ -measurable because both $x(t_n)$ and $\bar{x}_n$ are $\mathcal{F}_{t_n}$ -measurable.

Theorem 2. Under the assumptions (i)–(iv), there exists a constant $0 < \Delta^* < 1$ such that, for any $0 \leq \Delta \leq \Delta^*$, the local error of the compensated split-step θ method satisfies

$$\begin{aligned} \max_{0 \leq n \leq N-1} \|\mathbb{E}(\delta_{n+1} | \mathcal{F}_{t_n})\|_{L_2} &\leq H_1 \Delta^{3/2-1/p} \quad \text{as } \Delta \rightarrow 0, \\ \max_{0 \leq n \leq N-1} \|\delta_{n+1}\|_{L_2} &\leq H_2 \Delta^{1-1/p} \quad \text{as } \Delta \rightarrow 0, \end{aligned} \quad (18)$$

where the constants H_1 and H_2 are independent of Δ .

Proof. From (14a), (14b), (15a), and (15b), we have

$$\begin{aligned} \delta_{n+1} &= \int_{t_n}^{t_{n+1}} \left[f_\lambda(x(s), x(s - \tau(s)), \gamma_{N(s)+1}) \right. \\ &\quad \left. - f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right] ds \\ &\quad + \theta \int_{t_n}^{t_{n+1}} \left[f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right. \\ &\quad \left. - f_\lambda(\bar{x}_n^*, x(t_{\bar{m}}), \gamma_{N(t_n)+1}) \right] ds \\ &\quad + \int_{t_n}^{t_{n+1}} \left[g(x(s), x(s - \tau(s))) \right. \\ &\quad \left. - g(\bar{x}_n^*, x(t_m)) \right] dW(s) \\ &\quad + \int_{t_n}^{t_{n+1}} \left[h(x(s), x(s - \tau(s)), \gamma_{N(s)+1}) \right. \\ &\quad \left. - h(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1}) \right] d\bar{N}(s). \end{aligned} \quad (19)$$

Thus

$$\begin{aligned} &\left| \mathbb{E}[(x(t_{n+1}) - \bar{x}_{n+1}) | \mathcal{F}_{t_n}] \right| \\ &\leq \left| \mathbb{E} \left(\int_{t_n}^{t_{n+1}} \left(f_\lambda(x(s), x(s - \tau(s)), \gamma_{N(s)+1}) \right. \right. \right. \\ &\quad \left. \left. - f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right) ds \mid \mathcal{F}_{t_n} \right) \right| \\ &\quad + \theta \left| \mathbb{E} \left(\int_{t_n}^{t_{n+1}} \left(f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right. \right. \right. \right. \\ &\quad \left. \left. - f_\lambda(\bar{x}_n^*, x(t_{\bar{m}}), \gamma_{N(t_n)+1}) \right) ds \mid \mathcal{F}_{t_n} \right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq (1 + \lambda) K_2 \int_{t_n}^{t_{n+1}} \mathbb{E} (|x(s) - x(t_n)| \mid \mathcal{F}_{t_n}) ds \\
&\quad + (1 + \lambda) K_2 \int_{t_n}^{t_{n+1}} \mathbb{E} (|x(s - \tau(s)) - x(t_m)|) ds \\
&\quad + \theta (1 + \lambda) K_2 \int_{t_n}^{t_{n+1}} \mathbb{E} (|\bar{x}_n^* - x(t_n)| \mid \mathcal{F}_{t_n}) ds \\
&\quad + \theta (1 + \lambda) K_2 \int_{t_n}^{t_{n+1}} \mathbb{E} (|x(t_m) - x(t_m)|) ds \\
&\quad + \lambda K_2 \left| \mathbb{E} \left(\int_{t_n}^{t_{n+1}} (\gamma_{N(s)+1} - \gamma_{N(t_n)+1}) ds \mid \mathcal{F}_{t_n} \right) \right|.
\end{aligned} \tag{20}$$

To estimate $\mathbb{E}(|x(s - \tau(s)) - x(t_m)|)$, let us consider the following five possible cases.

(1) If $s - \tau(s) \geq [n\Delta - \tau(n\Delta)]\Delta \geq 0$, then

$$\begin{aligned}
s - \tau(s) - [n\Delta - \tau(n\Delta)]\Delta \\
\leq s - \tau(s) + \tau(n\Delta) - (n-1)\Delta \leq (K_1 + 2)\Delta.
\end{aligned} \tag{21}$$

From Theorem 1, it is obtained that

$$\begin{aligned}
&\mathbb{E} (|x(s - \tau(s)) - x(t_m)|) \\
&\leq C_2^{1/2} |s - \tau(s) - [n\Delta - \tau(n\Delta)]\Delta|^{1/2} \\
&\leq C_2^{1/2} (2 + K_1)^{1/2} \Delta^{1/2}.
\end{aligned} \tag{22}$$

(2) If $[n\Delta - \tau(n\Delta)]\Delta \geq s - \tau(s) \geq 0$, then

$$\begin{aligned}
&[n\Delta - \tau(n\Delta)]\Delta - s + \tau(s) \\
&\leq n\Delta - \tau(n\Delta) - s + \tau(s) \leq K_1\Delta.
\end{aligned} \tag{23}$$

Thus $\mathbb{E}(|x(s - \tau(s)) - x(t_m)|) \leq C_2^{1/2} K_1^{1/2} \Delta^{1/2}$.

(3) If $s - \tau(s) \leq 0$ and $[n\Delta - \tau(n\Delta)]\Delta \leq 0$, then

$$|s - \tau(s) - [n\Delta - \tau(n\Delta)]\Delta| \leq (K_1 + 2)\Delta. \tag{24}$$

Therefore,

$$\begin{aligned}
&\mathbb{E} (|x(s - \tau(s)) - x(t_m)|) \\
&= \mathbb{E} (|\xi(s - \tau(s)) - \xi(t_m)|) \\
&\leq K_4^{1/2} |s - \tau(s) - [n\Delta + \tau(n\Delta)]\Delta|^{1/2} \\
&\leq K_4^{1/2} (2 + K_1)^{1/2} \Delta^{1/2}.
\end{aligned} \tag{25}$$

(4) If $s - \tau(s) \geq 0$ and $[n\Delta - \tau(n\Delta)]\Delta \leq 0$, then

$$\begin{aligned}
s - \tau(s) &\leq s - \tau(s) - [n\Delta + \tau(n\Delta)]\Delta \\
&\leq (K_1 + 2)\Delta, \\
-[n\Delta + \tau(n\Delta)]\Delta &\leq s - \tau(s) - [n\Delta + \tau(n\Delta)]\Delta \\
&\leq (K_1 + 2)\Delta.
\end{aligned} \tag{26}$$

Thus

$$\begin{aligned}
&\mathbb{E} (|x(s - \tau(s)) - x(t_m)|) \\
&= \mathbb{E} (|x(s - \tau(s)) - \xi(t_m)|) \\
&\leq \mathbb{E} (|x(s - \tau(s)) - \xi(0)| + |\xi(0) - \xi(t_m)|) \\
&\leq K_4^{1/2} (|s - \tau(s)|^{1/2} + |[n\Delta + \tau(n\Delta)]\Delta|^{1/2}) \\
&\leq 2K_4^{1/2} (2 + K_1)^{1/2} \Delta^{1/2}.
\end{aligned} \tag{27}$$

(5) If $s - \tau(s) \leq 0$ and $[n\Delta - \tau(n\Delta)]\Delta \geq 0$, then

$$\begin{aligned}
&-[s - \tau(s)] \leq [n\Delta + \tau(n\Delta)] - [s - \tau(s)]\Delta \\
&\leq (K_1 + 2)\Delta, \\
&[n\Delta + \tau(n\Delta)]\Delta \leq [n\Delta + \tau(n\Delta)] - [s - \tau(s)] \\
&\leq (K_1 + 2)\Delta.
\end{aligned} \tag{28}$$

Thus

$$\begin{aligned}
&\mathbb{E} (|x(s - \tau(s)) - x(t_m)|) \\
&= \mathbb{E} (|\xi(s - \tau(s)) - x(t_m)|) \\
&\leq \mathbb{E} (|\xi(s - \tau(s)) - \xi(0)| + |\xi(0) - x(t_m)|) \\
&\leq 2K_4^{1/2} (2 + K_1)^{1/2} \Delta^{1/2}.
\end{aligned} \tag{29}$$

Combining the above five cases, it can be concluded that there exists a constant K such that $\mathbb{E}(|x(s - \tau(s)) - x(t_m)|) \leq K\Delta^{1/2}$.

Similarly, it can be proved that there exists a constant K' such that $\mathbb{E}(|x(s - \tau(s)) - x(t_m)|) \leq K'\Delta^{1/2}$. Therefore,

$$\begin{aligned}
&\mathbb{E} (|x(t_m) - x(t_m)|) \leq \mathbb{E} (|x(s - \tau(s)) - x(t_m)|) \\
&\quad + \mathbb{E} (|x(s - \tau(s)) - x(t_m)|) \\
&\leq (K + K')\Delta^{1/2}.
\end{aligned} \tag{30}$$

From (15a) and Theorem 1, we have, for any $\Delta < 1/[K_3\theta(1 + \lambda)]$, that there exists a constant K_6 such that

$$\begin{aligned}
&\mathbb{E} (|\bar{x}_n^*| \mid \mathcal{F}_{t_n}) \\
&= \mathbb{E} (|x(t_n) + [\theta f_\lambda(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1}) \\
&\quad + (1 - \theta) f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1})] \Delta| \mid \\
&\quad \mathcal{F}_{t_n}) \leq K_6.
\end{aligned} \tag{31}$$

Hence,

$$\begin{aligned}
& \mathbb{E} \left(|\bar{x}_n^* - x(t_n)| \mid \mathcal{F}_{t_n} \right) \\
&= \mathbb{E} \left(\left| \left[\theta f_\lambda \left(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1} \right) \right. \right. \right. \\
&\quad \left. \left. \left. + (1-\theta) f_\lambda \left(x(t_n), x(t_m), \gamma_{N(t_n)+1} \right) \right] \Delta \right| \mid \mathcal{F}_{t_n} \right) \\
&\leq \left[\theta (1+\lambda) K_3 \left(1 + \mathbb{E} \left(|\bar{x}_n^*| \mid \mathcal{F}_{t_n} \right) + \mathbb{E} \left(|x(t_m)| \mid \mathcal{F}_{t_n} \right) \right) \right. \\
&\quad \left. + (1-\theta) (1+\lambda) K_3 \right. \\
&\quad \cdot \left(1 + \mathbb{E} \left(|x(t_n)| \mid \mathcal{F}_{t_n} \right) + \mathbb{E} \left(|x(t_m)| \mid \mathcal{F}_{t_n} \right) \right) \\
&\quad \left. + \lambda K_3 \mathbb{E} \left(|\gamma_{N(t_n)+1}| \mid \mathcal{F}_{t_n} \right) \right] \Delta \\
&\leq \left[(1+\lambda) K_3 (1+2C_1+K_6\theta-C_1\theta) \right. \\
&\quad \left. + \lambda K_3 \mathbb{E} \left(|\gamma_{N(t_n)+1}| \mid \mathcal{F}_{t_n} \right) \right] \Delta. \tag{32}
\end{aligned}$$

Moreover, it is known from Theorem 3.4 in [12] that

$$\mathbb{E} \left(\int_{t_n}^{t_{n+1}} |\gamma_{N(s)+1} - \gamma_{N(t_n)+1}|^2 ds \right) \leq C \Delta^{2-2/p}. \tag{33}$$

Thus,

$$\begin{aligned}
& \left| \mathbb{E} \left(\int_{t_n}^{t_{n+1}} (\gamma_{N(s)+1} - \gamma_{N(t_n)+1}) ds \mid \mathcal{F}_{t_n} \right) \right| \\
&\leq C^{1/2} \Delta^{3/2-1/p}. \tag{34}
\end{aligned}$$

Summing up the above conclusions, we can obtain that, for any $\Delta < 1/[K_3\theta(1+\lambda)]$, there exists a constant H_1 such that

$$\mathbb{E} \left| \mathbb{E} \left[(x(t_{n+1}) - \bar{x}_{n+1}) \mid \mathcal{F}_{t_n} \right] \right|^2 \leq H_1^2 \Delta^{3-2/p}. \tag{35}$$

Otherwise, for any $\Delta < 1/[4K_3^2\theta^2(1+\lambda^2)]$, there exists a constant H_2 such that

$$\begin{aligned}
& \mathbb{E} \left(|x(t_{n+1}) - \bar{x}_{n+1}|^2 \right) \\
&\leq 4 \mathbb{E} \left(\left| \int_{t_n}^{t_{n+1}} \left[f_\lambda(x(s), x(s-\tau(s)), \gamma_{N(s)+1}) \right. \right. \right. \\
&\quad \left. \left. \left. - f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right] ds \right|^2 \right) \\
&\quad + 4\theta \mathbb{E} \left(\left| \int_{t_n}^{t_{n+1}} \left[f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right. \right. \right. \\
&\quad \left. \left. \left. - f_\lambda(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1}) \right] ds \right|^2 \right)
\end{aligned}$$

$$\begin{aligned}
& + 4 \mathbb{E} \left(\left| \int_{t_n}^{t_{n+1}} \left[g(x(s), x(s-\tau(s))) \right. \right. \right. \\
&\quad \left. \left. \left. - g(\bar{x}_n^*, x(t_m)) \right] dW(s) \right|^2 \right) \\
& + 4 \mathbb{E} \left(\left| \int_{t_n}^{t_{n+1}} \left[h(x(s), x(s-\tau(s)), \gamma_{N(s)+1}) \right. \right. \right. \\
&\quad \left. \left. \left. - h(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1}) \right] dN(s) \right|^2 \right) \\
&\leq 4K_2^2 \Delta \int_{t_n}^{t_{n+1}} \left[(1+\lambda) \mathbb{E} \left(|x(s) - x(t_n)|^2 \right) \right. \\
&\quad \left. + (1+\lambda) \mathbb{E} \left(|x(s-\tau(s)) - x(t_m)|^2 \right) \right] ds \\
&\quad + 4\theta K_2^2 \Delta \int_{t_n}^{t_{n+1}} \left[(1+\lambda) \mathbb{E} \left(|x(t_n) - \bar{x}_n^*|^2 \right) \right. \\
&\quad \left. + (1+\lambda) \mathbb{E} \left(|x(t_m) - x(t_m)|^2 \right) \right] ds \\
&\quad + 4K_2^2 (1+\lambda) \int_{t_n}^{t_{n+1}} \left[\mathbb{E} \left(|x(s) - \bar{x}_n^*|^2 \right) \right. \\
&\quad \left. + \mathbb{E} \left(|x(s-\tau(s)) - x(t_m)|^2 \right) \right] ds \\
&\quad + 4K_2^2 \lambda (1+\Delta) \int_{t_n}^{t_{n+1}} \left[\mathbb{E} \left(|\gamma_{N(s)+1} - \gamma_{N(t_n)+1}|^2 \right) \right] ds \\
&\leq H_2^2 \Delta^{2(1-1/p)}. \tag{36}
\end{aligned}$$

□

Theorem 3. Under assumptions (i)–(iv), there exists a constant $0 < \Delta^* < 1$ such that, for any $0 \leq \Delta \leq \Delta^*$, the global error of the compensated split-step θ method satisfies

$$\max_{0 \leq n \leq N-1} \|\epsilon_{n+1}\|_{L_2} \leq H_3 \Delta^{1/2-1/p} \quad \text{as } \Delta \rightarrow 0, \tag{37}$$

where H_3 is independent of Δ .

Proof. From the definitions of δ_n and ϵ_n , we have

$$\epsilon_{n+1} = \epsilon_n + u_n + \delta_{n+1}, \tag{38}$$

where

$$\begin{aligned}
u_n &= \theta \left[f_\lambda(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1}) - f_\lambda(x_n^*, x_m, \gamma_{N(t_n)+1}) \right] \Delta \\
&\quad + (1-\theta) \left[f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right. \\
&\quad \left. - f_\lambda(x_n, x_m, \gamma_{N(t_n)+1}) \right] \Delta \\
&\quad + [g(\bar{x}_n^*, x(t_m)) - g(x_n^*, x_m)] \Delta W_n \\
&\quad + [h(\bar{x}_n^*, x(t_m), \gamma_{N(t_n)+1}) - h(x_n^*, x_m, \gamma_{N(t_n)+1})] \Delta \bar{N}_n. \tag{39}
\end{aligned}$$

From (14a), (15a), and assumption (ii) it can be obtained that, for any $\Delta < 1/[K_2\theta(1+\lambda)]$,

$$\begin{aligned}
& \mathbb{E}(|\bar{x}_n^* - x_n^*| \mid \mathcal{F}_{t_n}) \\
&= \mathbb{E} \left(\left| x(t_n) - x_n \right. \right. \\
&\quad \left. \left. + \theta \left[f_\lambda(\bar{x}_n^*, x(t_{\bar{m}}), \gamma_{N(t_n)+1}) \right. \right. \right. \\
&\quad \left. \left. - f_\lambda(x_n^*, x_{\bar{m}}, \gamma_{N(t_n)+1}) \right] \Delta \right. \\
&\quad \left. + (1-\theta) \left[f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right. \right. \right. \\
&\quad \left. \left. - f_\lambda(x_n, x_m, \gamma_{N(t_n)+1}) \right] \Delta \mid \mathcal{F}_{t_n} \right) \\
&\leq \frac{1}{1 - K_2\theta(1+\lambda)\Delta} \\
&\quad \cdot \left[(1 + K_2(1-\theta)(1+\lambda)\Delta) \mathbb{E}(|\epsilon_n| \mid \mathcal{F}_{t_n}) \right. \\
&\quad \left. + K_2\theta(1+\lambda)\Delta \mathbb{E}(|\epsilon_{\bar{m}}|) \right. \\
&\quad \left. + K_2(1-\theta)(1+\lambda)\Delta \mathbb{E}(|\epsilon_m|) \right] \quad (40)
\end{aligned}$$

and, for any $\Delta < 1/[6K_2^2\theta^2(1+\lambda^2)]$,

$$\begin{aligned}
& \mathbb{E}(|\bar{x}_n^* - x_n^*|^2) \\
&= \mathbb{E} \left(\left| x(t_n) - x_n \right. \right. \\
&\quad \left. \left. + \theta \left[f_\lambda(\bar{x}_n^*, x(t_{\bar{m}}), \gamma_{N(t_n)+1}) \right. \right. \right. \\
&\quad \left. \left. - f_\lambda(x_n^*, x_{\bar{m}}, \gamma_{N(t_n)+1}) \right] \Delta \right. \\
&\quad \left. + (1-\theta) \left[f_\lambda(x(t_n), x(t_m), \gamma_{N(t_n)+1}) \right. \right. \right. \\
&\quad \left. \left. - f_\lambda(x_n, x_m, \gamma_{N(t_n)+1}) \right] \Delta \mid \mathcal{F}_{t_n} \right)^2 \quad (41) \\
&\leq \frac{1}{1 - 6K_2^2\theta^2(1+\lambda^2)\Delta} \\
&\quad \cdot \left[3(1 + 2K_2^2(1-\theta)^2(1+\lambda^2)\Delta^2) \mathbb{E}(|\epsilon_n|^2) \right. \\
&\quad \left. + 6K_2^2\theta^2(1+\lambda^2)\Delta^2 \mathbb{E}(|\epsilon_{\bar{m}}|^2) \right. \\
&\quad \left. + 6K_2^2(1-\theta)^2(1+\lambda^2)\Delta^2 \mathbb{E}(|\epsilon_m|^2) \right].
\end{aligned}$$

Since $\mathbb{E}(\Delta W_n) = 0$, $\mathbb{E}(\Delta W_n)^2 = \Delta$, $\mathbb{E}(\Delta \bar{N}_n) = 0$, and $\mathbb{E}(\Delta \bar{N}_n)^2 = \lambda\Delta$, we know from assumption (ii) that

$$\begin{aligned}
& |\mathbb{E}(u_n \mid \mathcal{F}_{t_n})| \\
&\leq K_2\theta(1+\lambda)\Delta \left[\mathbb{E}(|\bar{x}_n^* - x_n^*| \mid \mathcal{F}_{t_n}) + \mathbb{E}(|\epsilon_{\bar{m}}|) \right] \\
&\quad + K_2(1-\theta)(1+\lambda)\Delta \left[\mathbb{E}(|\epsilon_n| \mid \mathcal{F}_{t_n}) + \mathbb{E}(|\epsilon_m|) \right] \\
&\leq K_7\Delta \left[\mathbb{E}(|\epsilon_n| \mid \mathcal{F}_{t_n}) + \mathbb{E}(|\epsilon_m|) + \mathbb{E}(|\epsilon_{\bar{m}}|) \right],
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E}(|u_n|^2) \\
&\leq 4K_2^2\Delta(1+\lambda+2\theta^2(1+\lambda^2)\Delta) \mathbb{E}(|\bar{x}_n^* - x_n^*|^2) \\
&\quad + 8K_2^2(1-\theta)^2(1+\lambda^2)\Delta^2 \mathbb{E}(|\epsilon_n|^2) \\
&\quad + 8K_2^2\theta^2(1+\lambda^2)\Delta^2 \mathbb{E}(|\epsilon_{\bar{m}}|^2) \\
&\quad + 4K_2^2\Delta(1+\lambda+2(1-\theta)^2(1+\lambda^2)\Delta) \mathbb{E}(|\epsilon_m|^2) \\
&\leq K_8\Delta \left[\mathbb{E}(|\epsilon_n|^2) + \mathbb{E}(|\epsilon_{\bar{m}}|^2) + \mathbb{E}(|\epsilon_m|^2) \right], \quad (42)
\end{aligned}$$

where $K_7 = 4K_2(1+\lambda) + 3K_2^2(1+\lambda)^2$, $K_8 = 4K_2^2(1+\lambda+2(1+\lambda^2))(15K_2^2(1+\lambda^2)+3) + 4K_2^2(1+\lambda+6(1+\lambda^2))$. Hence

$$\begin{aligned}
& |\mathbb{E}\langle \epsilon_n, u_n \rangle| \\
&= \left| \mathbb{E} \left[\mathbb{E}(\langle \epsilon_n, u_n \rangle \mid \mathcal{F}_{t_n}) \right] \right| \\
&\leq \mathbb{E} \left[|\epsilon_n| \cdot |\mathbb{E}(u_n \mid \mathcal{F}_{t_n})| \right] \\
&\leq \left[\Delta^{-1} \left(\mathbb{E}|\mathbb{E}(u_n \mid \mathcal{F}_{t_n})|^2 \right) \right]^{1/2} (\Delta \mathbb{E}|\epsilon_n|^2)^{1/2} \quad (43) \\
&\leq \Delta^{-1} \left(\mathbb{E}|\mathbb{E}(u_n \mid \mathcal{F}_{t_n})|^2 \right) + \Delta \mathbb{E}(|\epsilon_n|^2) \\
&\leq K_7\Delta \left[\mathbb{E}(|\epsilon_n|^2) + \mathbb{E}(|\epsilon_{\bar{m}}|^2) + \mathbb{E}(|\epsilon_m|^2) \right] \\
&\quad + \Delta \mathbb{E}(|\epsilon_n|^2).
\end{aligned}$$

From Theorem 2, we have

$$\begin{aligned}
& |\mathbb{E}\langle \delta_{n+1}, \epsilon_n \rangle| \\
&= \left| \mathbb{E} \left[\mathbb{E}(\langle \delta_{n+1}, \epsilon_n \rangle \mid \mathcal{F}_{t_n}) \right] \right| \\
&\leq \mathbb{E} \left[|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})| \cdot |\epsilon_n| \right] \\
&\leq \left[\Delta^{-1} \left(\mathbb{E}|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})|^2 \right) \right]^{1/2} (\Delta \mathbb{E}|\epsilon_n|^2)^{1/2} \quad (44) \\
&\leq \Delta^{-1} \left(\mathbb{E}|\mathbb{E}(\delta_{n+1} \mid \mathcal{F}_{t_n})|^2 \right) + \Delta \mathbb{E}(|\epsilon_n|^2) \\
&\leq H_1\Delta^{2-2/p} + \Delta \mathbb{E}(|\epsilon_n|^2).
\end{aligned}$$

Combining the above conclusions, we have

$$\begin{aligned}
& \mathbb{E}(|\epsilon_{n+1}|^2) \\
&\leq \mathbb{E}(|\epsilon_n|^2) + 2\mathbb{E}(|\delta_{n+1}|^2) + 2\mathbb{E}(|u_n|^2) \\
&\quad + 2|\mathbb{E}\langle \delta_{n+1}, \epsilon_n \rangle| + 2|\mathbb{E}\langle \epsilon_n, u_n \rangle| \\
&\leq [1 + 2(1 + K_7 + K_8)\Delta] \mathbb{E}(|\epsilon_n|^2) \quad (45) \\
&\quad + 2\Delta(K_7 + K_8) \left(\mathbb{E}(|\epsilon_{[(n\Delta-\tau(n\Delta))/\Delta]}|^2) \right. \\
&\quad \left. + \mathbb{E}(|\epsilon_{[(n+1)\Delta-\tau((n+1)\Delta)/\Delta]}|^2) \right) \\
&\quad + (H_1 + 2H_2^2)\Delta^{2(1-1/p)}.
\end{aligned}$$

Additionally, when $-r \leq n \leq 0$, $\epsilon_n = 0$. From (45), it can be obtained that the following items are true.

(1) If $[(n\Delta - \tau(n\Delta))/\Delta]\Delta \leq 0$ and $[(n+1)\Delta - \tau((n+1)\Delta))/\Delta]\Delta \leq 0$, we have

$$\begin{aligned} \mathbb{E}(|\epsilon_{n+1}|^2) &\leq [1 + 2(1 + K_7 + K_8)\Delta] \mathbb{E}|\epsilon_n|^2 \\ &\quad + (H_1 + 2H_2^2)\Delta^{2(1-1/p)} \\ &\leq (H_1 + 2H_2^2)\Delta^{1-2/p} \sum_{i=0}^n [1 + 2(1 + K_7 + K_8)\Delta]^i \end{aligned} \quad (46)$$

$$\leq (H_1 + 2H_2^2) \frac{e^{2(1+K_7+K_8)T} - 1}{2(1 + K_7 + K_8)} \Delta^{1-2/p}.$$

(2) If $[(n\Delta - \tau(n\Delta))/\Delta]\Delta \geq 0$ and $[(n+1)\Delta - \tau((n+1)\Delta))/\Delta]\Delta \geq 0$, we have

$$\begin{aligned} \mathbb{E}(|\epsilon_{n+1}|^2) &\leq [1 + 2(1 + 3K_7 + 3K_8)\Delta] \max_{0 \leq i \leq n} (\mathbb{E}|\epsilon_i|^2) \\ &\quad + (H_1 + 2H_2^2)\Delta^{2(1-1/p)} \\ &\leq (H_1 + 2H_2^2) \frac{e^{2(1+3K_7+3K_8)T} - 1}{2(1 + 3K_7 + 3K_8)} \Delta^{1-2/p}. \end{aligned} \quad (47)$$

(3) If $[(n\Delta - \tau(n\Delta))/\Delta]\Delta \geq 0$ and $[(n+1)\Delta - \tau((n+1)\Delta))/\Delta]\Delta \leq 0$, we have

$$\begin{aligned} \mathbb{E}(|\epsilon_{n+1}|^2) &\leq [1 + 2(1 + 2K_7 + 2K_8)\Delta] \max_{0 \leq i \leq n} (\mathbb{E}|\epsilon_i|^2) \\ &\quad + (H_1 + 2H_2^2)\Delta^{2(1-1/p)} \\ &\leq (H_1 + 2H_2^2) \frac{e^{2(1+2K_7+2K_8)T} - 1}{2(1 + 2K_7 + 2K_8)} \Delta^{1-2/p}. \end{aligned} \quad (48)$$

Combining the above three cases, it can be obtained that

$$\max_{0 \leq n \leq N-1} (\mathbb{E}|\epsilon_{n+1}|^2)^{1/2} \leq H_3 \Delta^{1/2-1/p}, \quad (49)$$

where $H_3 = ((C_3 + 2C_4^2)(e^{2(1+3K_7+3K_8)T} - 1))/[2(1 + K_7 + K_8)]^{1/2}$. $\square$

4. Stability of the Analytical and Numerical Solutions for the SDEVDRJM

In this section, we will discuss the stability of the analytical solutions of the SDEVDRJM and the numerical method introduced in Section 2.

Consider the following scalar test equation:

$$\begin{aligned} dx(t) &= (a_1 x(t) + a_2 x(t - \tau(t))) dt \\ &\quad + (b_1 x(t) + b_2 x(t - \tau(t))) dW(t) \\ &\quad + \gamma(t) x(t) dN(t), \end{aligned} \quad (50)$$

where a_i and b_i are real constants. Define $\gamma(t) = \gamma_{N(t)+1} = \sum_j \gamma_{j+1} 1_{[\tau_j, \tau_{j+1}]}(t)$, where $\tau_0 = 0$ and τ_j ($j = 1, 2, \dots$) are the jump times.

4.1. Mean-Square Stability of the Analytical Solutions. In what follows, we give some sufficient conditions on the stability property of the analytical solutions of (50).

Theorem 4. Assume that the constants a_1, a_2, b_1, b_2 , and λ and the random variable γ satisfy

$$\begin{aligned} 2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| + b_2^2 \\ + 2|b_1||b_2| < 0. \end{aligned} \quad (51)$$

Then solution of (50) is mean-square stable. That is,

$$\lim_{t \rightarrow \infty} \mathbb{E}(|x(t)|^2) = 0. \quad (52)$$

Proof. For any $t > 0$ and $\delta > 0$, it follows from Itô formula that

$$\begin{aligned} |x(t + \delta)|^2 &= |x(t)|^2 + \int_t^{t+\delta} (2\langle x(s), a_1 x(s) + a_2 x(s - \tau(s)) \rangle \\ &\quad + |b_1 x(s) + b_2 x(s - \tau(s))|^2) ds \\ &\quad + \int_t^{t+\delta} 2\langle x(s), b_1 x(s) + b_2 x(s - \tau(s)) \rangle dW(s) \\ &\quad + \int_t^{t+\delta} (2\langle x(s), \gamma(s) x(s) \rangle + |\gamma(s) x(s)|^2) d\bar{N}(s) \\ &\quad + \lambda \int_t^{t+\delta} (2\langle x(s), \gamma(s) x(s) \rangle + |\gamma(s) x(s)|^2) ds. \end{aligned} \quad (53)$$

Taking the expectation yields

$$\begin{aligned} \mathbb{E}(|x(t + \delta)|^2) &= \mathbb{E}(|x(t)|^2) \\ &\quad + \int_t^{t+\delta} (2a_1 \mathbb{E}(|x(s)|^2) \\ &\quad + 2a_2 \mathbb{E}(|x(s)||x(s - \tau(s))|) \\ &\quad + \mathbb{E}(|b_1 x(s) + b_2 x(s - \tau(s))|^2)) ds \\ &\quad + \lambda \int_t^{t+\delta} (2\mathbb{E}(\gamma(s)|x(s)|^2) \\ &\quad + \mathbb{E}(|\gamma(s)x(s)|^2)) ds \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}(|x(t)|^2) \\
&+ \int_t^{t+\delta} \left[(2a_1 + b_1^2 + 2\lambda\mathbb{E}(\gamma) + \lambda\mathbb{E}(|\gamma|^2)) \right. \\
&\quad \cdot \mathbb{E}(|x(s)|^2) \\
&\quad + (2|a_2| + b_2^2 + 2|b_1||b_2|) \\
&\quad \cdot \sup_{-r \leq u \leq s} \mathbb{E}(|x(u)|^2) \Big] ds.
\end{aligned} \tag{54}$$

Let $v(t) = \mathbb{E}(|x(t)|^2)$, $\alpha = 2a_1 + b_1^2 + 2\lambda\mathbb{E}(\gamma) + \lambda\mathbb{E}(|\gamma|^2)$, and $\beta = 2|a_2| + b_2^2 + 2|b_1||b_2|$. It is obtained that

$$\begin{aligned}
D^+ v(t) &= \limsup_{\delta \searrow 0} \frac{v(t+\delta) - v(t)}{\delta} \\
&\leq \alpha v(t) + \beta \sup_{-r \leq u \leq t} v(u).
\end{aligned} \tag{55}$$

From (51), we know $0 < \beta < -\alpha$. Furthermore, according to lemma 1.1 in [19], there exist k and k' such that

$$v(t) \leq ke^{-k't}. \tag{56}$$

Thus, $v(t) \rightarrow 0$ when $t \rightarrow \infty$. $\square$

4.2. Mean-Square Stability of the Numerical Solutions. Applying the compensated split-step θ method to the test equation yields

$$\begin{aligned}
x_n^* &= x_n + \left[\theta(a_1 x_n^* + a_2 x_{\bar{m}}) + (1-\theta)(a_1 x_n + a_2 x_m) \right. \\
&\quad \left. + \lambda \gamma_{N(t_n)+1} x_n \right] \Delta,
\end{aligned} \tag{57a}$$

$$x_{n+1} = x_n^* + (b_1 x_n^* + b_2 x_m) \Delta W_n + \gamma_{N(t_n)+1} x_n^* \bar{\Delta} N_n. \tag{57b}$$

Note that a numerical method is said to be mean-square stable (MS-stable) if there exists a constant $\Delta^* > 0$ such that $\lim_{n \rightarrow \infty} \mathbb{E}(|x_n|^2) = 0$ for all $\Delta \in (0, \Delta^*)$ and a numerical method is said to be general mean-square stable (GMS-stable) if $\lim_{n \rightarrow \infty} \mathbb{E}(|x_n|^2) = 0$ holds for every time step size $\Delta > 0$. For notational simplicity, let

$$\begin{aligned}
A &= (b_1^2 + \lambda\mathbb{E}(\gamma^2) + |b_1 b_2|) \\
&\cdot \left[(1-\theta)^2 a_1^2 + \lambda^2 \mathbb{E}(\gamma^2) + |a_2|^2 + 2(1-\theta)a_1 \lambda \mathbb{E}(\gamma) \right. \\
&\quad \left. + 2|a_2|((1-\theta)|a_1| + \lambda\mathbb{E}(|\gamma|)) \right] + \theta^2 a_1^2 (b_2^2 + |b_1 b_2|),
\end{aligned}$$

$$\begin{aligned}
B &= (1-2\theta)a_1^2 + \lambda^2 \mathbb{E}(\gamma^2) + 2(1-\theta)a_1 \lambda \mathbb{E}(\gamma) \\
&\quad + 2|a_2|((1-\theta)|a_1| + \lambda\mathbb{E}(|\gamma|)) + |a_2|^2 \\
&\quad + 2(b_1^2 + \lambda\mathbb{E}(\gamma^2) + |b_1 b_2|)((1-\theta)a_1 + \lambda\mathbb{E}(\gamma) + 2|a_2|) \\
&\quad - 2a_1 \theta (b_2^2 + |b_1 b_2|), \\
\hat{\Delta} &= B^2 - 4A \left[2a_1 + b_1^2 + 2\lambda\mathbb{E}(\gamma) \right. \\
&\quad \left. + \lambda\mathbb{E}(\gamma^2) + 2|a_2| + b_2^2 + 2|b_1||b_2| \right].
\end{aligned} \tag{58}$$

Theorem 5. Assume that condition (51) holds. Then we have the following conclusions.

(i) If $A = 0$ and $B \leq 0$ or $A < 0$ and $B < 2(A[2a_1 + b_1^2 + 2\lambda\mathbb{E}(\gamma) + \lambda\mathbb{E}(\gamma^2) + 2|a_2| + b_2^2 + 2|b_1||b_2|])^{1/2}$, then the compensated split-step θ method for (50) is GMS-stable.

(ii) If $A = 0$ and $B > 0$, then there exists a constant $\Delta_0 > 0$ such that, for any $\Delta < \Delta_0$, the compensated split-step θ method for (50) is MS-stable.

(iii) If $A < 0$ and $B \geq 2(A[2a_1 + b_1^2 + 2\lambda\mathbb{E}(\gamma) + \lambda\mathbb{E}(\gamma^2) + 2|a_2| + b_2^2 + 2|b_1||b_2|])^{1/2}$, then there exists a constant Δ_1 such that, for any $\Delta < \Delta_1$, the compensated split-step θ method for (50) is MS-stable.

(iv) If $A > 0$, then there exists a constant Δ_2 such that, for any $\Delta < \Delta_2$, the compensated split-step θ method for (50) is MS-stable.

Here,

$$\Delta_0 = -\frac{2a_1 + b_1^2 + 2\lambda\mathbb{E}(\gamma) + \lambda\mathbb{E}(|\gamma|^2) + 2|a_2| + b_2^2 + 2|b_1||b_2|}{B}, \tag{59}$$

$$\Delta_1 = (-B - \sqrt{\hat{\Delta}})/2A \text{ and } \Delta_2 = (-B + \sqrt{\hat{\Delta}})/2A.$$

Proof. From (57a), if $\theta \neq 1/\Delta a_1$, we have

$$\begin{aligned}
x_n^* &= \frac{1}{1 - a_1 \theta \Delta} \left[(1 + (1-\theta)a_1 \Delta + \lambda \gamma_{N(t_n)+1} \Delta) x_n \right. \\
&\quad \left. + a_2 \theta \Delta x_{\bar{m}} + a_2 (1-\theta) \Delta x_m \right].
\end{aligned} \tag{60}$$

Then, it can be obtained that

$$\begin{aligned}
&(1 - a_1 \theta \Delta)^2 |x_n^*|^2 \\
&\leq (1 + (1-\theta)a_1 \Delta + \lambda \gamma_{N(t_n)+1} \Delta)^2 |x_n|^2 \\
&\quad + a_2^2 \theta^2 \Delta^2 |x_{\bar{m}}|^2 + a_2^2 (1-\theta)^2 \Delta^2 |x_m|^2 \\
&\quad + |a_2| \theta \Delta (1 + (1-\theta)|a_1| \Delta + \lambda |\gamma_{N(t_n)+1}| \Delta) \\
&\quad \cdot (|x_n|^2 + |x_{\bar{m}}|^2) + |a_2| (1-\theta) \\
&\quad \cdot \Delta (1 + (1-\theta)|a_1| \Delta + \lambda |\gamma_{N(t_n)+1}| \Delta) (|x_n|^2 + |x_m|^2) \\
&\quad + a_2^2 \theta (1-\theta) \Delta^2 (|x_{\bar{m}}|^2 + |x_m|^2)
\end{aligned}$$

$$\begin{aligned}
&\leq \left[(1 + (1 - \theta) a_1 \Delta + \lambda \gamma_{N(t_n)+1} \Delta)^2 \right. \\
&\quad + |a_2| \Delta (1 + (1 - \theta) |a_1| \Delta + \lambda |\gamma_{N(t_n)+1}| \Delta) \left. \right] |x_n|^2 \\
&\quad + [a_2^2 \theta \Delta^2 + |a_2| \theta \Delta \\
&\quad \cdot (1 + (1 - \theta) |a_1| \Delta + \lambda |\gamma_{N(t_n)+1}| \Delta)] |x_m|^2 \\
&\quad + [a_2^2 (1 - \theta) \Delta^2 + |a_2| (1 - \theta) \\
&\quad \cdot \Delta (1 + (1 - \theta) |a_1| \Delta + \lambda |\gamma_{N(t_n)+1}| \Delta)] |x_m|^2.
\end{aligned} \tag{61}$$

From (57b), it is true that

$$x_{n+1} = (1 + b_1 \Delta W_n + \gamma_{N(t_n)+1} \Delta \bar{N}_n) x_n^* + b_2 \Delta W_n x_m. \tag{62}$$

Hence,

$$\begin{aligned}
|x_{n+1}|^2 &= (1 + b_1 \Delta W_n + \gamma_{N(t_n)+1} \Delta \bar{N}_n)^2 |x_n^*|^2 \\
&\quad + b_2^2 (\Delta W_n)^2 |x_m|^2 + 2b_2 \Delta W_n \\
&\quad \cdot (1 + b_1 \Delta W_n + \gamma_{N(t_n)+1} \Delta \bar{N}_n) x_n^* x_m \\
&\leq (1 + b_1^2 (\Delta W_n)^2 + (\gamma_{N(t_n)+1})^2 (\Delta \bar{N}_n)^2 \\
&\quad + 2b_1 \Delta W_n + 2\gamma_{N(t_n)+1} \Delta \bar{N}_n \\
&\quad + 2b_1 \Delta W_n \gamma_{N(t_n)+1} \Delta \bar{N}_n) |x_n^*|^2 \\
&\quad + b_2^2 (\Delta W_n)^2 |x_m|^2 + |b_1 b_2| (\Delta W_n)^2 (|x_n^*|^2 + |x_m|^2) \\
&\quad + 2b_2 \Delta W_n (1 + \gamma_{N(t_n)+1} \Delta \bar{N}_n) x_n^* x_m.
\end{aligned} \tag{63}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}(|x_{n+1}|^2) \\
&\leq (1 + b_1^2 \Delta + \mathbb{E}(\gamma^2) \lambda \Delta + |b_1 b_2| \Delta) \mathbb{E}(|x_n^*|^2) \\
&\quad + (b_2^2 \Delta + |b_1 b_2| \Delta) \mathbb{E}(|x_m|^2) \\
&= \frac{1 + b_1 \Delta + \lambda \mathbb{E}(\gamma^2) \Delta + |b_1 b_2| \Delta}{(1 - a_1 \theta \Delta)^2} \\
&\quad \cdot \left[\mathbb{E}((1 + (1 - \theta) a_1 \Delta + \lambda \gamma_{N(t_n)+1} \Delta)^2) \right. \\
&\quad + |a_2| \Delta (1 + (1 - \theta) |a_1| \Delta + \lambda \mathbb{E}(|\gamma|) \Delta) \left. \right] \\
&\quad \cdot \mathbb{E}(|x_n|^2) + [a_2^2 \theta \Delta^2 + |a_2| \theta \Delta \\
&\quad \cdot (1 + (1 - \theta) |a_1| \Delta + \lambda \mathbb{E}(|\gamma|) \Delta)] \\
&\quad \cdot \mathbb{E}(|x_m|^2) + [a_2^2 (1 - \theta) \Delta^2 + |a_2| (1 - \theta) \Delta \\
&\quad \cdot (1 + (1 - \theta) |a_1| \Delta + \lambda \mathbb{E}(|\gamma|) \Delta)] \\
&\quad \cdot \mathbb{E}(|x_m|^2) + (b_2^2 \Delta + |b_1 b_2| \Delta) \mathbb{E}(|x_m|^2)
\end{aligned}$$

$$\begin{aligned}
&\leq P(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) \mathbb{E}(|x_n|^2) \\
&\quad + Q(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) \mathbb{E}(|x_m|^2) \\
&\quad + R(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) \mathbb{E}(|x_m|^2) \\
&\leq (P(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) \\
&\quad + Q(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) \\
&\quad + R(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma))) \max_{-r \leq s \leq n} \mathbb{E}(|x_s|^2),
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
P(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) &= \frac{1 + b_1^2 \Delta + \lambda \mathbb{E}(\gamma^2) \Delta + |b_1 b_2| \Delta}{(1 - a_1 \theta \Delta)^2} \\
&\quad \cdot [\mathbb{E}(1 + (1 - \theta) a_1 \Delta + \lambda \gamma_{N(t_n)+1} \Delta)^2 \\
&\quad + |a_2| \Delta (1 + (1 - \theta) |a_1| \Delta + \lambda |\mathbb{E}(\gamma)| \Delta)]; \\
Q(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) &= \frac{1 + b_1^2 \Delta + \lambda \mathbb{E}(\gamma^2) \Delta + |b_1 b_2| \Delta}{(1 - a_1 \theta \Delta)^2} \\
&\quad \cdot [a_2^2 \theta \Delta^2 + \theta |a_2| \Delta (1 + (1 - \theta) |a_1| \Delta + \lambda |\mathbb{E}(\gamma)| \Delta)]; \\
R(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) &= \frac{1 + b_1^2 \Delta + \lambda \mathbb{E}(\gamma^2) \Delta + |b_1 b_2| \Delta}{(1 - a_1 \theta \Delta)^2} \\
&\quad \cdot [a_2^2 (1 - \theta) \Delta^2 + (1 - \theta) |a_2| \Delta \\
&\quad \cdot (1 + (1 - \theta) |a_1| \Delta + \lambda |\mathbb{E}(\gamma)| \Delta)] \\
&\quad + b_2^2 \Delta + |b_1 b_2| \Delta.
\end{aligned} \tag{65}$$

Obviously, $P(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) + Q(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) + R(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) > 0$. Then, by recursive calculation, we conclude that $\lim_{n \rightarrow \infty} \mathbb{E}(|x_n|^2) = 0$ if and only if

$$\begin{aligned}
&P(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) + Q(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) \\
&\quad + R(a_1, a_2, b_1, b_2, \lambda, \mathbb{E}(\gamma)) < 1;
\end{aligned} \tag{66}$$

that is,

$$\begin{aligned}
& \left((b_1^2 + \lambda \mathbb{E}(\gamma^2) + |b_1 b_2|) \left[(1 - \theta)^2 a_1^2 + \lambda^2 \mathbb{E}(\gamma^2) + |a_2|^2 \right. \right. \\
& \quad \left. \left. + 2(1 - \theta) a_1 \lambda \mathbb{E}(\gamma) \right. \right. \\
& \quad \left. \left. + 2|a_2|((1 - \theta)|a_1| + \lambda \mathbb{E}(|\gamma|)) \right] \right. \\
& \quad \left. + \theta^2 a_1^2 (b_2^2 + |b_1 b_2|) \right) \Delta^2 \\
& \quad + \left((1 - 2\theta) a_1^2 + \lambda^2 \mathbb{E}(\gamma^2) \right. \\
& \quad \left. + 2(1 - \theta) a_1 \lambda \mathbb{E}(\gamma) + 2|a_2|((1 - \theta)|a_1| + \lambda \mathbb{E}(|\gamma|)) \right. \\
& \quad \left. + |a_2|^2 + 2(b_1^2 + \lambda \mathbb{E}(\gamma^2) + |b_1 b_2|) \right. \\
& \quad \left. \cdot ((1 - \theta) a_1 + \lambda \mathbb{E}(\gamma) + 2|a_2|) \right. \\
& \quad \left. - 2a_1 \theta (b_2^2 + |b_1 b_2|) \right) \Delta \\
& \quad + 2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| \\
& \quad + b_2^2 + 2|b_1| |b_2| < 0.
\end{aligned} \tag{67}$$

Let

$$\begin{aligned}
f(\Delta) = & A\Delta^2 + B\Delta + 2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) \\
& + 2|a_2| + b_2^2 + 2|b_1| |b_2|.
\end{aligned} \tag{68}$$

Then, we obtain the following conclusions.

(1) If $A = 0$, then (68) can be written as

$$\begin{aligned}
f(\Delta) = & B\Delta + 2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| \\
& + b_2^2 + 2|b_1| |b_2|.
\end{aligned} \tag{69}$$

Furthermore,

- (a) if $B \leq 0$, we have $f(\Delta) < 0$ for all $\Delta > 0$, which means that (66) is true for all $\Delta > 0$;
- (b) if $B > 0$, we have $f(\Delta) < 0$ for any $\Delta < -(2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| + b_2^2 + 2|b_1| |b_2|)/B$. Thus inequality (66) is hold for any

$$\Delta < -\frac{2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| + b_2^2 + 2|b_1| |b_2|}{B}, \tag{70}$$

which implies that (66) holds for such Δ .

(2) If $A < 0$, then we have the following:

- (a) if $B \leq 0$, we have $f(\Delta) < 0$ for all $\Delta > 0$, which means that (66) is true for all $\Delta > 0$;
- (b) if $B > 0$ and $\hat{\Delta} = B^2 - 4A[2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| + b_2^2 + 2|b_1| |b_2|] < 0$, we have $f(\Delta) < 0$ for all $\Delta > 0$, which means that (66) is true for all $\Delta > 0$;

- (c) if $B > 0$ and $\hat{\Delta} = B^2 - 4A[2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) + 2|a_2| + b_2^2 + 2|b_1| |b_2|] \geq 0$, we have $f(\Delta) < 0$ for any $\Delta < (-B - \sqrt{\hat{\Delta}})/2A$.

(3) If $A > 0$, we have

$$\begin{aligned}
\hat{\Delta} = & B^2 - 4A[2a_1 + b_1^2 + 2\lambda \mathbb{E}(\gamma) + \lambda \mathbb{E}(|\gamma|^2) \\
& + 2|a_2| + b_2^2 + 2|b_1| |b_2|] > 0.
\end{aligned} \tag{71}$$

Therefore, we have $f(\Delta) < 0$ for any $\Delta < (-B + \sqrt{\hat{\Delta}})/2A$. $\square$

The above results show that the compensated split-step θ method is of the mean-square stability for the test equations. Furthermore, it is obtained that when the parameters a_1 , a_2 , b_1 , b_2 , and λ make the problem stable, there exists θ such that the method is stable for any time step size Δ , which implies that the proposed numerical method is of the A-stable property.

5. Conclusions

In this paper, A compensated split-step θ method is formulated to numerically solve SDEVDRJMs. It is proved that the numerical solutions converge in mean-square to their analytical solutions with such a rate that is arbitrarily close to 1/2. Furthermore, the stability of the compensated split-step θ method is investigated via a test equation and the results show that the proposed numerical method is of not only mean-square stability but also the A-stable property at least for the test equation. The results demonstrate that the proposed compensated split-step θ method is of satisfactory performance for numerically solving SDEVDRJMs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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