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Research Article

Study on $\mathscr{H}_-$ Index of Stochastic Linear Continuous-Time Systems

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This paper studies the $\mathscr{H}_-$ index problem. We obtain a necessary and sufficient condition of $\mathscr{H}_-$ index larger than $\gamma > 0$. A generalized differential equation is introduced and it is proved that its solvability and the feasibility of the $\mathscr{H}_-$ index are equivalent. We extend the deterministic cases to the stochastic models. Our results can be used to fault detection filter analysis. Finally, the effectiveness of the proposed results is illustrated by an example.

1. Introduction

It is well known that many control and filtering problems have been discussed based on a certain performance index of a system, such as $\mathscr{H}_2$ norm, $\mathscr{H}_\infty$ norm, and $\mathscr{H}_-$ index; see [1–9]. $\mathscr{H}_\infty$ norm is the measure of the worst-case disturbance inputs on the controlled outputs [1–4]. The $\mathscr{H}_-$ index is a measure of the minimum sensitivity of system outputs to system inputs. $\mathscr{H}_\infty$ norm and $\mathscr{H}_-$ index with specific application to fault detection filter have been carried out in [10–17]. To ensure robustness, $\mathscr{H}_-$ index should be maximized and $\mathscr{H}_\infty$ norm should be minimized. Using $\mathscr{H}_-/\mathscr{H}_\infty$ performance can make certain that the residual signal is maximally sensitive to faults and highly robust to disturbance inputs; see [16, 17].

In [12], $\mathscr{H}_-$ index was defined as the minimum nonzero singular value in zero frequency. In [10], the authors extended the results of [12] to all frequency range. By means of LMIs, a necessary and sufficient condition was given for the infinite frequency range. The case for finite frequency range was concluded through frequency weighting. In recent decades, a great deal of attention has been attracted to $\mathscr{H}_-$ index in time domain. A fault residual generator was designed to maximize the fault sensitivity in the finite time domain [16–20]. Based on $\mathscr{H}_-$ index, results on optimal fault detection can be found in [17, 18] and the references. The lower bound of $\mathscr{H}_-$ index for linear time-varying systems was proposed in [19, 20]. A sliding mode observer was designed for sensor fault diagnosis of nonlinear time-delay systems; see [21]. In [22], a fault-tolerant controller was projected to compensate nonlinear faults by using a fuzzy adaptive fault observer.

Although there is much work on the $\mathscr{H}_-$ index problem, to the best of our knowledge, very little work was concerned with the $\mathscr{H}_-$ index in stochastic systems. In this paper, the $\mathscr{H}_-$ index for stochastic linear continuous-time systems is discussed. The definition of the $\mathscr{H}_-$ index is extended to the stochastic case. We present a necessary and sufficient condition of the $\mathscr{H}_-$ index. A generalized differential equation is introduced and it is proved that its solvability and the feasibility of the $\mathscr{H}_-$ index are equivalent. Comparing our results with the bounded real lemma [2, 9], it shows that the $\mathscr{H}_-$ index is not completely dual to $\mathscr{H}_\infty$ norm. The $\mathscr{H}_-$ index discussed in this paper is only for *tall* or *square* systems. The reason for this is that $\mathscr{H}_-$ index is zero for *wide* systems. But bounded real lemma for $\mathscr{H}_\infty$ is applicable to any systems. Finally, the effectiveness of the given methods is illustrated by numerical example.

The outline of the paper is organized as follows. In Section 2, some efficient criteria are given for the $\mathscr{H}_-$ index of stochastic linear systems in finite horizon. Section 3 contains an example provided to show the efficiency of the proposed results. Finally, we conclude this paper in Section 4.

Notations. R is the field of real numbers. $R^{m\times n}$ is the vector space of all $m \times n$ matrices with entries in R. $\mathcal{S}_n(R)$ is the set of all real symmetric matrices $R^{n\times n}$. A' is the transpose of matrix A. A^{-1} is the inverse of A. Given positive semidefinite (positive definite) matrix A, we denote it by $A \geq 0$ ($A > 0$). E is the mathematical expectation. I is identity matrix. 0_n is $n\times n$ zero matrix. $\mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$ is the space of nonanticipative stochastic process $y(t) \in R^p$ with respect to increasing σ-algebras $\mathscr{F}_t$ ($t \geq 0$) satisfying $\|y(t)\|^2_{[0,T]} < \infty$, where $\|y(t)\|^2_{[0,T]} = E\int_0^T y(t)'y(t)dt = E\int_0^T \|y(t)\|^2 dt$. A *square* (*wide* or *tall*) system denotes a system when the number of inputs equals (is more than or less than) the outputs number.

2. Finite Horizon Stochastic $\mathscr{H}_-$ Index

In this section, we will discuss the $\mathscr{H}_-$ index problem of stochastic linear continuous-time systems. We give a necessary and sufficient condition of the $\mathscr{H}_-$ index larger than $\gamma > 0$ in finite horizon.

Consider the following stochastic linear time-varying system $\mathscr{G}$:

$$\begin{aligned} dx(t) &= [A(t)x(t) + B(t)v(t)]\,dt \\ &\quad + [A_0(t)x(t) + B_0(t)v(t)]\,d\omega(t), \\ z(t) &= C(t)x(t) + D(t)v(t), \quad x(0) = x_0. \end{aligned} \tag{1}$$

In the above, $\omega(t)$ is the one-dimensional standard Wiener process defined on the complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$, with the natural filter $\mathscr{F}_t$ generated by $\omega(t)$ up to time t. Consider $x(t) \in R^n$, $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, and $z(t) \in R^l$ are the system state, control input, and regulated output, respectively. $A(t)$, $B(t)$, $A_0(t)$, $B_0(t)$, $C(t)$, and $D(t)$ are coefficients with appropriate dimensions. For any $0 < T < \infty$ and $(v(t), x_0) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p) \times R^n$, there exists unique solution $x(t) = x(t; v, x_0) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^n)$ of (1).

The finite horizon stochastic $\mathscr{H}_-$ index of system (1) can be stated as follows.

Definition 1. For stochastic system (1), given $0 < T < \infty$, its $\mathscr{H}_-$ index in $[0,T]$ is defined as

$$\begin{aligned} \|\mathscr{G}\|_-^{[0,T]} &= \inf_{v\neq 0, x_0=0} \frac{\|z(t)\|_{[0,T]}}{\|v(t)\|_{[0,T]}} \\ &= \inf_{v\neq 0, x_0=0} \frac{\left\{E\int_0^T z(t)'z(t)\,dt\right\}^{1/2}}{\left\{E\int_0^T v(t)'v(t)\,dt\right\}^{1/2}}, \end{aligned} \tag{2}$$

where $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$.

Remark 2. If v is fault signal and z is the residual, then the $\mathscr{H}_-$ index describes the smallest fault sensitivity of system (1). In this paper, we suppose that system (1) is *tall* or *square* because the $\mathscr{H}_-$ index is zero for *wide* system.

Given $\gamma > 0$ and $0 < T < \infty$, let

$$\begin{aligned} J_T^\gamma(x_0, v) &= \|z(t)\|^2_{[0,T]} - \gamma^2 \|v(t)\|^2_{[0,T]} \\ &= E\int_0^T \left[z(t)'z(t) - \gamma^2 v(t)'v(t)\right] dt. \end{aligned} \tag{3}$$

We will study the following optimal problem:

$$\min_{v\in\mathscr{L}^2_{\mathscr{F}}([0,T],R^p)} J_T^\gamma(x_0, v). \tag{4}$$

Remark 3. It can be shown that $\|\mathscr{G}\|_-^{[0,T]} > \gamma$ is equivalent to the following inequality

$$\begin{aligned} J_T^\gamma(0, v) &= \|z(t)\|^2_{[0,T]} - \gamma^2 \|v(t)\|^2_{[0,T]} \\ &= E\int_0^T \left[z(t)'z(t) - \gamma^2 v(t)'v(t)\right] dt > 0, \end{aligned} \tag{5}$$

$\forall v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, $v(t) \neq 0$.

Remark 4. When $T = \infty$, (2) corresponds to the infinite horizon case.

Lemma 5. *Suppose $P(t) : [0,T] \mapsto \mathcal{S}^n(R)$ is continuously differentiable, $T > 0$. Then, for every $x_0 \in R^n$, $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$,*

$$\begin{aligned} J_T^\gamma(x_0, v) &= x_0' P(0) x_0 - E\left[x(T)'P(T)x(T)\right] \\ &\quad + E\int_0^T \left[\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{M}(t, P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right] dt, \end{aligned} \tag{6}$$

where $\mathbb{M}(t, P(t)) = \begin{bmatrix} L(P(t)) + \dot{P}(t) & K(P(t)) \\ K(P(t))' & H^\gamma(P(t)) \end{bmatrix} \in \mathcal{S}^{n+l}(R)$,

$$\begin{aligned} L(P(t)) &= P(t)A(t) + A(t)'P(t) \\ &\quad + A_0(t)'P(t)A_0(t) + C(t)'C(t), \end{aligned}$$

$$K(P(t)) = P(t)B(t) + A_0(t)'P(t)B_0(t) + C(t)'D(t),$$
$$H^\gamma(P(t)) = B_0(t)'P(t)B_0(t) + D(t)'D(t) - \gamma^2 I. \tag{7}$$

Proof. Let $x_0 \in R^n$, $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, and $x(t) = x(t;v,x_0)$ denote the corresponding solution of (1). Applying Ito's formula to $x(t)'P(t)x(t)$ and taking expectations, we have that, for any $T > 0$,

$$\begin{aligned} &E\left[x(T)'P(T)x(T)\right] - \left[x(0)'P(0)x(0)\right] \\ &= E\int_0^T d\left[x(t)'P(t)x(t)\right] \\ &= E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{Q}(t,P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt, \end{aligned} \tag{8}$$

where

$$\mathbb{Q}(t,P(t)) = \begin{bmatrix} P(t)A(t) + A(t)'P(t) + A_0(t)'P(t)A_0(t) + \dot{P}(t) & P(t)B(t) + A_0(t)'P(t)B_0(t) \\ B(t)'P(t) + B_0(t)'P(t)A_0(t) & B_0(t)'P(t)B_0(t) \end{bmatrix}. \tag{9}$$

So

$$\begin{aligned} J_T^\gamma(x_0,v) &= E\int_0^T \left[\|Cx(t) + Dv(t)\|^2 - \gamma^2\|v(t)\|^2\right] dt \\ &\quad + x(0)'P(0)x(0) \\ &\quad - E\left[x(T)'P(T)x(T)\right] \\ &\quad + E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{Q}(t,P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt \\ &= x(0)'P(0)x(0) - E\left[x(T)'P(T)x(T)\right] \\ &\quad + E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{M}(t,P(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt, \end{aligned} \tag{10}$$

which ends the proof. □

Below, we prove the following theorem which is necessary in this paper.

Theorem 6. *For (1) and some given $\gamma > 0$, if the following differential Riccati equation*

$$\begin{aligned} &L(P(t)) + \dot{P}(t) = K(P(t))H^\gamma(P(t))^{-1}K(P(t))', \\ &H^\gamma(P(t)) > 0, \\ &P(T) = 0 \end{aligned} \tag{11}$$

admits solution $P_T(t)$ on $[0,T]$, then $\|\mathscr{G}\|_-^{[0,T]} > \gamma$.

Proof. By Lemma 5, for every $v(t) \in \mathscr{L}^2_{\mathscr{F}}([0,T],R^p)$, $v \neq 0$, $x_0 = 0$, we conclude that

$$J_T^\gamma(0,v) = E\int_0^T \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathbb{M}(t,P_T(t)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} dt. \tag{12}$$

By using completion of squares argument and the first equality in (11), we have

$$\begin{aligned} J_T^\gamma(0,v) &= E\int_0^T x(t)'\left[L(P_T(t)) + \dot{P}_T(t) - K(P_T(t))\right. \\ &\quad \left.\cdot H^\gamma(P_T(t))^{-1}K(P_T(t))'\right]x(t)\,dt \\ &\quad + E\int_0^T \left\{\left[v(t) + H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t)\right]'\right. \\ &\quad \cdot H^\gamma(P_T(t)) \\ &\quad \left.\cdot\left[v(t) + H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t)\right]\right\}dt \\ &= E\int_0^T \left\{\left[v(t) - v^*(t)\right]'H^\gamma(P_T(t))\right. \\ &\quad \left.\cdot\left[v(t) - v^*(t)\right]\right\}dt, \end{aligned} \tag{13}$$

where $v^*(t) = -H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t)$.

From $H^\gamma(P_T(t)) > 0$, $J_T^\gamma(0,v) \geq 0$, to show $J_T^\gamma(0,v) > 0$, we define the operator $\mathscr{L}$: $\mathscr{L}\mathscr{V}(t) = v(t) - v^*(t)$ with its realization:

$$\begin{aligned} dx(t) &= (A(t)x(t) + B(t)v(t))\,dt \\ &\quad + [A_0(t)x(t) + B_0(t)v(t)]\,d\omega(t), \\ x(0) &= 0, \end{aligned} \tag{14}$$

$$v(t) - v^*(t) = v(t) + H^\gamma(P_T(t))^{-1} K(P_T(t))' x(t).$$

Then $\mathscr{L}^{-1}$ exists, which is determined by

$$\begin{aligned} dx(t) &= \Big[A(t) - B(t)H^\gamma(P_T(t))^{-1}\big(B(t)'P_T(t) \\ &\quad + B_0(t)'P_T(t)A_0(t) + D(t)'C(t)\big)\Big]x(t)\,dt \\ &\quad + \Big[A_0(t) - B_0(t)H^\gamma(P_T(t))^{-1}\big(B(t)'P_T(t) \\ &\quad + B_0'(t)P_T(t)A_0(t) + D(t)'C(t)\big)\Big]x(t)\,d\omega(t) \\ &\quad + B(t)(v(t) - v^*(t))\,dt + B_0(t)(v(t) - v^*(t))\,d\omega(t), \\ x_0 &= 0, \end{aligned} \tag{15}$$

where $v(t) = -H^\gamma(P_T(t))^{-1}K(P_T(t))'x(t) + (v(t) - v^*(t))$.

We assume that $H^\gamma(P_T(t)) \geq \epsilon I$, $\epsilon > 0$, so there exists constant $C_0 > 0$, such that

$$\begin{aligned} &J_T^\gamma(0,v) \\ &\quad = E\int_0^T [v(t) - v^*(t)]' H^\gamma(P_T(t))[v(t) - v^*(t)]\,dt \\ &\quad \geq \epsilon \|v(t) - v^*(t)\|_{[0,T]}^2 = \epsilon \|\mathscr{L}\mathscr{V}(t)\|_{[0,T]}^2 \\ &\quad \geq C_0 \|v(t)\|_{[0,T]}^2 > 0. \end{aligned} \tag{16}$$

That is, $\|\mathscr{G}\|_-^{[0,T]} > \gamma$.

Now, we consider the following equation:

$$\begin{aligned} &\dot{X}(t) + L(X(t)) + K(X(t))F(t) + F(t)'K(X(t))' \\ &\quad + F(t)'H^\gamma(X(t))F(t) = 0, \quad t \in [0,T], \\ &X(T) = 0, \end{aligned} \tag{17}$$

where $F(t) \in C[0,T]$ and this equation has unique solution $X(t) = P_F^\gamma(t)$, $t \in [0,T]$.

It is easy to see that (17) satisfies the following equation:

$$\begin{aligned} &\dot{P}_F^\gamma(t) + \begin{bmatrix} I \\ F(t) \end{bmatrix}' \begin{bmatrix} L(P_F^\gamma(t)) & K(P_F^\gamma(t)) \\ K(P_F^\gamma(t))' & H^\gamma(P_F^\gamma(t)) \end{bmatrix} \begin{bmatrix} I \\ F(t) \end{bmatrix} \\ &\quad = 0, \quad t \in [0,T], \\ &P_F^\gamma(T) = 0. \end{aligned} \tag{18}$$

□

Lemma 7. *Suppose $F(t) \in C[0,T]$ and $P_F^\gamma(t)$ is the solution of (18). Then if $v(t) \in \mathscr{L}_{\mathscr{F}}^2([0,T],R^p)$, one obtains*

$$\begin{aligned} J_T^\gamma(x_0, v + Fx_F) &= x_0'P_F^\gamma(0)x_0 \\ &\quad + E\int_0^T \Big[v(t)'G(t)x_F(t) + x_F(t)'G(t)'v(t) \\ &\quad + v(t)'H^\gamma(P_F^\gamma(t))v(t)\Big]\,dt, \end{aligned} \tag{19}$$

where $x_F(t) = x(t, F(t)x_F(t) + v(t), x_0)$ is the solution of

$$\begin{aligned} dx_F(t) &= (A(t) + B(t)F(t))x_F(t)\,dt \\ &\quad + (A_0(t) + B_0(t)F(t))x_F(t)\,d\omega(t) \\ &\quad + B_0(t)v(t)\,d\omega(t) + B(t)v(t)\,dt \end{aligned} \tag{20}$$

with $x_F(0) = x_0$ and

$$G(t) = K(P_F^\gamma(t))' + H^\gamma(P_F^\gamma(t))F(t). \tag{21}$$

As $v = 0$, then

$$J_T^\gamma(x_0, Fx_F) = x_0'P_F^\gamma(0)x_0. \tag{22}$$

Proof. In terms of Lemma 5 with $P(t) = P_F^\gamma(t)$ and $F(t)x_F(t) + v(t)$ for $v(t)$,

$$\begin{aligned} J_T^\gamma(x_0, v + Fx_F) &= x_0'P_F^\gamma(0)x_0 \\ &\quad + E\int_0^T \left\{\begin{bmatrix} x_F(t) \\ F(t)x_F(t) + v(t) \end{bmatrix}' \mathbb{M}(t, P_F^\gamma(t)) \right. \\ &\quad \left. \cdot \begin{bmatrix} x_F(t) \\ F(t)x_F(t) + v(t) \end{bmatrix}\right\} dt = x_0'P_F^\gamma(0)x_0 \\ &\quad + E\int_0^T \left\{x_F(t)' \begin{bmatrix} I \\ F(t) \end{bmatrix}' \right. \\ &\quad \left. \cdot \begin{bmatrix} L(P_F^\gamma(t)) + \dot{P}_F^\gamma(t) & K(P_F^\gamma(t)) \\ K(P_F^\gamma(t))' & H^\gamma(P_F^\gamma(t)) \end{bmatrix} \begin{bmatrix} I \\ F(t) \end{bmatrix} x_F(t)\right\} dt \end{aligned}$$

$$+E\int_0^T \left\{v(t)'G(t)x_F(t)+x_F(t)'G(t)'v(t)\right.$$
$$\left.+v(t)'H^\gamma(P_F^\gamma(t))v(t)\right\}dt = x_0'P_F^\gamma(0)x_0$$
$$+E\int_0^T \left\{v(t)'G(t)x_F(t)+x_F(t)'G(t)'v(t)\right.$$
$$\left.+v(t)'H^\gamma(P_F^\gamma(t))v(t)\right\}dt. \tag{23}$$

This means that (19) holds. Let $v = 0$ in (19); we obtain (22).

Now we are in a position to prove that $H^\gamma(P_F^\gamma(t))$ is invertible for $t \in [0,T]$. □

Lemma 8. *For system (1), if $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ for some given $\gamma > 0$, $F(t) \in C[0,T]$, $T > 0$, and $P_F^\gamma(t)$ satisfies (18). Then,*

$$H^\gamma(P_F^\gamma(t)) \geq \left[\left(\|\mathcal{G}\|_-^{[0,T]}\right)^2 - \gamma^2\right] I > 0, \quad t \in [0,T]. \tag{24}$$

Proof. Let us first prove that $H^\gamma(P_F^\gamma(t)) \geq 0$. Suppose this is false; then there exists $t^* \in [0,T)$, $u \in R^l$, $\|u\| = 1$ such that $u'H^\gamma(P_F^\gamma(t^*))u \leq -\eta$ for some $\eta > 0$. Then, for sufficiently small $\delta > 0$,

$$u'H^\gamma(P_F^\gamma(t))u \leq -\frac{\eta}{2}, \quad t \in [t^*, t^*+\delta] \subset [0,T]. \tag{25}$$

Define

$$v(t) = \begin{cases} 0, & t \in [0,t^*) \cup (t^*+\delta, T], \\ u, & t \in [t^*, t^*+\delta]. \end{cases} \tag{26}$$

Using Lemma 7 with this $v(t)$ and $x_0 = 0$, we can derive that $x_F(t) = 0$ for $t \in [0,t^*]$ and

$$\begin{aligned} J_T^\gamma(0,v) &= E\int_0^T \left[\|C(t)x_F(t)+D(t)v(t)\|^2\right. \\ &\quad \left.-\gamma^2\|v(t)\|^2\right]dt = E\int_0^T \left[v(t)'G(t)x_F(t)\right. \\ &\quad \left.+x_F(t)'G(t)'v(t)+v(t)'H^\gamma(P_F^\gamma(t))v(t)\right]dt \\ &\leq E\int_{t^*}^{t^*+\delta}\left(2\|G(t)'u\|\|x_F(t)\|-\frac{\eta}{2}\right)dt. \end{aligned} \tag{27}$$

Since $x_F(t)$ is continuous and $x_F(t^*) = 0$, (27) is negative. Moreover, the condition $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ implies $J_T^\gamma(0,v) \geq 0$. As a result, this is a contradiction. If $t^* = T$, we can replace $[t^*, t^*+\delta]$ by $[T-\delta, T]$ and use a similar proof.

Next, let $\|\mathcal{G}\|_-^{[0,T]} > (\gamma^2+\rho^2)^{1/2}$ for any $\rho > 0$ and $\lambda = (\gamma^2+\rho^2)^{1/2}$. Replacing γ with λ in (18), we obtain the corresponding solution $P_F^\lambda(t)$. Applying the previous step, we can deduce that $H^\lambda(P_F^\lambda(t)) \geq 0$. For any $t_0 \in [0,T)$, set $F_{t_0} = F(t+t_0)$, $t \in [0, T-t_0]$. Let $P_{F_{t_0}}^\lambda(t)$ be the solution of (18) with γ replaced by λ and F replaced by F_{t_0} on $[0, T-t_0]$. Then, $P_{F_{t_0}}^\lambda(t) = P_F^\lambda(t+t_0)$, $t \in [0, T-t_0]$. By (22), for any $t_0 \in [0,T)$, $x_0 \in R^n$,

$$\begin{aligned} x_0'P_F^\lambda(t_0)x_0 &= x_0'P_{F_{t_0}}^\lambda(0)x_0 = J_{T-t_0}^\lambda\left(x_0, F_{t_0}x_{F_{t_0}}\right) \\ &\leq J_{T-t_0}^\gamma\left(x_0, F_{t_0}x_{F_{t_0}}\right) = x_0'P_F^\gamma(t_0)x_0, \end{aligned} \tag{28}$$

and so $H^\lambda(P_F^\gamma(t_0)) \geq H^\lambda(P_F^\lambda(t_0)) \geq 0$. By continuity, $H^\gamma(P_F^\gamma(t)) \geq \rho^2 I$ for all $t \in [0,T]$. As this holds for arbitrary $\rho^2 < (\|\mathcal{G}\|_-^{[0,T]})^2 - \gamma^2$, it follows that $H^\gamma(P_F^\gamma(t)) \geq [(\|\mathcal{G}\|_-^{[0,T]})^2 - \gamma^2]I > 0$. This completes the proof. □

Remark 9. When $t = T$, (24) becomes $H^\gamma(P_F^\gamma(T)) = D(T)'D(T) - \gamma^2 I > 0$. If system (1) is time-invariant, then

$$D'D - \gamma^2 I > 0. \tag{29}$$

Remark 10. By the equality $A(I-BA)^{-1} = (I-AB)^{-1}A$, we have that $C'[I - D(D'D-\gamma^2 I)^{-1}D']C = C'(I-\gamma^{-2}DD')^{-1}C$. If system (1) is time-invariant and square, by (29),

$$\begin{aligned} &C'\left[I - D\left(D'D-\gamma^2 I\right)^{-1}D'\right]C \\ &\quad = C'\left(I-\gamma^{-2}DD'\right)^{-1}C \leq 0. \end{aligned} \tag{30}$$

Now, we present the following theorem which is important in this paper.

Theorem 11. *Suppose system (1) is time-invariant and square and satisfies $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ for given $\gamma \leq 0$. Then (11) has a unique solution $P_T(t) \leq 0$ on $[0,T]$ for every $T > 0$. Moreover, $J_T^\gamma(x_0, v)$ is minimized by the feedback control:*

$$\begin{aligned} v^*(t) &= F_T(t)x_{F_T}(t), \\ F_T(t) &= -H^\gamma(P_T(t))^{-1}K(P_T(t))', \end{aligned} \tag{31}$$

where $x_{F_T}(t)$ satisfies

$$\begin{aligned} dx_{F_T}(t) &= \left(A+BF_T(t)\right)x_{F_T}(t)\,dt \\ &\quad +\left[A_0+B_0F_T(t)\right]x_{F_T}(t)\,dw(t), \\ x_{F_T}(0) &= x_0 \end{aligned} \tag{32}$$

and the optimal cost is

$$\min_{v\in\mathcal{L}_{\mathcal{F}}^2([0,T],R^p)} J_T^\gamma(x_0,v) = x_0'P_T(0)x_0. \tag{33}$$

Proof. We prove that $\|\mathcal{G}\|_{-}^{[0,T]} > \gamma$ implies the existence of solution $P_T(t)$ of (11) on $[0,T]$. Using a contradiction argument, we suppose that (11) does not admit a solution. By the standard theory of differential equations, there exists unique solution $P_T(t)$ backward in time on maximal interval $[T_0,T]$ $(T_0 \geq 0)$, and as $t \to T_0$, $P_T(t)$ becomes unbounded.

Let $0 < \delta < T - T_0$, $x(T_0+\delta) = x_{T_0,\delta} \in R^n$, $Q(t) = B'P_T(t) + B_0'P_T(t)A_0 + D'C$, $R(t) = B_0'P_T(t)B_0 + D'D - \gamma^2 I$, by completing the squares; then

$$\begin{aligned}
J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right) &= E\int_{T_0+\delta}^{T}\left[z(t)'z(t)\right.\\
&\left.-\gamma^2 v(t)'v(t)\right]dt = E\int_{T_0+\delta}^{T}\left[z(t)'z(t)\right.\\
&\left.-\gamma^2 v(t)'v(t)\right]dt + E\int_{T_0+\delta}^{T} d\left(x(t)'P_T(t)x(t)\right)dt\\
&+ x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta} = E\int_{T_0+\delta}^{T} x(t)'\left[C'C\right.\\
&+ A'P_T(t) + P_T(t)A + A_0'P_T(t)A_0 + \dot{P}_T(t)\\
&\left.-Q(t)'R(t)^{-1}Q(t)\right]x(t)\,dt + E\int_{T_0+\delta}^{T}\left[v(t)\right.\\
&\left.+R(t)^{-1}Q(t)x(t)\right]'R(t)\left[v(t)\right.\\
&\left.+R(t)^{-1}Q(t)x(t)\right]dt + x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta}\\
&= E\int_{T_0+\delta}^{T}\left[v(t)+R(t)^{-1}Q(t)x(t)\right]'R(t)\left[v(t)\right.\\
&\left.+R(t)^{-1}Q(t)x(t)\right]dt + x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta}.
\end{aligned} \tag{34}$$

Obviously,

$$\begin{aligned}
&\min_{v\in\mathcal{L}_{\mathcal{F}}^2([T_0+\delta,T],R^p)} J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right)\\
&\quad = J^\gamma\left(x,v^*,x_{T_0,\delta},T_0+\delta\right) = x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta},
\end{aligned} \tag{35}$$

where $v^*(t) = -R(t)^{-1}Q(t)x(t)$.

Furthermore, we can see that

$$\begin{aligned}
J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right) &= E\int_{T_0+\delta}^{T}\left[z(t)'z(t)\right.\\
&\left.-\gamma^2 v(t)'v(t)\right]dt\\
&= E\int_{T_0+\delta}^{T}\left\{[Cx(t)+Dv(t)]'[Cx(t)+Dv(t)]\right.\\
&\left.-\gamma^2 v(t)'v(t)\right\}dt = E\int_{T_0+\delta}^{T} x(t)'C'\left[I\right.\\
&\left.-D\left(D'D-\gamma^2 I\right)^{-1}D'\right]Cx(t)\,dt + E\int_{T_0+\delta}^{T}\left[v(t)\right.\\
&+\left(D'D-\gamma^2 I\right)^{-1}D'Cx(t)\Big]'\left(D'D-\gamma^2 I\right)\left[v(t)\right.\\
&\left.+\left(D'D-\gamma^2 I\right)^{-1}D'Cx(t)\right]dt.
\end{aligned} \tag{36}$$

From Remark 10,

$$\begin{aligned}
J^\gamma\left(x,\tilde{v},x_{T_0,\delta},T_0+\delta\right) &= E\int_{T_0+\delta}^{T} x(t)'\\
&\cdot C'\left[I-D\left(D'D-\gamma^2 I\right)^{-1}D'\right]Cx(t)\,dt \leq 0,
\end{aligned} \tag{37}$$

where $\tilde{v}(t) = -(D'D-\gamma^2 I)^{-1}D'Cx(t)$. Considering (35), we get

$$x_{T_0,\delta}'P_T(T_0+\delta)x_{T_0,\delta} \leq J^\gamma\left(x,\tilde{v},x_{T_0,\delta},T_0+\delta\right) \leq 0, \tag{38}$$

which implies that

$$P_T(T_0+\delta) \leq 0. \tag{39}$$

By linearity, the solution of (1) with initial state $x_{T_0,\delta}$ satisfies

$$\begin{aligned}
x\left(t,v,x_{T_0,\delta},T_0+\delta\right) &= x\left(t,0,x_{T_0,\delta},T_0+\delta\right)\\
&+ x\left(t,v,0,T_0+\delta\right).
\end{aligned} \tag{40}$$

Suppose $\Phi(t)$ is the solution of

$$\begin{aligned}
C'C + A'\Phi(t) + \Phi(t)A + A_0'\Phi(t)A_0 + \dot{\Phi}(t) &= 0,\\
\Phi(T) &= 0,
\end{aligned} \tag{41}$$

and we have

$$\begin{aligned}
&J^\gamma\left(x,v,x_{T_0,\delta},T_0+\delta\right) - J^\gamma\left(x,v,0,T_0+\delta\right)\\
&\quad = x_{T_0,\delta}'\Phi(T_0+\delta)x_{T_0,\delta} + E\int_{T_0+\delta}^{T}\left[v(t)'\right.\\
&\quad\left.\cdot\left(B_0'\Phi(t)+D'C\right)x\left(t,0,x_{T_0,\delta},T_0+\delta\right)\right]dt\\
&\quad + E\int_{T_0+\delta}^{T}\left[x\left(t,0,x_{T_0,\delta},T_0+\delta\right)'\left(B_0'\Phi(t)+D'C\right)'\right.\\
&\quad\left.\cdot v(t)\right]dt.
\end{aligned} \tag{42}$$

Take $\|\mathcal{G}\|_{-}^{[0,T]} \geq (\gamma^2\epsilon^2)^{1/2}$,

$$\bar{v}(t) = \begin{cases} \bar{v}(t) = -\left(D'D-\gamma^2 I\right)^{-1} D'Cx(t), & t \in [0, T_0+\delta] \\ v(t), & t \in (T_0+\delta, T], \end{cases} \tag{43}$$

and it is easy to show that

$$\begin{aligned} J^{\gamma}(x, v, 0, T_0+\delta) &= E\int_0^T \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt \\ &- E\int_0^{T_0+\delta} \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt \\ &= E\int_0^T \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt - E\int_0^{T_0+\delta} x(t)' \\ &\cdot C'\left[I - D\left(D'D-\gamma^2 I\right)^{-1} D'\right] Cx(t)\, dt \\ &- E\int_0^{T_0+\delta} \left[\bar{v}(t) + \left(D'D-\gamma^2 I\right)^{-1} D'Cx(t)\right]' \\ &\cdot \left(D'D-\gamma^2 I\right) \\ &\cdot \left[\bar{v}(t) + \left(D'D-\gamma^2 I\right)^{-1} D'Cx(t)\right] dt \\ &\geq E\int_0^T \left(\|z(t)\|^2 - \gamma^2\|\bar{v}(t)\|^2\right) dt \geq \epsilon^2\|\bar{v}(t)\|_{[0,T]}^2 \\ &\geq \epsilon^2\|v(t)\|_{[T_0+\delta,T]}^2. \end{aligned} \tag{44}$$

It follows that

$$\begin{aligned} J^{\gamma}\left(x, v, x_{T_0,\delta}, T_0+\delta\right) &\geq E\int_{T_0+\delta}^T \epsilon^2\|v(t)\|^2 dt \\ &+ x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} + E\int_{T_0+\delta}^T \left[v(t)'\left(B_0'\Phi(t)\right.\right. \\ &\left.\left.+ D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right] dt \\ &+ E\int_{T_0+\delta}^T \left[x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)'\left(B_0'\Phi(t) + D'C\right)'\right. \\ &\left.\cdot v(t)\right] dt = x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} + E\int_{T_0+\delta}^T \left\|\epsilon\left[v\right.\right. \\ &\left.\left.- \epsilon^{-2}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right]\right\|^2 dt \\ &- E\int_{T_0+\delta}^T \left\|\epsilon^{-1}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0\right.\right. \\ &\left.\left.+\delta\right)\right\|^2 dt \geq x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} \\ &- E\int_{T_0+\delta}^T \left\|\epsilon^{-1}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0\right.\right. \\ &\left.\left.+\delta\right)\right\|^2 dt. \end{aligned} \tag{45}$$

It is obvious that there exists constant $C_0 > 0$ such that

$$C_2\left\|x_{T_0,\delta}\right\|^2 \geq E\int_{T_0+\delta}^T \left\|x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right\|^2 dt. \tag{46}$$

So, there is constant $C_1 > 0$ such that

$$\begin{aligned} &E\int_{T_0+\delta}^T \left\|\epsilon^{-1}\left(B_0'\Phi(t) + D'C\right) x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right)\right\|^2 dt \\ &\leq C_1\left\|x_{T_0,\delta}\right\|^2. \end{aligned} \tag{47}$$

In addition,

$$\begin{aligned} &x'_{T_0,\delta}\Phi(T_0+\delta)\, x_{T_0,\delta} \\ &= -E\int_{T_0+\delta}^T d\left(x(t)'\Phi(t)\, x(t)\right) \\ &= E\int_{T_0+\delta}^T \left\|x\left(t, 0, x_{T_0,\delta}, T_0+\delta\right) C\right\|^2 dt \geq 0. \end{aligned} \tag{48}$$

From (45), we have

$$J^{\gamma}\left(x, v, x_{T_0,\delta}, T_0+\delta\right) \geq -C_1\left\|x_{T_0,\delta}\right\|^2. \tag{49}$$

In view of (35) and (39), it yields

$$-C_1 \leq P_T(T_0+\delta) \leq 0. \tag{50}$$

So, $P_T(T_0+\delta)$ can not become unbounded as $\delta \to 0$, which means that (11) has unique solution $P_T(t)$ on $[0, T]$.

Setting $F(t) = F_T(t)$, $t \in [0, T]$, in (17), from (31), we obtain

$$\begin{aligned} &\dot{P}_T(t) + L\left(P_T(t)\right) + K\left(P_T(t)\right) F(t) \\ &\quad + F(t)' K\left(P_T(t)\right)' + F(t)' H^{\gamma}\left(P_T(t)\right) F(t) = 0. \end{aligned} \tag{51}$$

Hence $P_T(t)$ satisfies (17), or equivalently (18). So

$$P_{F_T}^{\gamma}(t) = P_T(t), \quad t \in [0, T]. \tag{52}$$

By (31),

$$G(t) = K\left(P_T(t)\right)' + H^{\gamma}\left(P_T(t)\right) F_T(t) = 0, \tag{53}$$

and, in terms of Lemma 7,

$$J_T^\gamma(x_0, v + F_T x) = x_0' P_T(0) x_0 + E\int_0^T \left[v(t)' H^\gamma(P_T(t)) v(t)\right] dt. \tag{54}$$

But by Lemma 8,

$$H^\gamma(P_T(t)) = H^\gamma\left(P_{F_T}^\gamma(t)\right) \succeq \left[\left(\|\mathcal{G}\|_-^{[0,T]}\right)^2 - \gamma^2\right] I \succ 0, \quad t \in [0,T]. \tag{55}$$

Hence, $v^*(t) = F_T(t)x(t)$ minimizes $J_T^\gamma(x_0, v)$ and $\min_{v\in\mathcal{L}^2_{\mathcal{F}}([0,T],R^p)} J_T^\gamma(x_0, v) = x_0' P_T(0) x_0$. □

According to Theorems 6 and 11, we get the following theorem.

Theorem 12. *If system (1) is time-invariant and square, for given $\gamma > 0$, the following are equivalent:*

(i) *Consider $\|\mathcal{G}\|_-^{[0,T]} > \gamma$.*

(ii) *The following equation*

$$\begin{aligned} &P(t)A + A'P(t) + A_0'P(t)A_0 + C'C + \dot{P}(t) \\ &\quad = \left(P(t)B + A_0'P(t)B_0 + C'D\right) \\ &\quad \cdot \left(B_0'P(t)B_0 + D'D - \gamma^2 I\right)^{-1} \\ &\quad \cdot \left(P(t)B + A_0'P(t)B_0 + C'D\right)', \\ &B_0'P(t)B_0 + D'D - \gamma^2 I > 0, \\ &P(T) = 0 \end{aligned} \tag{56}$$

has unique solution $P_T(t) \le 0$ on $[0,T]$. Moreover, $\min_{v\in\mathcal{L}^2_{\mathcal{F}}([0,T],R^p)} J_T^\gamma(x_0, v) = x_0' P_T(0) x_0$.

Remark 13. For given $\gamma > 0$, if we replace B, C, D, and $v(t)$ with $B_\delta = [B \;\; 0_{n\times n}]$, $C_\delta = [C \;\; \delta I_n]'$, $D_\delta = \begin{bmatrix} D & 0_{l\times n} \\ 0_{n\times l} & 0_{n\times n}\end{bmatrix}$, and $v_\delta(t) = [v(t) \;\; 0_{n\times n}]'$, respectively, and $z(t)$ with $z_\delta(t)$ in (1), we deduce the corresponding $\mathcal{H}_-$ index $\|\mathcal{G}\|_{\delta-}^{[0,T]}$ and

$$\begin{aligned} J_{T,\delta}^\gamma(x_0, v) &= E\int_0^T \left\{\|z_\delta(t)\|^2 - \gamma^2 \|v_\delta(t)\|^2\right\} dt \\ &= E\int_0^T \left\{\|z(t)\|^2 - \gamma^2 \|v(t)\|^2 + \delta^2 I\right\} dt. \end{aligned} \tag{57}$$

When $\|\mathcal{G}\|_-^{[0,T]} > \gamma$, then $\|\mathcal{G}\|_{\delta-}^{[0,T]} > \gamma$. Using Theorem 12 to the modified data, it is easy to see that the following equation

$$\begin{aligned} &P(t)A + A'P(t) + A_0'P(t)A_0 + C'C + \delta^2 I + \dot{P}(t) \\ &\quad = \left(P(t)B + A_0'P(t)B_0 + C'D\right) \\ &\quad \cdot \left(B_0'P(t)B_0 + D'D - \gamma^2 I\right)^{-1} \\ &\quad \cdot \left(P(t)B + A_0'P(t)B_0 + C'D\right)', \\ &B_0'P(t)B_0 + D'D - \gamma^2 I > 0, \\ &P(T) = 0 \end{aligned} \tag{58}$$

has unique solution $P_{\delta,T}(t) \le 0$ on $[0,T]$. Moreover, $\min_{v\in\mathcal{L}^2_{\mathcal{F}}([0,T],R^p)} J_{T,\delta}^\gamma(x_0, v) = x_0' P_{\delta,T}(0) x_0$.

Now, we are to show what happens as T increases.

Theorem 14. *If system (1) is time-invariant and square, $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ for some $\gamma > 0$. Then $P_T(t)$ in (56) decreases as T increases for every $t \in [0,T]$.*

Proof. Suppose $\overline{T} > T$, $t \in [0,T]$, and $x_0 \in R^n$. Let v_{T-t}^* be optimal for x_0 on $[0, T-t]$, and set

$$v(\tau) = \begin{cases} v_{T-t}^*(\tau), & \tau \in [0, T-t] \\ -\left(D'D - \gamma^2 I\right)^{-1} D'Cx(\tau), & \tau \in \left(T-t, \overline{T}-t\right]. \end{cases} \tag{59}$$

By the time invariance of $P_T(t)$, $P_{T-t}(0) = P_T(t)$. Then,

$$\begin{aligned} x_0' P_{\overline{T}}(t) x_0 &= x_0' P_{\overline{T}-t}(0) x_0 \le J_{\overline{T}-t}^\gamma(x_0, v) \\ &= J_{T-t}^\gamma(x_0, v_{T-t}^*) + E\int_{T-t}^{\overline{T}-t} \left\{\|z(\tau)\|^2 - \gamma^2 \|v(\tau)\|^2\right\} d\tau \\ &= J_{T-t}^\gamma(x_0, v_{T-t}^*) + E\int_{T-t}^{\overline{T}-t} \Big\{x'(\tau) \\ &\quad \cdot C'\left[I - D\left(D'D - \gamma^2 I\right)^{-1} D'\right] Cx(\tau)\Big\} d\tau \\ &\quad + E\int_{T-t}^{\overline{T}-t} \Big\{\left[v(\tau) + \left(D'D - \gamma^2 I\right)^{-1} D'Cx(\tau)\right]' \\ &\quad \cdot \left(D'D - \gamma^2 I\right)\left[v(\tau) + \left(D'D - \gamma^2 I\right)^{-1} D'Cx(\tau)\right]\Big\} d\tau \end{aligned}$$

$$= J_{T-t}^{\gamma}\left(x_0, v_{T-t}^{*}\right) + E\int_{T-t}^{\overline{T}-t}\left\{x'(\tau)\right.$$

$$\left.\cdot C'\left[I - D\left(D'D - \gamma^2 I\right)^{-1} D'\right] Cx(\tau)\right\} d\tau$$

$$\leq J_{T-t}^{\gamma}\left(x_0, v_{T-t}^{*}\right) = x_0' P_T(t)\, x_0. \tag{60}$$

This means that $P_T(t)$ decreases as T increases for every $t \in [0, T]$. □

3. A Numerical Example

Below, we give a numerical example to illustrate the rightness of Theorems 12 and 14.

Example 1. In system (1), we consider a two-dimensional linear stochastic system with the following parameters:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},$$
$$B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{61}$$
$$A_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Set $\gamma = 0.5$, $T = 2, 3$; by solving (56), we can obtain the solutions of

$$P_2(t) = \begin{bmatrix} p_2^{11}(t) & p_2^{12}(t) \\ p_2^{12}(t) & p_2^{22}(t) \end{bmatrix},$$
$$P_3(t) = \begin{bmatrix} p_3^{11}(t) & p_3^{12}(t) \\ p_3^{12}(t) & p_3^{22}(t) \end{bmatrix}, \tag{62}$$

for which their trajectories are shown in Figure 1. If we set $t = 1$, then it yields

$$P_2(1) = \begin{bmatrix} -0.0892 & -0.0953 \\ -0.0953 & -0.1446 \end{bmatrix},$$
$$P_3(1) = \begin{bmatrix} -0.1044 & -0.0889 \\ -0.0889 & -0.1489 \end{bmatrix}. \tag{63}$$

It is easy to see that $P_2(1) > P_3(1)$, which verifies the rightness of Theorem 14.

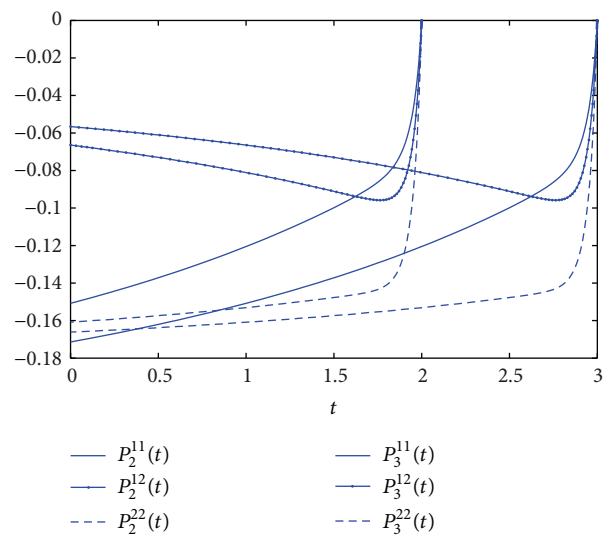

Figure 1: The trajectories of $P_2(t)$ and $P_3(t)$.

4. Conclusion

In this paper, we have solved the $\mathscr{H}_-$ index problem where both stochastic and deterministic perturbations are present. Necessary and sufficient condition for the lower bound of $\mathscr{H}_-$ index is given by means of the solvability of a generalized differential equation. The proposed results are not completely dual to $\mathscr{H}_\infty$ norm, and the effectiveness of the given methods is illustrated by numerical example.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] K. Zhou, J. C. Doyle, and K. Clover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, USA, 1996.

[2] B.-S. Chen and W. Zhang, "Stochastic $\mathscr{H}_2/\mathscr{H}_\infty$ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[3] W. H. Zhang and B.-S. Chen, "State feedback $\mathscr{H}_\infty$ control for a class of nonlinear stochastic systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 1973–1991, 2006.

[4] W. Zhang, H. Zhang, and B.-S. Chen, "Generalized Lyapunov equation approach to state-dependent stochastic stabilization/detectability criterion," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1630–1642, 2008.

[5] Q. X. Zhu, "Asymptotic stability in the pth moment for stochastic differential equations with Lévy noise," *Journal of*

Mathematical Analysis and Applications, vol. 416, no. 1, pp. 126–142, 2014.

[6] Q. Zhu and J. Cao, "Stability of Markovian jump neural networks with impulse control and time varying delays," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 5, pp. 2259–2270, 2012.

[7] Q. X. Zhu, J. D. Cao, and R. Rakkiyappan, "Exponential input-to-state stability of stochastic Cohen-Grossberg neural networks with mixed delays," *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1085–1098, 2015.

[8] Q. X. Zhu and J. D. Cao, "Mean-square exponential input-to-state stability of stochastic delayed neural networks," *Neurocomputing*, vol. 131, pp. 157–163, 2014.

[9] D. Hinrichsen and A. J. Pritchard, "Stochastic $\mathscr{H}_\infty$," *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pp. 1504–1538, 1998.

[10] J. Liu, J. L. Wang, and G.-H. Yang, "An LMI approach to minimum sensitivity analysis with application to fault detection," *Automatica*, vol. 41, no. 11, pp. 1995–2004, 2005.

[11] J. L. Wang, G. H. Yang, and J. Liu, "An LMI approach to $\mathscr{H}_-$ index and mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design," *Automatica*, vol. 43, pp. 1656–1665, 2007.

[12] J. Chen and R. J. Pattor, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, Boston, Mass, USA, 1999.

[13] I. Hwang, S. Kim, Y. Kim, and C. E. Seah, "A survey of fault detection, isolation, and reconfiguration methods," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 3, pp. 636–653, 2010.

[14] S. Yin, S. X. Ding, A. Haghani, H. Hao, and P. Zhang, "A comparison study of basic data-driven fault diagnosis and process monitoring methods on the benchmark Tennessee Eastman process," *Journal of Process Control*, vol. 22, no. 9, pp. 1567–1581, 2012.

[15] D. Wang, P. Shi, and W. Wang, "Robust fault detection for continuous-time switched delay systems: an linear matrix inequality approach," *IET Control Theory and Applications*, vol. 4, no. 1, pp. 100–108, 2010.

[16] S. Yin, H. Luo, and S. X. Ding, "Real-time implementation of fault-tolerant control systems with performance optimization," *IEEE Transactions on Industrial Electronics*, vol. 61, no. 5, pp. 2402–2411, 2014.

[17] K. Iftikha, P. Q. Khan, and M. Abid, "Optimal fault detection filter design for switched linear systems," *Nonlinear Analysis*, vol. 15, pp. 132–144, 2015.

[18] M. Zhong, S. X. Ding, and E. L. Ding, "Optimal fault detection for linear discrete time-varying systems," *Automatica*, vol. 46, no. 8, pp. 1395–1400, 2010.

[19] X. B. Li and H. H. T. Liu, "Characterization of $\mathscr{H}_-$ index for linear time-varying systems," *Automatica*, vol. 49, no. 3, pp. 1449–1457, 2013.

[20] X. Li and H. H. T. Liu, "Minimum system sensitivity study of linear discrete time systems for fault detection," *Mathematical Problems in Engineering*, vol. 2013, Article ID 276987, 13 pages, 2013.

[21] M. Chen, C.-S. Jiang, and Q.-X. Wu, "Sensor fault diagnosis for a class of time delay uncertain nonlinear systems using neural network," *International Journal of Automation and Computing*, vol. 5, no. 4, pp. 401–405, 2008.

[22] K. Zhang, B. Jiang, and P. Shi, "A new approach to observer-based fault-tolerant controller design for Takagi-Sugeno fuzzy systems with state delay," *Circuits, Systems, and Signal Processing*, vol. 28, no. 5, pp. 679–697, 2009.