Research Article

Attractor of Beam Equation with Structural Damping under Nonlinear Boundary Conditions

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Simultaneously, considering the viscous effect of material, damping of medium, and rotational inertia, we study a kind of more general Kirchhoff-type extensible beam equation $u_{tt} - u_{xxtt} + u_{xxxx} - \sigma(\int_0^L (u_x)^2 dx)u_{xx} - \phi(\int_0^L (u_x)^2 dx)u_{xxt} = q(x)$, in $[0, L] \times \mathbb{R}^+$ with the structural damping and the rotational inertia term. Little attention is paid to the longtime behavior of the beam equation under nonlinear boundary conditions. In this paper, under nonlinear boundary conditions, we prove not only the existence and uniqueness of global solutions by prior estimates combined with some inequality skills, but also the existence of a global attractor by the existence of an absorbing set and asymptotic compactness of corresponding solution semigroup. In addition, the same results also can be proved under the other nonlinear boundary conditions.

1. Introduction

In this paper, we will consider a kind of more general Kirchhoff-type beam equation. The physical origin of the problem lies in the theory of vibrations of an extensible beam of length L ; moreover, during vibration, the elements of a beam not only perform a translatory motion but also rotate.

A mathematical model for this problem is an initial boundary value problem for the nonlinear Kirchhoff-type beam equation

$$\begin{aligned} u_{tt} - u_{xxtt} + u_{xxxx} - \sigma\left(\int_0^L (u_x)^2 dx\right)u_{xx} \\ - \phi\left(\int_0^L (u_x)^2 dx\right)u_{xxt} \\ = q(x), \quad (x, t) \in (0, L) \times \mathbb{R}^+. \end{aligned} \quad (1)$$

We assume the nonlinear boundary conditions

$$u(0, t) = u_x(L, t) = u_{xx}(0, t) = 0, \quad (2)$$

$$u_{xxx}(L, t) = f(u(L, t)) + g(u_t(L, t)) \quad (3)$$

and the initial conditions

$$u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in (0, L). \quad (4)$$

Here the unknown function $u(x, t)$ is the elevation of the surface of beam, $u^0(x)$ and $u^1(x)$ are the given initial value functions, and the subscript t and x denote derivative with respect to t and x , respectively. $-u_{xxtt}$ expresses the rotational inertia, and nonlinear terms $\sigma(\int_0^L (u_x)^2 dx)u_{xx}$ and $\phi(\int_0^L (u_x)^2 dx)u_{xxt}$ represent the extensibility effects and the structural damping, respectively. $q(x)$ is a static load. Moreover the assumptions on nonlinear functions $\sigma(\cdot)$, $\phi(\cdot)$, $g(\cdot)$, and $f(\cdot)$ and the external force function $q(x)$ will be specified later.

In (1), when the structural damping term and the rotational inertia term are absent, (1) is a model for vibrations of tensile beam. This was proposed by Woinowsky-Krieger [1] in the form

$$u_{tt} + u_{xxxx} - \left(\alpha + \beta \int_0^L u_x^2 dx\right)u_{xx} = 0. \quad (5)$$

One of the first mathematical analyses for equation

$$u_{tt} + u_{xxxx} - M \left(\int_0^L (u_x)^2 dx \right) u_{xx} = 0 \quad (6)$$

was done by Ball [2] which was later extended to an abstract setting by defining a linear operator A by Medeiros [3]. In [4], Patcheu obtained the decay of the energy for above equation when a nonlinear damping $g(u_t)$ was effective in Ω . In addition, the attractor on extensible beams with null boundary conditions was considered by several authors. We quote, for instance, [5–7], and so on. But the longtime behavior of the beam equation with nonlinear boundary conditions was paid little attention. We also refer the reader to a few works. One of the first studies in this direction was done by Pazoto and Menzala [8], where stabilization of a thermoelastic extensible beam was considered. Motivated by the result, Ma proved the existence of global solutions and the existence of a global attractor in [9] and [10], respectively, for the Kirchhoff-type beam equation

$$u_{tt} + u_{xxxx} - M \left(\int_0^L (u_x)^2 dx \right) u_{xx} = 0, \quad (7)$$

with the absence of the structural damping and the rotational inertia, subjected to the nonlinear boundary conditions

$$\begin{aligned} u(0, t) &= u_x(0, t) = u_{xx}(L, t) = 0, \\ u_{xxx}(L, t) - M \left(\int_0^L (u_x)^2 dx \right) u_x(L, t) \\ &= f(u(L, t)) + g(u_t(L, t)). \end{aligned} \quad (8)$$

In the following, we mentioned some results on longtime behavior of beam equation with the rotational inertia term. Under null boundary conditions, Geredeli and Lasiecka [11] considered the existence of a compact attractor of beam

$$\begin{aligned} u_{tt} - \alpha \Delta u_{tt} + \Delta^2 u + d(x) g(u_t) \\ - \alpha \operatorname{div}(d(x)) g(\nabla u_t) - F(u) = p(x) \end{aligned} \quad (9)$$

with a rotational inertia term. Under nonlinear boundary conditions

$$u = 0, \quad \Delta u = -g \left(\frac{\partial u_t}{\partial \nu} \right), \quad (10)$$

Ji and Lasiecka [12] considered the semilinear Kirchhoff equation

$$u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + f(u) = 0 \quad (11)$$

with rotational inertia, and they showed that the above problem is uniformly stabilized with uniform energy decay rates.

In addition, we also mentioned some results on longtime behavior of the equation with the structural damping term. Chueshov [13] studied the global attractor with a structural damping of the form $\sigma(\|\nabla u\|^2)(-\Delta)^\theta u_t$ with $1/2 \leq \theta \leq 1$.

Chueshov [14] and Yang et al. [15] considered the global attractor for the Kirchhoff-type equation

$$u_{tt} - \sigma(\|\nabla u\|^2) \Delta u_t - \phi(\|\nabla u\|^2) \Delta u + f(u) = h(x) \quad (12)$$

with structural damping under null boundary conditions, respectively.

On (1) under the following other nonlinear boundary conditions

$$\begin{aligned} u(L, t) &= u_x(0, t) = u_{xx}(L, t) = 0, \\ u_{xxx}(0, t) &= f(u(0, t)) + g(u_t(0, t)) \end{aligned} \quad (13)$$

we also can get the same on the existence of global solutions and the existence of global attractor.

Our fundamental assumptions on $\sigma(\cdot)$, $\phi(\cdot)$, $g(\cdot)$, $f(\cdot)$, and $q(x)$ are given as follows.

Assumption 1. We assume that $\sigma(\cdot)$, $\phi(\cdot) \in C^1(R)$ are all nondecreasing and satisfy

$$\sigma(z)z \geq \widehat{\sigma}(z) \geq 0, \quad \forall z \geq 0, \quad (14)$$

where $\widehat{\sigma}(z) = \int_0^z \sigma(s)ds \geq 0$. Moreover

$$\sigma(0) = 0, \quad \phi(0) = 0, \quad (15)$$

$$\sigma(s) \geq \phi(s) \geq \alpha + \beta s^\gamma \quad (\alpha, \beta > 0, \gamma \geq 1), \quad \forall s \in R^+. \quad (16)$$

Assumption 2. The function $g(\cdot) : R \rightarrow R$ is of class $C^1(R)$ and satisfies $g(0) = 0$, and there exist constants k_1, k_2 and $r \geq 0$ such that

$$(g(u) - g(v))(u - v) \geq k_1 |u - v|^2, \quad \forall u, v \in R, \quad (17)$$

$$|g(u) - g(v)| \leq k_2 (1 + |u|^r + |v|^r) |u - v|, \quad \forall u, v \in R. \quad (18)$$

Assumption 3. The function $f(\cdot) : R \rightarrow R$ is of class $C^1(R)$ and satisfies $f(0) = 0$, and there exist constants k and $q \geq 0$ such that

$$|f(u) - f(v)| \leq k (1 + |u|^q + |v|^q) |u - v|, \quad \forall u, v \in R, \quad (19)$$

$$-L_0 \leq \widehat{f}(u) \leq f(u)u + L_1, \quad (20)$$

where $\widehat{f}(z) = \int_0^z f(s)ds \geq 0$.

Assumption 4. Consider $q(x) \in L^2(0, L)$.

Under the above assumptions, we prove the existence of global solutions and the existence of a global attractor of extensible beam equation system (1)–(4). And the paper is organized as follows. In Section 2, we introduce some Sobolev spaces. In Section 3, we discuss the existence of global strong and weak solutions. In Section 4, we establish the result of the existence of a global attractor in $V \times U$.

2. Preliminaries

Our analysis is based on the following Sobolev spaces. Let

$$\begin{aligned} U &= \{u \in H^1(0, L); u(0) = 0\}, \\ V &= \{u \in H^2(0, L); u(0) = u_x(L) = 0\}, \\ W &= \{u \in H^4(0, L) \cap V; u_{xx} = 0\}. \end{aligned} \quad (21)$$

Motivated by the boundary condition (3) we assume, for regular solutions, that data (u^0, u^1) satisfies the following compatibility condition:

$$u_{xxx}^0(L) = f(u^0(L)) + g(u^1(L)). \quad (22)$$

Then for regular solutions we consider the phase space

$$H_1 = \{(u^0, u^1) \in W \times W; \text{ satisfies condition (22)}\}. \quad (23)$$

In the case of weak solutions we consider the phase space

$$H_0 = V \times U = \overline{H_1}, \quad (24)$$

which guarantees that, for regular data, the nonlinear condition (22) holds. In H_0 we adopt the norm defined by

$$\|u, v\|_{H_0}^2 = \|(u, v)\|_{V \times U}^2 = \|u_{xx}\|^2 + \|v_x\|^2. \quad (25)$$

3. The Existence of Global Solutions

Firstly, using the classical Galerkin method, we can establish the existence and uniqueness of regular solution to problem (1)–(4). We state it as follows.

Theorem 1. Assume Assumptions 1–4 and (22)–(23) hold, for any initial data $(u^0, u^1) \in H_1$; then problem (1)–(4) has a unique regular solution $u(t)$ with

$$u \in L_{\text{loc}}^\infty(R^+, W) \cap C^0([0, \infty); V) \cap C^1([0, \infty); U). \quad (26)$$

Moreover,

$$\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2 + \hat{\sigma}(\|u_x\|^2) \leq M_1, \quad (27)$$

where $M_1 > 0$ depends on the initial data and q , but not on $t > 0$.

Proof. Let us consider the variational problem associated with (1)–(4): find $u(t) \in W$ such that

$$\begin{aligned} & \int_0^L u_{tt} \omega dx + \int_0^L u_{xtt} \omega_x dx + \int_0^L u_{xx} \omega_{xx} dx \\ & + \sigma(\|u_x\|^2) \int_0^L u_x \omega_x dx + \phi(\|u_x\|^2) \int_0^L u_{xt} \omega_x dx \\ & + f(u(L, t)) \omega(L) + g(u_t(L, t)) \omega(L) \\ & = \int_0^L q(x) \omega dx \end{aligned} \quad (28)$$

for all $\omega \in V$. This is done with the Galerkin approximation methods which is standard. Here we denote the approximate solution by $u^m(t)$. We can get the theorem by proving the existence of approximation solution, the estimate of approximation solution, convergence, uniqueness, and $u \in C^0([0, \infty); V) \cap C^1([0, \infty); U)$. In the following we give the estimates of approximation solution, the proof of uniqueness of solution, and the proof of $u \in C^0([0, \infty); V) \cap C^1([0, \infty); U)$. $\square$

Estimate 1. In approximate equation of (28), putting $\omega = u_t^m(t)$ and considering $\hat{\sigma}(z) = \int_0^z \sigma(s) ds$, $\hat{f}(z) = \int_0^z f(s) ds$, using Schwarz inequality, and then integrating from 0 to $t < t_m$, we see that

$$\begin{aligned} & \|u_t^m\|^2 + \|u_{xt}^m\|^2 + \|u_{xx}^m\|^2 + \hat{\sigma}(\|u_x^m\|^2) \\ & + \hat{f}(u^m(L, t)) + 2 \int_0^t \phi(\|u_x^m\|^2) \|u_{xt}^m\|^2 ds \\ & + 2 \int_0^t g(u_t^m(L, t)) u_t^m(L, t) ds \\ & \leq \int_0^t \|q(x)\|^2 ds + \int_0^t \|u_{tt}^m\|^2 ds + \|u_t^m(0)\|^2 \\ & + \|u_{xt}^m(0)\|^2 + \|u_{xx}^m(0)\|^2 + \hat{\sigma}(\|u_x^m(0)\|^2) \\ & + \hat{f}(u^m(L, 0)). \end{aligned} \quad (29)$$

Taking into account the assumptions $\int_0^t g(u_t^m(L, t)) u_t^m(L, t) ds \geq 0$, $\phi(\cdot) \geq 0$, and $\hat{f}, \hat{\sigma} \geq 0$ of f, g, σ , and ϕ , we see that there exists $M_1 > 0$ such that

$$\|u_t^m\|^2 + \|u_{xt}^m\|^2 + \|u_{xx}^m\|^2 + \hat{\sigma}(\|u_x^m\|^2) \leq M_1 \quad (30)$$

for all $t \in [0, T]$ and for all $m \in N$.

Estimate 2. In approximate equation of (28), integrating by parts with $\omega = u_{tt}^m(0)$ and $t = 0$ and considering the compatibility condition (22) and then using Schwarz inequality and the mean value inequality, we see that there exists $M_2 > 0$ such that

$$\|u_{tt}^m(0)\|^2 + \|u_{xtt}^m(0)\|^2 \leq M_2 \quad (31)$$

for all $t \in [0, T]$ and for all $m \in N$.

Estimate 3. Let us fix $t, \xi > 0$ such that $\xi < T - t$. Taking the difference of approximate equation of (28) with $t = t + \xi$ and $t = t$ and replacing ω by $u_t^m(t + \xi) - u_t^m(t)$, we can find a constant $M_3 > 0$, depending only on T , such that

$$\|u_{tt}^m\|^2 + \|u_{xtt}^m\|^2 + \|u_{xxt}^m\|^2 \leq M_3, \quad \forall m \in N, \quad \forall t \in [0, T]. \quad (32)$$

Uniqueness. Let u, v be two solutions of (1)–(4) with the same initial data. Then writing $z = u - v$ and taking the difference (28) with $u = u$ and $u = v$ and replacing ω by z_t and

then using mean value theorem and the Young inequalities combined with Estimates 1 and 3, we deduce that, for some constant $C > 0$,

$$\begin{aligned} & \frac{d}{dt} (\|z_t^m\|^2 + \|z_{xt}^m\|^2 + \|z_{xx}^m\|^2) \\ & \leq C (\|z_t^m\|^2 + \|z_{xt}^m\|^2 + \|z_{xx}^m\|^2), \quad \forall t \in (0, T). \end{aligned} \quad (33)$$

Then from Gronwall's lemma we see that $u = v$.

The Proof of $u \in C^0([0, \infty); V) \cap C^1([0, \infty); U)$. Since $u_{xx}, u_{xxt} \in L^2(0, \infty; L^2(0, L))$, we get $u(x, t) \in C^0([0, \infty); V)$. Similarly, $u(x, t) \in C^1([0, \infty); U)$. The proof of Theorem 1 is completed.

Theorem 2. Assume the assumptions of Theorem 1 and (24) hold; if the initial data $(u^0, u^1) \in H_0$, then there exists a unique weak solution $u(t)$ of problem (1)–(4) which depends continuously on initial data with respect to the norm of $V \times U$.

Proof. Let us consider $\{u^0, u^1\} \in \overline{W \times W} = V \times L^2$, and since $W \times W$ is dense in $V \times U$, there exists $\{u_\mu^0, u_\mu^1\} \subset W \times W$, such that

$$\begin{aligned} u_\mu^0 & \longrightarrow u^0, \quad \text{in } V; \\ u_\mu^1 & \longrightarrow u^1, \quad \text{in } U. \end{aligned} \quad (34)$$

We observe that, for each $\mu \in N$, there exists u_μ , smooth solution of the initial boundary value problem (1)–(4) which satisfies

$$\begin{aligned} & u_{tt\mu} - u_{xxt\mu} + u_{xxx\mu} - \sigma \left(\int_0^L (u_{x\mu})^2 dx \right) u_{xx\mu} \\ & - \phi \left(\int_0^L (u_{x\mu})^2 dx \right) u_{xxt\mu} = q(x), \quad (x, t) \in (0, L) \times \mathbb{R}^+, \\ & u_\mu(x, 0) = u_\mu^0(x), \quad u_{t\mu}(x, 0) = u_\mu^1(x), \quad x \in (0, L), \\ & u_\mu(0, t) = u_{x\mu}(L, t) = u_{xx\mu}(0, t) = 0, \quad t \in (0, \infty), \\ & u_{xxx\mu}(L, t) = f(u_\mu(L, t)) + g(u_{t\mu}(L, t)), \quad t \in (0, \infty). \end{aligned} \quad (35)$$

Considering the arguments used in the estimate of the existence of solution, we obtain

$$\|u_{t\mu}\|^2 + \|u_{xt\mu}\|^2 + \|u_{xx\mu}\|^2 \leq C_0, \quad (36)$$

where C_0 is a positive constant independent of $\mu \in N$.

Defining $Z_{\mu, \sigma} = u_\mu - u_\sigma$; $\mu, \sigma \in N$ where u_μ and u_σ are regular solutions of (35), following the steps already used in the uniqueness of regular solution for (1)–(4) and considering the convergence given in (34) ($u_\mu^0 \rightarrow u^0$ in V and $u_\mu^1 \rightarrow u^1$ in U), we deduce that there exists u such that

$$\begin{aligned} u_\mu & \longrightarrow u \quad \text{strongly in } C([0, T]; V), \\ u_{t\mu} & \longrightarrow u_t \quad \text{strongly in } C([0, T]; U). \end{aligned} \quad (37)$$

From the above convergence, we can pass to the limit using standard arguments in order to obtain

$$\begin{aligned} & u_{tt} - u_{xxt} + u_{xxx} - \sigma \left(\int_0^L (u_x)^2 dx \right) u_{xx} \\ & - \phi \left(\int_0^L (u_x)^2 dx \right) u_{xxt} \\ & = q(x), \quad (x, t) \in (0, L) \times \mathbb{R}^+, \\ & u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in (0, L). \end{aligned} \quad (38)$$

Theorem 2 is proved. $\square$

Remark 3. Theorem 2 implies that problem (1)–(4) defines a nonlinear C_0 -semigroup $S(t)$ on H_0 . Indeed, let us set $S(t)(u^0, u^1) = (u(t), u_t(t))$, where u is the unique solution corresponding to initial data $(u^0, u^1) \in H_0$. Moreover, the operator $S(t)$ defined in H_0 maps H_0 into itself and it enjoys the usual semigroup properties

$$\begin{aligned} S(t + \tau) &= S(t) + S(\tau), \quad \forall t, \tau \in \mathbb{R}, \\ S(0) &= I. \end{aligned} \quad (39)$$

And it is obvious that the map $\{S(t), t \geq 0\}$ is continuous in space H_0 .

4. The Existence of Global Attractor

In this section, we give the existence of a global attractor.

A global attractor for a C_0 -semigroup $S(t)$ defined on a complete metric space H is a bounded closed subset $\mathbb{A} \subset H$ which is positive fully invariant, that is, $S(t)\mathbb{A} = \mathbb{A}$, for all $t \geq 0$, and uniformly attracting, that is,

$$\text{dist}(S(t)B, \mathbb{A}) = \sup_{x \in S(t)B} \inf_{y \in \mathbb{A}} d(x, y) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty, \quad (40)$$

for any bounded set $B \subset H$.

A bounded set $\mathbb{B} \subset H$ is an absorbing set for $S(t)$ if, for any bounded set $B \subset H$, there exists $t_B = t(B) \geq 0$ such that

$$S(t)B \subset \mathbb{B}, \quad \forall t \geq t_B, \quad (41)$$

which defines $(H, S(t))$ as a dissipative dynamical system.

Theorem 4. Assume the hypotheses of Theorem 2 and $q = r = 0$, $k > 0$ is sufficiently small, and then the corresponding semigroup $S(t)$ of problem (1)–(4) has an absorbing set in H_0 .

Proof. Now, we show that semigroup $S(t)$ has an absorbing set in H_0 . Firstly, we can calculate the total energy functional

$$\begin{aligned} E(t) &= \frac{1}{2} (\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2 + \hat{\sigma}(\|u_x\|^2)) \\ &+ \hat{f}(u(L, t)) - \int_0^L qu(t) dx. \end{aligned} \quad (42)$$

Let us fix an arbitrary bounded set $B \subset H_0$ and consider the solutions of problem (1)–(4) given by $(u(t), u_t(t)) = S(t)(u^0, u^1)$ with $(u^0, u^1) \in B$. Our analysis is based on the modified energy functional

$$\tilde{E}(t) = E(t) + L^4 \|q\|^2 + L_0. \quad (43)$$

It is easy to see that $(d/dt)E(t) = (d/dt)\tilde{E}(t)$.

Indeed, since $u(0) = 0$, we have

$$\begin{aligned} |u(L, t)|^2 &= |u(L, t) - u(0, t)|^2 \\ &= \left| \int_0^L u_x dx \right|^2 \\ &\leq \int_0^L dx \int_0^L u_x^2 dx = L \|u_x\|^2. \end{aligned} \quad (44)$$

In a similar way, since $u(0, t) = u_x(L, t) = u_{xx}(0, t) = 0$, the following inequalities hold

$$\begin{aligned} \|u\| &\leq \sqrt{L} \|u_x\|, \quad \|u_x\| \leq L \|u_{xx}\|, \\ \|u\| &\leq L^2 \|u_{xx}\|. \end{aligned} \quad (45)$$

Now let us define

$$\varphi(t) = \int_0^L (u_t - u_{xxt}) u dx. \quad (46)$$

By multiplying (1) by u_t and integrating over $(0, L)$, we have

$$\frac{d}{dt} E(t) + \phi(\|u_x\|^2) \|u_{xt}\|^2 + g(u_t(L, t)) u_t(L, t) = 0. \quad (47)$$

Then, multiplying (1) by u and integrating over $(0, L)$, we obtain

$$\begin{aligned} \frac{d}{dt} \varphi(t) - \|u_t\|^2 - \|u_{xt}\|^2 + \|u_{xx}\|^2 + u(L) u_{xxx}(L) \\ + \sigma(\|u_x\|^2) \|u_x\|^2 + \phi(\|u_x\|^2) \int_0^L u_x u_{xt} dx = \int_0^L qu dx. \end{aligned} \quad (48)$$

Taking into account the boundary condition (3), we get

$$\begin{aligned} \frac{d}{dt} \varphi(t) - \|u_t\|^2 - \|u_{xt}\|^2 + \|u_{xx}\|^2 \\ + u(L) [f(u(L, t)) + g(u_t(L, t))] \\ + \sigma(\|u_x\|^2) \|u_x\|^2 + \phi(\|u_x\|^2) \int_0^L u_x u_{xt} dx \\ = \int_0^L qu dx. \end{aligned} \quad (49)$$

Taking the sum of (47) with ε times (49) and using the Schwarz inequality, we have

$$\begin{aligned} \frac{d}{dt} E(t) + \varepsilon \frac{d}{dt} \varphi(t) + \varepsilon E(t) \\ + \frac{\varepsilon}{2} \|u_t\|^2 + \frac{\varepsilon}{2} \|u_{xt}\|^2 + \frac{\varepsilon}{2} \|u_{xx}\|^2 \\ + \varepsilon \sigma(\|u_x\|^2) \|u_x\|^2 - \frac{\varepsilon}{2} \hat{\sigma}(\|u_x\|^2) \\ + \phi(\|u_x\|^2) \|u_{xt}\|^2 + g(u_t(L, t)) u_t(L, t) \\ = 2\varepsilon (\|u_t\|^2 + \|u_{xt}\|^2) - \varepsilon \phi(\|u_x\|^2) \int_0^L u_x u_{xt} dx \\ + \varepsilon \hat{f}(u(L, t)) - \varepsilon f(u(L, t)) u(L, t) \\ - \varepsilon g(u_t(L, t)) u(L, t). \end{aligned} \quad (50)$$

So, using (15), we obtain

$$g(u_t(L, t)) u_t(L, t) \geq k_1 |u_t(L, t)|^2 \quad (51)$$

and using (16) and (44), we obtain

$$\begin{aligned} \varepsilon g(u_t(L, t)) u(L, t) &\leq 3\varepsilon k_2 |u_t(L, t)| |u(L, t)| \\ &\leq 3L^{3/2} \varepsilon k_2 \|u_{xx}\| |u_t(L, t)| \\ &\leq 9\varepsilon k_2^2 L^3 |u_t(L, t)|^2 + \frac{\varepsilon}{4} \|u_{xx}\|^2. \end{aligned} \quad (52)$$

Taking into account (18) and (44)–(45), we have

$$\begin{aligned} |\varepsilon \hat{f}(u(L, t)) - \varepsilon f(u(L, t)) u(L, t)| \\ \leq \left| \frac{\varepsilon}{2} f(u(L, t)) u(L, t) \right| + \varepsilon L_1 \\ \leq \frac{\varepsilon}{2} 3k \|u\|_\infty^2 + \varepsilon L_1 \\ \leq \frac{\varepsilon}{2} 3k L^3 \|u_{xx}\|^2 + \varepsilon L_1. \end{aligned} \quad (53)$$

Using the mean value inequality, we get

$$\begin{aligned} \left| -\varepsilon \phi(\|u_x\|^2) \int_0^L u_x u_{xt} dx \right| \\ \leq \frac{\varepsilon}{2} \phi(\|u_x\|^2) (\|u_x\|^2 + \|u_{xt}\|^2). \end{aligned} \quad (54)$$

Then inserting (51)–(54) into (50), we obtain

$$\begin{aligned}
& \frac{d}{dt}E(t) + \varepsilon \frac{d}{dt}\varphi(t) + \varepsilon E(t) + \frac{\varepsilon}{2} \|u_t\|^2 \\
& + \frac{\varepsilon}{2} \|u_{xt}\|^2 + \frac{\varepsilon}{2} \|u_{xx}\|^2 \\
& + \varepsilon \sigma(\|u_x\|^2) \|u_x\|^2 - \frac{\varepsilon}{2} \widehat{\sigma}(\|u_x\|^2) \\
& + \phi(\|u_x\|^2) \|u_{xt}\|^2 + k_1 |u_t(L)|^2 \\
& \leq 2\varepsilon (\|u_t\|^2 + \|u_{xt}\|^2) + \frac{\varepsilon}{2} \phi(\|u_x\|^2) \|u_{xt}\|^2 \\
& + \frac{\varepsilon}{2} \phi(\|u_x\|^2) \|u_x\|^2 \\
& + \varepsilon 3kL^3 \|u_{xx}\|^2 + \varepsilon L_1 + 9\varepsilon k_2^2 L^3 |u_t(L, t)|^2 + \frac{\varepsilon}{4} \|u_{xx}\|^2.
\end{aligned} \tag{55}$$

Taking $0 < \varepsilon \leq \min\{k_1/k_2^2 L^3, 1\} = \varepsilon_0$ small enough, we get

$$k_1 |u_t(L, t)|^2 > 9\varepsilon k_2^2 L^3 |u_t(L, t)|^2. \tag{56}$$

Since with $k > 0$ sufficiently small we have $1 - 6kL^3 \geq 0$, we get

$$\frac{\varepsilon}{2} \|u_{xx}\|^2 \geq \frac{1}{2} \varepsilon 3kL^3 \|u_{xx}\|^2 + \frac{\varepsilon}{4} \|u_{xx}\|^2. \tag{57}$$

Using $\phi(s) \geq \alpha + \beta s^\gamma$, ($\gamma \geq 1$, for all $s \in R$), we get

$$\begin{aligned}
\frac{1}{2} \phi(\|u_x\|^2) \|u_{xt}\|^2 & \geq \frac{\alpha}{2} \|u_{xt}\|^2 + \frac{\beta}{2} \|u_x\|^{2\gamma} \|u_{xt}\|^2 \\
& \geq \frac{\alpha}{2} \|u_{xt}\|^2 \\
& \geq \frac{\alpha}{4\sqrt{L}} \|u_t\|^2 + \frac{\alpha}{4} \|u_{xt}\|^2.
\end{aligned} \tag{58}$$

So setting $0 < \varepsilon \leq \min\{\varepsilon_0, \alpha/8\sqrt{L}\}$ small enough,

$$\frac{\alpha}{4\sqrt{L}} \|u_t\|^2 + \frac{\alpha}{4} \|u_{xt}\|^2 \geq 2\varepsilon (\|u_t\|^2 + \|u_{xt}\|^2). \tag{59}$$

Considering that $\sigma(z)z - \widehat{\sigma}(z) \geq 0$ and $\sigma(s) \geq \phi(s)$, for all $s \in R$, we obtain

$$\varepsilon \sigma(\|u_x\|^2) \|u_x\|^2 - \frac{\varepsilon}{2} \widehat{\sigma}(\|u_x\|^2) - \frac{\varepsilon}{2} \phi(\|u_x\|^2) \|u_x\|^2 \geq 0. \tag{60}$$

Therefore with (56)–(60), (55) is transformed into

$$\frac{d}{dt}E(t) + \varepsilon \frac{d}{dt}\varphi(t) + \varepsilon E(t) \leq \varepsilon L_1. \tag{61}$$

Adding $\varepsilon L_0 + \varepsilon L^4 \|h\|^2$ on both sides of inequality (61) and taking into account that $(d/dt)E(t) = (d/dt)\tilde{E}(t)$, we obtain

$$\frac{d}{dt}(\tilde{E}(t) + \varepsilon \varphi(t)) + \varepsilon \tilde{E}(t) \leq \varepsilon (L_0 + L_1 + L^4 \|q\|^2). \tag{62}$$

Now, let us set

$$\tilde{\tilde{E}}_\varepsilon(t) = \tilde{E}(t) + \varepsilon \varphi(t). \tag{63}$$

Then since from (45)– $\int_0^L qu \, dx \geq -L^2 \|q\| \|u_{xx}\| \geq -L^4 \|q\|^2 - (1/4) \|u_{xx}\|^2$, also $\widehat{f}(u(L, t)) \geq -L_0$ and $\widehat{\sigma}(\|u_x\|^2) \geq 0$, we can get

$$\begin{aligned}
\tilde{E}(t) & \geq \frac{1}{4} (\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2) \\
& = \frac{1}{4} \|(u(t), u_t(t))\|_{V \times U}^2.
\end{aligned} \tag{64}$$

Thus, $\tilde{E}(t)$ dominates $\|(u(t), u_t(t))\|_{V \times U}$. Also, from (45), we have

$$\begin{aligned}
|\tilde{\tilde{E}}_\varepsilon(t) - \tilde{E}(t)| & = |\varepsilon \varphi(t)| \\
& = \left| \varepsilon \int_0^L (u_t - u_{xxt}) u \, dx \right| \\
& \leq \frac{\varepsilon (L^2 + L^4)}{2} (\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2) \\
& \leq 2\varepsilon (L^2 + L^4) \tilde{E}(t)
\end{aligned} \tag{65}$$

which implies that, for ε sufficiently small enough,

$$\begin{aligned}
(1 - 2\varepsilon (L^2 + L^4)) \tilde{E}(t) & \leq \tilde{\tilde{E}}_\varepsilon(t) \\
& \leq (1 + 2\varepsilon (L^2 + L^4)) \tilde{E}(t).
\end{aligned} \tag{66}$$

Inserting (66) into (62), we get

$$\begin{aligned}
\frac{d}{dt} \tilde{\tilde{E}}_\varepsilon(t) + \left(\frac{1}{1 + 2\varepsilon (L^2 + L^4)} \right) \tilde{\tilde{E}}_\varepsilon(t) \\
\leq \varepsilon (L_0 + L_1 + L^4 \|q\|^2).
\end{aligned} \tag{67}$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned}
\tilde{\tilde{E}}_\varepsilon(t) & \leq \tilde{\tilde{E}}_\varepsilon(0) \exp\left(-\frac{1}{1 + 2\varepsilon (L^2 + L^4)} t\right) \\
& = \left| \varepsilon \int_0^L (u_t - u_{xxt}) u \, dx \right| \\
& + \varepsilon (L_0 + L_1 + L^4 \|q\|^2) (1 + 2\varepsilon (L^2 + L^4)) \\
& \cdot \left(1 - \exp\left(-\frac{1}{1 + 2\varepsilon (L^2 + L^4)} t\right) \right).
\end{aligned} \tag{68}$$

Since the given invariant set B is bounded, $\tilde{\tilde{E}}_\varepsilon(0)$ is also bounded. Then there exists $t_B > 0$ large enough such that

$$\begin{aligned}
[1 - 2\varepsilon (L^2 + L^4)] \tilde{E}(t) \\
\leq \tilde{\tilde{E}}_\varepsilon(t) \leq \varepsilon (L_0 + L_1 + L^4 \|q\|^2) (1 + 2\varepsilon (L^2 + L^4)), \\
\forall t > t_B.
\end{aligned} \tag{69}$$

Then from (64) we have

$$\begin{aligned} & \| (u(t), u_t(t)) \|_{V \times U}^2 \\ & \leq \frac{4\varepsilon (L_0 + L_1 + L^4 \|q\|^2) (1 + 2\varepsilon (L^2 + L^4))}{1 - 2\varepsilon (L^2 + L^4)}, \quad (70) \\ & \quad \forall t > t_B. \end{aligned}$$

This shows that

$$\begin{aligned} \mathbb{B} = & \left\{ (u, u_t) \in H_0 : \right. \\ & \| (u, u_t) \|_{H_0}^2 \\ & \left. \leq \frac{4\varepsilon (L_0 + L_1 + L^4 \|q\|^2) (1 + 2\varepsilon (L^2 + L^4))}{1 - 2\varepsilon (L^2 + L^4)} \right\} \quad (71) \end{aligned}$$

is an absorbing set for $S(t)$ in H_0 . The proof of Theorem 4 is ended. $\square$

A semigroup $S(t)$ is asymptotically smooth in H if for any bounded positive invariant set $B \subset H$, there exists a compact set $K \subset B$ such that

$$\text{dist}(S(t)B, K) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty. \quad (72)$$

Then the following lemma is well known.

Lemma 5 (see [16], Theorem 2.3). *Let $S(t)$ be a dissipative C_0 -semigroup defined on a metric space H ; then $S(t)$ has a compact global attractor in H if and only if it is asymptotically smooth in H .*

The asymptotic smoothness can be verified from a result by Khanmamedov [17] and Chueshov and Lasiecka [16]. Assume that H is a Banach space.

Lemma 6 (see [16], Proposition 2.10). *Assume that for any bounded positive invariant set $B \subset H$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\| \leq \varepsilon + \phi_T(x, y), \quad \forall x, y \in B, \quad (73)$$

where $\phi_T : H \times H \rightarrow \mathbb{R}$ satisfies

$$\liminf_{k \rightarrow \infty} \liminf_{l \rightarrow \infty} \phi_T(z_k, z_l) = 0 \quad (74)$$

for any sequence (z_n) of B . Then $S(t)$ is asymptotically smooth in H .

Theorem 7. *Assume the hypotheses of Theorem 2 and $\varrho = r = 0$; then the corresponding semigroup $S(t)$ of problem (1)–(4) is asymptotic compactness.*

Proof. We are going to apply Lemma 6 to prove the asymptotic compactness. Given initial data (u^0, u^1) and (v^0, v^1) in a bounded invariant set $B \subset H_0$, let u, v be the corresponding

weak solutions of problem (1)–(4). Then the difference $w = u - v$ is a weak solution of

$$\begin{aligned} & w_{tt} - w_{xxtt} + w_{xxxx} - \sigma(\|u_x\|^2) w_{xx} - \Delta \sigma v_{xx} \\ & - \phi(\|u_x\|^2) w_{xxt} - \Delta \phi v_{xxt} = 0, \quad (75) \\ & w(0, t) = w_x(L, t) = w_{xx}(0, t) = 0, \\ & w_{xxx}(L, t) = \Delta f + \Delta g, \end{aligned}$$

where

$$\begin{aligned} \Delta \sigma &= \sigma(\|u_x\|^2) - \sigma(\|v_x\|^2), \\ \Delta \phi &= \phi(\|u_x\|^2) - \phi(\|v_x\|^2), \\ \Delta f &= f(u(L, t)) - f(v(L, t)), \\ \Delta g &= g(u_t(L, t)) - g(v_t(L, t)). \end{aligned} \quad (76)$$

Let us assume

$$F(t) = \frac{1}{2} \|w_t\|^2 + \frac{1}{2} \|w_{xt}\|^2 + \frac{1}{2} \|w_{xx}\|^2 + \frac{1}{2} \sigma(\|u_x\|^2) \|w_x\|^2 \quad (77)$$

and define

$$\psi(t) = \int_0^L (w_t - w_{xxt}) w dx. \quad (78)$$

As before, by density, we can assume formally that w is sufficiently regular. Then, multiplying the first equation in (75) by w_t and integrating over $(0, L)$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|w_t\|^2 + \|w_{xt}\|^2 + \|w_{xx}\|^2 + \sigma(\|u_x\|^2) \|w_x\|^2) \\ & + \phi(\|u_x\|^2) \|w_{xt}\|^2 + w_{xxx}(L, t) w_t(L, t) \\ & = \sigma'(\|u_x\|^2) \int_0^L u_x u_{xt} dx \|w_x\|^2 + \Delta \sigma \int_0^L v_{xx} w_t dx \\ & + \Delta \phi \int_0^L v_{xxt} w_t dx. \end{aligned} \quad (79)$$

Taking into account the third equation in (75) we see that

$$\begin{aligned} & \frac{d}{dt} F(t) + \phi(\|u_x\|^2) \|w_{xt}\|^2 + \Delta g w_t(L, t) \\ & = \sigma'(\|u_x\|^2) \int_0^L u_x u_{xt} dx \|w_x\|^2 + \Delta \sigma \int_0^L v_{xx} w_t dx \\ & + \Delta \phi \int_0^L v_{xxt} w_t dx - \Delta f w_t(L, t). \end{aligned} \quad (80)$$

By multiplying first equation in (75) by w and integrating over $(0, L)$, we obtain that

$$\begin{aligned} & \frac{d}{dt} \phi(t) - \|w_t\|^2 - \|w_{xt}\|^2 + \|w_{xx}\|^2 \\ & + \sigma(\|u_x\|^2) \|w_x\|^2 \\ & + \phi(\|u_x\|^2) \int_0^L w_x w_{xt} dx + w_{xxx} w(L, t) \\ & - \Delta \sigma \int_0^L v_{xx} w dx - \Delta \phi \int_0^L v_{xxt} w dx = 0. \end{aligned} \quad (81)$$

Also taking into account the third equation in (75) we see that

$$\begin{aligned} & \frac{d}{dt} \phi(t) + 2F(t) \\ & = -\phi(\|u_x\|^2) \int_0^L w_x w_{xt} dx - (\Delta f + \Delta g) w(L, t) \\ & + \Delta \sigma \int_0^L v_{xx} w dx \\ & - \Delta \phi \int_0^L v_{xxt} w dx + 2\|w_t\|^2 + 2\|w_{xt}\|^2. \end{aligned} \quad (82)$$

Then summing (80) with η times (82) we obtain that

$$\begin{aligned} & \frac{d}{dt} F(t) + \eta \frac{d}{dt} \phi(t) + \eta F(t) + \eta F(t) \\ & + \phi(\|u_x\|^2) \|w_{xt}\|^2 + \Delta g w_t(L, t) \\ & = \sigma'(\|u_x\|^2) \int_0^L u_x u_{xt} dx \|w_x\|^2 + \Delta \sigma \int_0^L v_{xx} w_t dx \\ & + \Delta \phi \int_0^L v_{xxt} w_t dx - \Delta f w_t(L, t) \\ & - \eta \phi(\|u_x\|^2) \int_0^L w_x w_{xt} dx - \eta (\Delta f + \Delta g) w(L, t) \\ & + \eta \Delta \sigma \int_0^L v_{xx} w dx \\ & - \eta \Delta \phi \int_0^L v_{xxt} w dx + 2\eta \|w_t\|^2 + 2\eta \|w_{xt}\|^2. \end{aligned} \quad (83)$$

In view of assumption (17) of the function g , we obtain that

$$\Delta g w_t(L, t) \geq k_1 |w_t(L, t)|^2. \quad (84)$$

Also since $\phi(s) \geq \alpha + \beta s^\gamma$ ($\alpha, \beta > 0$, $\gamma \geq 1$), we see that

$$\phi(\|u_x\|^2) \|w_{xt}\|^2 \geq (\alpha + \beta \|u_x\|^{2\gamma}) \|w_{xt}\|^2. \quad (85)$$

In the following, let us estimate the right hand side of (83). We recall that u , v , and w satisfy the estimate $\|u_t\|^2 + \|u_{xt}\|^2 + \|u_{xx}\|^2 + \hat{\sigma}(\|u_x\|^2) \leq M_1$, then denoting by C_0 a generic positive constant which depends only on B we can simplify several notations.

Firstly, since $\sigma(\cdot), \phi(\cdot) \in C^1(R)$, $\phi(0) = 0$, and $\|u_x\|^2 \leq C_0$, we get $\sigma'(\|u_x\|^2) \leq C_0$, $\phi'(\|u_x\|^2) \leq C_0$; then we have

$$\begin{aligned} & \sigma'(\|u_x\|^2) \int_0^L u_x u_{xt} dx \|w_x\|^2 \leq C_0 \|w_x\|^2; \\ & -\eta \phi(\|u_x\|^2) \int_0^L w_x w_{xt} dx \leq C_0 \|w_x\| \|w_{xt}\|. \end{aligned} \quad (86)$$

From the mean value theorem and noting that $\sigma(a^2) - \sigma(b^2) \leq \sigma'(\sup\{a^2, b^2\})|a + b||a - b|$ and $\phi(a^2) - \phi(b^2) \leq \phi'(\sup\{a^2, b^2\})|a + b||a - b|$, we have

$$\begin{aligned} & \Delta \sigma \int_0^L v_{xx} w_t dx \leq C_0 \|w_x\| \|w_t\|, \\ & \eta \Delta \sigma \int_0^L v_{xx} w dx \leq C_0 \|w_x\|^2, \\ & \Delta \phi \int_0^L v_{xxt} w_t dx \leq C_0 \|w_x\| \|w_{xt}\|, \\ & -\eta \Delta \phi \int_0^L v_{xxt} w_x dx \leq C_0 \|w_x\|^2. \end{aligned} \quad (87)$$

From assumption (19) of the function f and $\varrho = 0$ and inequalities (44), we get

$$\begin{aligned} & |-\Delta f w_t(L)| \leq 3k \|w(t)\|_\infty |w_t(L, t)| \\ & \leq C_0 \|w_x\| |w_t(L, t)| \\ & \leq C_0 \|w_x\|^2 + \frac{k}{2} |w_t(L, t)|^2, \\ & |-\eta w(L) \Delta f| \leq \eta 3k \|w\|_\infty |w(L)| \\ & \leq \eta 3k \|w\|_\infty \sqrt{L} \|w_x\| \\ & \leq C_0 \|w_x\|^2. \end{aligned} \quad (88)$$

From assumption (18) of the function g and $r = 0$ and inequalities (44), we get

$$\begin{aligned} & |-\eta w(L, t) \Delta g| \leq \eta 3k_2 |w(L, t)| |w_t(L, t)| \\ & \leq \eta \sqrt{L} \|w_x\| 3k_2 |w_t(L, t)| \\ & \leq \frac{9\eta^2 L k_2^2}{2k_1} \|w_x\|^2 + \frac{k_1}{2} |w_t(L, t)|^2. \end{aligned} \quad (89)$$

Substituting (84)–(89) into (83) and using Schwarz inequality, we obtain that

$$\begin{aligned} & \frac{d}{dt} F(t) + \eta \frac{d}{dt} \phi(t) + 2\eta F(t) + (\alpha + \beta \|u_x\|^{2\gamma}) \|w_{xt}\|^2 \\ & \leq \left(5C_0 + \frac{9\eta^2 L k_2^2}{2k_1} + \frac{2C_0^2}{\eta} \right) \|w_x\|^2 + (3\eta L + 3\eta) \|w_{xt}\|^2. \end{aligned} \quad (90)$$

With $\eta \leq \alpha/(3L+3)$ sufficiently small enough, we have

$$\begin{aligned} \frac{d}{dt}F(t) + \eta \frac{d}{dt}\phi(t) + 2\eta F(t) \\ \leq \left(5C_0 + \frac{9\eta^2 L k_2^2}{2k_1} + \frac{2C_0^2}{\eta}\right) \|w_x\|^2. \end{aligned} \quad (91)$$

Defining $F_\eta(t) = F(t) + \eta\phi(t)$ and considering (44)-(45), we have

$$\begin{aligned} |F_\eta(t) - F(t)| &= |\eta\phi(t)| \\ &= \left| \eta \int_0^L (w_t - w_{xxt}) w dx \right| \\ &\leq \eta \|w_t\| \|w\| + \eta \|w_{xt}\| \|w_x\| \\ &\leq \eta \|w_t\| L^2 \|w_{xx}\| + \eta \|w_{xt}\| \sqrt{L} \|w_{xx}\| \\ &\leq \eta L^2 (\|w_t\|^2 + \|w_{xt}\|^2 + \|w_{xx}\|^2) \\ &\leq \eta L^2 F(t), \end{aligned} \quad (92)$$

which implies for $\eta < \min\{1/L^2, \alpha/(3L+3)\}$ sufficiently small enough that

$$(1 - \eta L^2) F(t) \leq F_\eta(t) \leq (1 + \eta L^2) F(t). \quad (93)$$

Let $C_1 = 5C_0 + 9\eta^2 L k_2^2 / 2k_1 + 2C_0^2 / \eta$; then inserting (93) into (91), we obtain that

$$\frac{d}{dt}F_\eta(t) + \frac{2\eta}{1 + \eta L^2} F_\eta(t) \leq C_1 \|w_x\|^2, \quad t \geq 0. \quad (94)$$

From Gronwall's lemma, we get

$$\begin{aligned} F_\eta(t) &\leq F_\eta(0) e^{-(2\eta/(1+\eta L^2))t} \\ &\quad + C_1 \int_0^t e^{-(\eta/(1+\eta L^2))(t-s)} \|w_x(s)\|^2 ds. \end{aligned} \quad (95)$$

On the other hand, we have

$$\begin{aligned} F_\eta(t) &\geq (1 - \eta L^2) F(t) \\ &= \frac{(1 - \eta L^2)}{2} (\|w_t\|^2 + \|w_{xt}\|^2 + \|w_{xx}\|^2 \\ &\quad + \sigma(\|u_x\|^2) \|w_x\|^2) \\ &\geq \frac{1 - \eta L^2}{2} (\|w_t\|^2 + \|w_{xt}\|^2 + \|w_{xx}\|^2). \end{aligned} \quad (96)$$

Therefore, combining (95) and (96), we can fix a constant $C_B > 0$, depending on the size of B but not on $t > 0$, such that $E_1(0) \leq C_B$; so

$$\begin{aligned} \|(w(t), w_t(t))\|_{V \times U} &\leq \frac{\sqrt{2}}{\sqrt{1 - \eta L^2}} C_B \exp\left(-\frac{\eta}{1 + \eta L^2} t\right) \\ &\quad + \sqrt{\frac{2C_1}{1 - \eta L^2}} \left[\int_0^t \|w_x(s)\|^2 ds \right]^{1/2}. \end{aligned} \quad (97)$$

Given $\varepsilon > 0$, we choose T large such that

$$\sqrt{\frac{2}{1 - \eta L^2}} C_B \exp\left(-\frac{\eta}{1 + \eta L^2} t\right) \leq \varepsilon \quad (98)$$

and define $\phi_T : H_0 \times H_0 \rightarrow \mathbb{R}$ as

$$\begin{aligned} \phi_T((u^0, u^1), (v^0, v^1)) \\ = \sqrt{\frac{2C_1}{1 - \eta L^2}} \left[\int_0^T \|u_x(t) - v_x(t)\|^2 dt \right]^{1/2}. \end{aligned} \quad (99)$$

Then from (97)-(99), we get

$$\begin{aligned} \|S(T)(u^0, u^1) - S(T)(v^0, v^1)\|_{H_0} \\ \leq \varepsilon + \phi_T((u^0, u^1), (v^0, v^1)) \end{aligned} \quad (100)$$

for all $(u^0, u^1), (v^0, v^1) \in B$.

Let (u_n^0, u_n^1) be a given sequence of initial data in B . Then the corresponding sequence $(u_n(t), u_{nt})$ of solutions of problem (1)-(4) is uniformly bounded in H_0 , because B is bounded and positively invariant. So $\{u_n\}$ is bounded in $C([0, \infty), V) \cap C^1([0, \infty), U)$. Since $V \hookrightarrow H_0^1(\Omega)$ compactly, there exists a subsequence u_{nk} which converges strongly in $C([0, T], H_0^1(\Omega))$. Therefore

$$\begin{aligned} \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \int_0^T \|u_{xnk}(s) - u_{xnl}(s)\|^2 ds &= 0, \\ \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} \phi_T((u_{nk}^0, u_{nk}^1), (u_{nl}^0, u_{nl}^1)) &= 0. \end{aligned} \quad (101)$$

So $S(t)$ is asymptotically smooth in H_0 . That is, Lemma 6 holds. Thus Theorem 7 is proved. $\square$

Remark 8. The novelty and difficulty of Theorems 4 lie in the appropriate definition on the functions $\varphi(t) = \int_0^L (u_t - u_{xxt})u dx$, $\psi(t) = \int_0^L (w_t - w_{xxt})w dx$ and the relationship between $\tilde{E}_\varepsilon(t)$ and the energy function $\tilde{E}(t)$ and the relationship between $F_\eta(t)$ and the function $F(t)$.

Lemma 9 ([18], Theorem 1.1 in Chapter I). *One assumes that H is a metric space and that the operators $\{S(t)\}_{t \geq 0}$ are given and satisfy the usual group properties and $S(t)$ is continuous operator from H into itself and the semigroup $\{S(t)\}_{t \geq 0}$ is asymptotically compact. One also assumes that there exist an open set U and a bounded set $\mathbb{B}$ of U such that U is absorbing in $\mathbb{B}$. Then $\omega(\mathbb{B})$ is a compact attractor.*

By view of Lemma 9, with Theorems 4 and 7, the main result of a global attractor reads as follows.

Theorem 10. *The corresponding semigroup $S(t)$ of problem (1)-(4) has a compact global attractor in the phase space H_0 .*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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