Research Article

Legendre Polynomials Operational Matrix Method for Solving Fractional Partial Differential Equations with Variable Coefficients

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A numerical method for solving a class of fractional partial differential equations with variable coefficients based on Legendre polynomials is proposed. A fractional order operational matrix of Legendre polynomials is also derived. The initial equations are transformed into the products of several matrixes by using the operational matrix. A system of linear equations is obtained by dispersing the coefficients and the products of matrixes. Only a small number of Legendre polynomials are needed to acquire a satisfactory result. Results obtained using the scheme presented here show that the numerical method is very effective and convenient for solving fractional partial differential equations with variable coefficients.

1. Introduction

The subject of fractional calculus was found over 300 years ago. The theory of integrals and derivatives of noninteger order goes back to Leibnitz, Liouville, and Letnikov. In recent years, fractional derivative and fractional differential equations have played a very significant role in many areas in fluid flow, physics, mechanics, and other applications. A lot of practical problems can be elegantly modeled with the help of the fractional derivative [1–5]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Due to the increasing applications, a lot of attention has been paid to numerical and exact solution of fractional differential equations and fractional partial equations. The analytical solutions of fractional differential equations are still in a preliminary stage. Except in a limited number of these equations, we have difficulty in seeking their analytical as well as numerical solutions. Thus there have been attempts to develop the methods for getting analytical and numerical solutions of fractional differential equations. Recently, some methods have drawn attention, such as Adomian decomposition method (ADM) [6, 7], variational iteration method (VIM) [8], generalized

differential transform method (GDTM) [9–11], finite difference method (FDM) [12], and wavelet method [13, 14].

In this paper, our study focuses on a class of fractional partial differential equations as follows:

$$a(x) \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} + b(x) \frac{\partial^\beta u(x, t)}{\partial t^\beta} = f(x, t), \quad (1)$$

such that the initial conditions

$$\begin{aligned} u(x, 0) &= g(x), \\ u(0, t) &= h(t), \end{aligned} \quad (2)$$

where $\partial^\alpha u(x, t)/\partial x^\alpha$ and $\partial^\beta u(x, t)/\partial t^\beta$ are fractional derivatives of Caputo sense, $0 < \alpha, \beta \leq 1$, $f(x, t)$, $a(x)$, and $b(x)$ are the known and $u(x, t)$ is the unknown.

There have been several methods for solving the fractional partial differential equation. Doha et al. used Jacobi tau approximation to solve the numerical solution of the space fractional diffusion equation [15]. Yi et al. [16] applied block pulse functions method to obtain the fractional partial equations. Podlubny [17] obtained the numerical solution of the fractional partial differential equations with constant coefficients by using Laplace transform method.

2. Definitions of Fractional Derivatives and Integrals

Definition 1. Riemann-Liouville fractional integral of order α , ($\alpha \geq 0$) is defined as follows [17]:

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, \quad t > 0, \quad (3)$$

$$I^0 u(t) = u(t),$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$ is the gamma function.

The Riemann-Liouville fractional integral satisfies the following properties:

$$I^\alpha I^\beta u(t) = I^{\beta} I^\alpha u(t),$$

$$I^\alpha I^\beta u(t) = I^{\alpha+\beta} u(t), \quad (4)$$

$$I^\alpha t^r = \frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)} t^{\alpha+r}.$$

Definition 2. Caputo's fractional derivative of order α , ($\alpha \geq 0$) is defined as follows [17]:

$$D_*^\alpha u(t) = \frac{1}{\Gamma(r-\alpha)} \int_0^t \frac{u^r(\tau)}{(t-\tau)^{\alpha-r+1}} d\tau, \quad (5)$$

$$0 \leq r-1 < \alpha < r.$$

Particularly, the operator D_*^α satisfies the following properties (c is a constant):

$$D_*^\alpha c = 0,$$

$$D_*^\alpha t^\beta = \begin{cases} 0, & \beta \in \mathbb{N}_0, \beta < [\alpha], \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \beta \in \mathbb{N}_0, \beta \geq [\alpha]; \beta \notin \mathbb{N}, \beta > [\alpha], \end{cases} \quad (6)$$

$$D_*^\alpha I^\alpha u(t) = u(t),$$

$$I^\alpha D_*^\alpha u(t) = u(t) - \sum_{k=0}^{r-1} u^{(k)}(0^+) \frac{t^k}{k!}, \quad t > 0,$$

$$D_*^\alpha (\lambda f(t) + \mu g(t)) = \lambda D_*^\alpha (f(t)) + \mu D_*^\alpha (g(t)).$$

3. Legendre Polynomials and Some of Their Properties

The Legendre basis polynomials of degree n in $[0, 1]$ (see [18]) are defined by

$$P_{i+1}(t) = \frac{(2i+1)(2t-1)}{(i+1)} P_i(t) - \frac{i}{i-1} P_{i-1}(t), \quad (7)$$

$$i = 1, 2, \dots,$$

where $P_0(t) = 1$, $P_1(t) = 2t - 1$. The Legendre polynomials of degree i can be also written as

$$P_i(t) = \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)!}{(i-k)!} \frac{t^k}{(k!)^2}. \quad (8)$$

Let

$$\Phi(t) = [P_0(t), P_1(t), \dots, P_n(t)]^T. \quad (9)$$

The Legendre polynomials given by (7) can be expressed in the matrix form

$$\Phi(t) = \mathbf{A} \mathbf{T}_n(t), \quad (10)$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & (-1)^2 2! & 0 & \dots & 0 \\ (-1)^2 & (-1)^3 \frac{3!}{1!} & (-1)^4 \frac{4!}{2!} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^n & (-1)^{n+1} \frac{(n+1)!}{(n-1)!} & (-1)^{n+1} \frac{(n+2)!}{(n-2)!2!} & \dots & (-1)^{2n} \frac{(2n)!}{n!} \end{bmatrix}, \quad (11)$$

$$\mathbf{T}_n(t) = [1, t, \dots, t^n]^T.$$

Obviously

$$\mathbf{T}_n(t) = \mathbf{A}^{-1} \Phi(t). \quad (12)$$

A function $u(t) \in L^2(0, 1)$ can be expressed in terms of the Legendre basis. In practice, only the first $(n+1)$ term of Legendre polynomials is considered. Hence

$$u(t) \cong \sum_{i=0}^n c_i P_i(t) = \mathbf{c}^T \Phi(t), \quad (13)$$

where $\mathbf{c} = [c_0, c_1, \dots, c_n]^T$, c_i ($i = 0, 1, 2, \dots, n$) are called Legendre coefficients.

We extend the notion to two-dimensional space and define two-dimensional Legendre polynomials of order $\widehat{m} = n+1$ as a product function of two Legendre polynomials:

$$P_{\Pi}(x, t) = P_a(x) P_b(t), \quad (14)$$

$$\Pi = \widehat{m}a + b + 1, \quad a, b = 0, 1, 2, \dots, n.$$

For the function $u(x, t) \in L^2([0, 1] \times [0, 1])$, we can also get its approximation by using Legendre polynomials:

$$u(x, t) \cong \sum_{i=0}^n \sum_{j=0}^n u_{ij} P_i(x) P_j(t) = \Phi^T(x) \mathbf{U} \Phi(t), \quad (15)$$

where

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix}. \quad (16)$$

Theorem 3 (see [19]). *If a continuous function $u(x, t)$, defined on $[0, 1] \times [0, 1]$, has bounded mixed fourth partial derivative $\partial^4 u(x, t)/\partial^2 x \partial t^2$, then the Legendre expansion of the function converges uniformly to the function.*

For sufficiently smooth function $u(x, t)$ on $[0, 1] \times [0, 1]$, the error of the approximation is given by

$$\|u(x, t) - P_{\Pi}(x, t)\|_2 \leq \left(C_1 + C_2 + C_3 \frac{1}{\widehat{m}^{\widehat{m}+1}} \right) \frac{1}{\widehat{m}^{\widehat{m}+1}}, \quad (17)$$

where

$$\begin{aligned} C_1 &= \frac{1}{4} \max_{(x,t) \in [0,1] \times [0,1]} \left| \frac{\partial^{\widehat{m}+1} u(x, t)}{\partial x^{\widehat{m}+1}} \right|, \\ C_2 &= \frac{1}{4} \max_{(x,t) \in [0,1] \times [0,1]} \left| \frac{\partial^{\widehat{m}+1} u(x, t)}{\partial t^{\widehat{m}+1}} \right|, \\ C_3 &= \frac{1}{16} \max_{(x,t) \in [0,1] \times [0,1]} \left| \frac{\partial^{2\widehat{m}+2} u(x, t)}{\partial x^{\widehat{m}+1} \partial t^{\widehat{m}+1}} \right|. \end{aligned} \quad (18)$$

We refer the reader to [20] for the proof of the above result.

4. Numerical Solution of the Fractional Partial Differential Equation

Consider the fractional partial differential equation with variable coefficients equation (1). If we approximate the function $u(x, t)$ with the Legendre polynomials, it can be written as (15). Then we have

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial x^\alpha} &\simeq \frac{\partial^\alpha (\Phi^T(x) \mathbf{U} \Phi(t))}{\partial x^\alpha} = \left(\frac{\partial^\alpha \Phi(x)}{\partial x^\alpha} \right)^T \mathbf{U} \Phi(t) = \left(\frac{\partial^\alpha \mathbf{T}_n(x)}{\partial x^\alpha} \right)^T \mathbf{A}^T \mathbf{U} \Phi(t) \\ &= \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{1-\alpha} & \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha} \end{bmatrix} \mathbf{A}^T \mathbf{U} \Phi(t) \\ &= \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} x^{-\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{-\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{-\alpha} \end{bmatrix} \mathbf{A}^T \mathbf{U} \Phi(t) = \Phi^T(x) (\mathbf{A}^{-1})^T \mathbf{M} \mathbf{A}^T \mathbf{U} \Phi(t), \\ \frac{\partial^\beta u(x, t)}{\partial t^\beta} &\simeq \frac{\partial^\beta (\Phi^T(x) \mathbf{U} \Phi(t))}{\partial t^\beta} = \Phi^T(x) \mathbf{U} \frac{\partial^\beta \Phi(t)}{\partial t^\beta} = \Phi^T(x) \mathbf{U} \mathbf{A} \frac{\partial^\beta \mathbf{T}_n(t)}{\partial t^\beta} \\ &= \Phi^T(x) \mathbf{U} \mathbf{A} \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\beta)} t^{1-\beta} & \frac{\Gamma(3)}{\Gamma(3-\beta)} t^{2-\beta} & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} t^{n-\beta} \end{bmatrix}^T \\ &= \Phi^T(x) \mathbf{U} \mathbf{A} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta)} t^{-\beta} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\beta)} t^{-\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)} t^{-\beta} \end{bmatrix} \mathbf{T}_n(t) = \Phi^T(x) \mathbf{U} \mathbf{A} \mathbf{N} \mathbf{A}^{-1} \Phi(t). \end{aligned} \quad (19)$$

Let

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{-\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{-\alpha} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{-\alpha} \end{bmatrix}, \quad (20)$$

$$\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta)}t^{-\beta} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\beta)}t^{-\beta} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)}t^{-\beta} \end{bmatrix}.$$

Substituting (19) into (1), we have

$$a(x)\Phi^T(x)(\mathbf{A}^{-1})^T\mathbf{M}\mathbf{A}^T\mathbf{U}\Phi(t) + b(x)\Phi^T(x)\mathbf{U}\mathbf{A}\mathbf{N}\mathbf{A}^{-1}\Phi(t) = f(x, t). \quad (21)$$

Dispersing (21) by the points (x_i, t_j) ($i = 1, 2, \dots, n_x; j = 1, 2, \dots, n_t$), we can obtain $\mathbf{U}$ which is unknown.

5. Error Analysis

In this part, in order to illustrate the effectiveness of $\partial^\alpha u(x, y)/\partial x^\alpha \cong \Phi^T(x)\mathbf{U}\Phi(y)$, we have given the following theorem. Let $\partial^\alpha u_n(x, y)/\partial x^\alpha$ be the following approximation of $\partial^\alpha u(x, y)/\partial x^\alpha$:

$$\frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} = \sum_{i=1}^n \sum_{j=1}^n u_{ij} P_i(x) P_j(y). \quad (22)$$

Then we have

$$\frac{\partial^\alpha u(x, y)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} = \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} u_{ij} P_i(x) P_j(y). \quad (23)$$

Theorem 4. Suppose that the function $\partial^\alpha u_n(x, y)/\partial x^\alpha$ obtained by using Legendre polynomials is the approximation of $\partial^\alpha u(x, y)/\partial x^\alpha$, and $u(x, y)$ has bounded mixed fractional partial derivative $|\partial^{4+\alpha+\beta} u(x, y)/\partial x^{2+\alpha} \partial y^{2+\beta}| \leq \widehat{M}$; then we have the following upper bound of error:

$$\left\| \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} \right\|_E \leq \frac{\widehat{M}}{8} \left(\frac{\Gamma'(n-0.5)}{\Gamma(n-0.5)} \right)''', \quad (24)$$

where $\|u(x, y)\|_E = (\int_{-1}^1 \int_{-1}^1 u^2(x, y) dx dy)^{1/2}$ and

$$u_{ij} = \left(\frac{2i+1}{2} \right) \left(\frac{2j+1}{2} \right) \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} P_i(x) P_j(y) dx dy. \quad (25)$$

Proof. The property of the sequence $\{P_i(x)\}$ on $[-1, 1]$ implies that

$$\int_{-1}^1 P_i(x) P_j(x) dx = \begin{cases} \frac{2}{2i+1}, & i = j; \\ 0, & i \neq j; \end{cases} \quad (26)$$

then

$$\begin{aligned} & \left\| \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} \right\|_E^2 \\ &= \int_{-1}^1 \int_{-1}^1 \left[\frac{\partial^\alpha u(x, y)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} \right]^2 dx dy \\ &= \int_{-1}^1 \int_{-1}^1 \left[\sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} u_{ij} P_i(x) P_j(y) \right]^2 dx dy \\ &= \int_{-1}^1 \int_{-1}^1 \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} u_{ij}^2 P_i^2(x) P_j^2(y) dx dy \\ &= \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} u_{ij}^2 \int_{-1}^1 P_i^2(x) dx \int_{-1}^1 P_j^2(y) dy \\ &= \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} u_{ij}^2 \frac{2}{2i+1} \frac{2}{2j+1}. \end{aligned} \quad (27)$$

The Legendre polynomials coefficients of function $\partial^\alpha u(x, y)/\partial x^\alpha$ are given by

$$u_{ij} = \left(\frac{2i+1}{2} \right) \left(\frac{2j+1}{2} \right) \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} P_i(x) P_j(y) dx dy. \quad (28)$$

Therefore, we obtain

$$\begin{aligned} u_{ij} &= \frac{2j+1}{4} \int_{-1}^1 \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} [P_{i+1}(x) - P_{i-1}(x)] \\ &\quad \cdot P_j(y) \Big|_{-1}^1 dy - \frac{2j+1}{4} \\ &\quad \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^{\alpha+1} u(x, y)}{\partial x^{\alpha+1}} [P_{i+1}(x) - P_{i-1}(x)] P_j(y) dx dy \\ &= -\frac{2j+1}{4} \int_{-1}^1 \int_{-1}^1 \frac{\partial^{\alpha+1} u(x, y)}{\partial x^{\alpha+1}} [P_{i+1}(x) \\ &\quad - P_{i-1}(x)] P_j(y) dx dy = -\frac{2j+1}{4} \end{aligned}$$

$$\begin{aligned}
& \cdot \int_{-1}^1 \frac{\partial^{\alpha+1} u(x, y)}{\partial x^{\alpha+1}} \left[\frac{P_{i+2}(x) - P_i(x)}{2i+3} \right. \\
& \left. - \frac{P_i(x) - P_{i-2}(x)}{2i-1} \right] P_j(y) \Big|_{-1}^1 dy + \frac{2j+1}{4} \\
& \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^{\alpha+2} u(x, y)}{\partial x^{\alpha+2}} \left[\frac{P_{i+2}(x) - P_i(x)}{2i+3} \right. \\
& \left. - \frac{P_i(x) - P_{i-2}(x)}{2i-1} \right] P_j(y) dx dy = \frac{2j+1}{4} \\
& \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^{\alpha+2} u(x, y)}{\partial x^{\alpha+2}} \left[\frac{P_{i+2}(x) - P_i(x)}{2i+3} \right. \\
& \left. - \frac{P_i(x) - P_{i-2}(x)}{2i-1} \right] P_j(y) dx dy.
\end{aligned} \tag{29}$$

Now, let $\tau_i(x) = (2i-1)P_{i+2}(x) - 2(2i+1)P_i(x) + (2i+3)P_{i-2}(x)$; then we have

$$\begin{aligned}
u_{ij} &= \frac{2j+1}{4(2i-1)(2i+3)} \\
& \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^{2+\alpha} u(x, y)}{\partial x^{2+\alpha}} \tau_i(x) P_j(y) dx dy.
\end{aligned} \tag{30}$$

By solving this equation, we have

$$\begin{aligned}
u_{ij} &= \frac{1}{4(2i-1)(2i+3)(2j-1)(2j+3)} \\
& \cdot \int_{-1}^1 \int_{-1}^1 \frac{\partial^{4+\alpha+\beta} u(x, y)}{\partial x^{2+\alpha} \partial y^{2+\beta}} \tau_i(x) \tau_j(y) dx dy.
\end{aligned} \tag{31}$$

So we have

$$\begin{aligned}
|u_{ij}| &\leq \frac{1}{4(2i-1)(2i+3)(2j-1)(2j+3)} \\
& \cdot \int_{-1}^1 \int_{-1}^1 \left| \frac{\partial^{4+\alpha+\beta} u(x, y)}{\partial x^{2+\alpha} \partial y^{2+\beta}} \right| |\tau_i(x)| |\tau_j(y)| dx dy \\
&\leq \frac{\widehat{M}}{4(2i-1)(2i+3)(2j-1)(2j+3)} \\
& \cdot \int_{-1}^1 |\tau_i(x)| dx \int_{-1}^1 |\tau_j(y)| dy.
\end{aligned} \tag{32}$$

Moreover, it was easily obtained that

$$\int_{-1}^1 |\tau_m(t)| dt \leq \sqrt{24} \frac{2i+3}{\sqrt{2i-3}}; \tag{33}$$

thus, we have

$$\begin{aligned}
|u_{ij}| &\leq \frac{24\widehat{M}}{4(2i-1)(2i+3)(2j-1)(2j+3)} \frac{(2i+3)}{\sqrt{2i-3}} \\
& \cdot \frac{(2j+3)}{\sqrt{2j-3}} \leq \frac{6\widehat{M}}{(2i-3)^{3/2}(2j-3)^{3/2}}.
\end{aligned} \tag{34}$$

Namely

$$|u_{ij}|^2 \leq \frac{36\widehat{M}^2}{(2i-3)^3(2j-3)^3}. \tag{35}$$

Therefore, we have

$$\begin{aligned}
& \left\| \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} \right\|_E^2 \\
& \leq \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} u_{ij}^2 \frac{2}{2i+1} \frac{2}{2j+1} \\
& \leq \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{144\widehat{M}^2}{(2i-3)^3(2j-3)^3(2i+1)(2j+1)} \\
& \leq \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} \frac{144\widehat{M}^2}{(2i-3)^4(2j-3)^4} \\
& = \left[\sum_{i=n+1}^{\infty} \frac{12\widehat{M}}{(2i-3)^4} \right]^2 = \left[\frac{\widehat{M}}{8} \left(\frac{\Gamma'(n-0.5)}{\Gamma(n-0.5)} \right)''' \right]^2;
\end{aligned} \tag{36}$$

thus

$$\left\| \frac{\partial^\alpha u(x, y)}{\partial x^\alpha} - \frac{\partial^\alpha u_n(x, y)}{\partial x^\alpha} \right\|_E \leq \frac{\widehat{M}}{8} \left(\frac{\Gamma'(n-0.5)}{\Gamma(n-0.5)} \right)'''. \tag{37}$$

This completes the proof. $\square$

6. Numerical Examples

Example 1. Consider the following nonhomogeneous partial differential equation:

$$x^{1/3} \frac{\partial^{1/2} u(x, t)}{\partial x^{1/2}} + x^{1/2} \frac{\partial^{1/2} u(x, t)}{\partial t^{1/2}} = f(x, t) \tag{38}$$

$$(x, t) \in [0, 1] \times [0, 1], \quad u(x, 0) = 10x^2(1-x), \quad u(0, t) = u(1, t) = 0,$$

where $f(x, t) = -40\sqrt{t}(3+2t)(-1+x)x^{7/3}/3\sqrt{\pi} - 16(1+t)^2x^2(-5+6x)/3\sqrt{\pi}$. The exact solution of this equation is

$u(x, t) = 10x^2(1-x)(1+t)^2$. Tables 1–3 show the absolute errors for $t = 1/4s$, $t = 1/2s$, and $t = 3/4s$ of different n .

TABLE 1: Absolute error for $t = 1/4$ s and different values of n .

x	$n = 2$	$n = 3$	$n = 4$
0.1	0.3246	$3.2831e - 015$	$3.3582e - 016$
0.2	0.2351	$4.2734e - 015$	$5.2745e - 015$
0.3	0.4820	$4.1237e - 015$	$5.6521e - 015$
0.4	0.3274	$5.7320e - 015$	$6.2742e - 016$
0.5	0.8231	$4.7381e - 015$	$7.1640e - 016$
0.6	0.9127	$7.3722e - 015$	$3.2356e - 016$
0.7	1.2188	$6.3276e - 015$	$4.2745e - 015$
0.8	1.5181	$1.2374e - 014$	$3.7224e - 015$
0.9	0.8364	$3.1744e - 015$	$8.8874e - 015$

TABLE 2: Absolute error for $t = 1/2$ s and different values of n .

x	$n = 2$	$n = 3$	$n = 4$
0.1	0.2821	$4.2137e - 015$	$3.4325e - 016$
0.2	0.4375	$5.8711e - 015$	$6.5332e - 016$
0.3	0.1021	$4.2210e - 015$	$5.2435e - 016$
0.4	0.2387	$3.1016e - 016$	$4.9722e - 016$
0.5	0.8277	$3.4762e - 016$	$5.3478e - 015$
0.6	0.9322	$5.3265e - 015$	$1.3241e - 016$
0.7	1.3846	$4.7632e - 015$	$9.2371e - 016$
0.8	1.6654	$4.2346e - 016$	$8.4812e - 015$
0.9	0.9144	$2.0437e - 016$	$6.3273e - 016$

TABLE 3: Absolute error for $t = 3/4$ s and different values of n .

x	$n = 2$	$n = 3$	$n = 4$
0.1	0.3126	$5.0832e - 016$	$5.3281e - 016$
0.2	0.8978	$4.1845e - 016$	$2.3258e - 015$
0.3	0.2374	$2.3448e - 016$	$4.0112e - 016$
0.4	0.2951	$3.2155e - 016$	$5.1223e - 016$
0.5	0.3327	$5.5518e - 015$	$6.3274e - 015$
0.6	1.3267	$6.2440e - 016$	$5.7421e - 015$
0.7	0.8723	$3.5220e - 016$	$5.2871e - 016$
0.8	0.9229	$4.2301e - 016$	$4.8810e - 016$
0.9	1.1327	$5.2310e - 015$	$6.1138e - 016$

From Tables 1–3, we can see that the absolute error is very small when $n \geq 3$. Also, when n is fixed, the more points we take, the more accurate numerical solutions we obtain. Figures 1–3 show the fact that $n_{x,t}$ is the number of x_i, t_j .

Example 2. Consider the following fractional partial differential equation:

$$x \frac{\partial^{1/4} u(x, t)}{\partial x^{1/4}} + x^{2/3} \frac{\partial^{1/3} u(x, t)}{\partial t^{1/3}} = f(x, t), \quad (39)$$

$$u(x, 0) = x(x-1)(x^2+1), \quad u(0, t) = 0,$$

FIGURE 1: Numerical solution of $n_{x,t} = 3$. FIGURE 2: Numerical solution of $n_{x,t} = 6$.

FIGURE 3: Exact solution.

where

$$f(x, t) = \frac{3t^{2/3} (55 + 66t + 81t^3) (-1 + x) x^2 (1 + x^2)}{110\Gamma(2/3)} + \frac{4(1 + t + t^2 + t^4) x^{17/12} [-385 + 8x(55 - 60x + 64x^2)]}{1155\Gamma(3/4)}. \quad (40)$$

FIGURE 4: The numerical solutions for Example 2 when $n = 4$.

FIGURE 6: Exact solution for Example 2.

FIGURE 5: The numerical solutions for Example 2 when $n = 5$. FIGURE 7: The numerical solutions for Example 3 when $n = 3$.

The exact solution is $u(x, t) = x(x-1)(x^2+1)(1+t+t^2+t^4)$. The numerical solutions for $n = 4$, $n = 5$ are displayed in Figures 4 and 5 and the exact solution is shown in Figure 6.

Example 3. Consider this equation:

$$x^{1/2} \frac{\partial^{1/3} u(x, t)}{\partial x^{1/3}} + x^{2/3} \frac{\partial^{1/3} u(x, t)}{\partial t^{1/3}} = f(x, t), \quad (41)$$

$$u(x, 0) = x^2 - 1, \quad u(0, t) = -1 - (t-1)^2 - (t-1)^3,$$

where

$$f(x, t) = \frac{9(1+t-2t^2+t^3)x^{13/6}}{5\Gamma(2/3)} + \frac{3t^{2/3}(-2+3t)(-10+9t)(-1+x^2)x^{2/3}}{40\Gamma(2/3)}. \quad (42)$$

The exact solution is $u(x, t) = (x^2 - 1)[1 + (t-1)^2 + (t-1)^3]$. The numerical solutions for $n = 3$, $n = 4$ are displayed in Figures 7 and 8. The absolute errors are shown in Figures 9 and 10 when $n = 3$ and $n = 4$.

FIGURE 8: The numerical solutions for Example 3 when $n = 4$.

From Examples 1–3, we can see that the method in this paper can be effectively used to solve the numerical solution of fractional partial differential equation with variable coefficients. From the above results, the absolute errors between

FIGURE 9: The absolute errors for Example 3 when $n = 3$.

FIGURE 10: The absolute errors for Example 3 when $n = 4$.

the numerical solutions and the exact solution are rather small. What is more, due to the absolute error in this paper is about 10^{-15} , the Legendre polynomials method can reach higher degree of accuracy by comparing the approximations obtained by block pulse method [16].

7. Conclusion

In this paper, we use the Legendre polynomials method to solve a class of fractional partial differential equations with variable coefficients. The Legendre polynomials operational matrix of fractional differentiation is derived from the property of Legendre polynomials. The initial equation is translated into the product of some relevant matrixes, which can also be regarded as the system of linear equations. The error analysis of Legendre polynomials is also given. The numerical results show that numerical solutions obtained by our method are in very good agreement with the exact solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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