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Research Article

Minimum Phase Property of Chebyshev-Sharpened Cosine Filters

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We prove that the Chebyshev sharpening technique, recently introduced in literature, provides filters with a Minimum Phase (MP) characteristic when it is applied to cosine filters. Additionally, we demonstrate that cascaded expanded Chebyshev-Sharpened Cosine Filters (CSCFs) are also MP filters, and we show that they achieve a lower group delay for similar magnitude characteristics in comparison with traditional cascaded expanded cosine filters. The importance of the characteristics of cascaded expanded CSCFs is also elaborated. The developed examples show improvements in the group delay ranged from 23% to 47% at the cost of a slight increase of usage of hardware resources. For an application of a low-delay decimation filter, the proposed scheme exhibits a 24% lower group delay, with 35% less computational complexity (estimated in Additions per Output Sample) and slightly less usage of hardware elements.

1. Introduction

A Minimum Phase (MP) digital filter has all zeros on or inside the unit circle [1]. We consider in this paper MP Finite Impulse Response (FIR) filters, which find applications in cases where a long delay, usually introduced by Linear Phase (LP) FIR filters, is not allowed. Examples of these cases include data communication systems or speech and audio processing systems [2–4].

The basic building block analyzed in this paper, the *cosine filter*, is a simple FIR filter whose transfer function and frequency response are, respectively, given by

$$H(z) = \frac{1}{2}\left[1 + z^{-1}\right], \tag{1}$$

$$H\left(e^{j\omega}\right) = \left[\cos\left(\frac{\omega}{2}\right)\right] e^{-j\omega/2}. \tag{2}$$

This filter is of special interest because of the following main reasons:

(a) It has MP property because its zero lies on the unit circle.

(b) It has a low computational complexity because it does not require multipliers, which are the most costly and power-consuming elements in a digital filter [5].

(c) It has a low usage of hardware elements, which can be translated into a low demand of chip area for implementation.

The fact that a cosine filter has the Minimum Phase characteristic becomes significant because these basic building blocks can be used to design filters with a low delay and simultaneously a low computational complexity and a low usage of hardware resources.

Due to the aforementioned characteristics, cascaded expanded cosine filters were investigated in [6]. A *cascaded* *expanded cosine filter* is defined as a filter with transfer function and frequency response, respectively, given by

$$H_A(z) = \left[\prod_{k=1}^{K} H\left(z^k\right)\right]^L, \quad (3)$$

$$H_A\left(e^{j\omega}\right) = \left[\prod_{k=1}^{K}\left[\cos\left(k \cdot \frac{\omega}{2}\right)\right]^L\right] e^{-j\omega L \sum_{k=1}^{K} k/2}, \quad (4)$$

with $H(z)$ being the cosine filter given in (1), whereas K and L are arbitrary integers. In method [6], Rouche's theorem was employed to demonstrate that when cascaded expanded cosine filters are sharpened with a modified version of the technique from [7] (originally devised for LP FIR filters), the result is an overall FIR filter that has all its zeros on or inside the unit circle; that is, it satisfies the MP characteristic. Nevertheless, the resulting filter still has a high group delay because a large number of cascaded expanded cosine filters are needed to meet a given attenuation specification, as we see in the examples from [6].

On the other hand, the Chebyshev sharpening technique was recently introduced in [8] to design LP FIR filters based on comb subfilters for decimation applications. In that method the sharpening is performed with an Nth degree Chebyshev polynomial of first kind, defined as

$$P(x) = \sum_{n=0}^{N} c_n \cdot x^n, \quad (5)$$

where c_n are integers [9]. When applied to comb filters, Chebyshev sharpening provides solutions with advantages like a simple and elegant design method, a low-complexity resulting LP FIR filter, and improved attenuation characteristics in the resulting filter. However, filters from [8] are not guaranteed to have MP characteristic.

From the aforementioned literature we can extract the following observations:

(a) In MP FIR filters the reduction of the group delay is a priority.

(b) The use of cosine filters results in low-complexity multiplierless MP FIR filters.

(c) The recent Chebyshev sharpening method from [8] can improve the attenuation of cosine filters and is a potentially useful approach to preserve a simple multiplierless solution with a lower group delay in comparison with simple cascaded expanded cosine filters.

Motivated by the remarks listed above, we present in this paper the following contributions:

(1) The mathematical proof that the use of Chebyshev sharpening in cosine filters, which produces *Chebyshev-Sharpened Cosine Filters* (CSCFs), guarantees resulting multiplierless filters with all of their zeros placed on the unit circle, that is, with MP property: this demonstration hinges upon the factorization of the transfer function of the CSCF into second-order sections, taking advantage of the antisymmetry of the roots of the Chebyshev polynomial.

(2) The mathematical proof that *cascaded expanded CSCFs* have also all of their zeros placed on the unit circle: this demonstration is a simple extension of aforementioned proof for CSCFs.

(3) The explanation of how cascaded expanded CSCFs can be efficiently employed in the design of MP FIR filters.

(4) Examples where it is shown that cascaded expanded CSCFs are useful to obtain the same stopband attenuation as cascaded expanded cosine filters but with a lower group delay: from these examples we see an improvement from 23% to 47% in the reduction of the group delay, at the cost of a slight increase of the usage of hardware resources. For an application in a decimation filter embedded into a low-delay oversampled Analog-to-Digital Converter (ADC), the proposed scheme exhibits a 24% lower group delay referred to high rate, with 35% less computational complexity (estimated in Additions per Output Sample) and slightly less usage of hardware elements.

Following this introduction, Section 2 presents the definition of CSCFs and cascaded expanded CSCFs. The proofs of MP characteristic in CSCFs and cascaded expanded CSCFs are given in Sections 3 and 4, respectively. In Section 5 we provide details on the characteristics and applications of the cascaded expanded CSCFs. Then, Section 6 presents examples and discussion of results. Finally, concluding remarks are given in Section 7.

2. Definition of Chebyshev-Sharpened Cosine Filter (CSCF) and Cascaded Expanded CSCF

We define the transfer function and the frequency response of an Nth order Chebyshev-Sharpened Cosine Filter (CSCF), respectively, as

$$H_{C,N}(z,\gamma) = \sum_{n=0}^{N} z^{-(N-n)/2} \cdot c_n \cdot [\gamma H(z)]^n, \quad (6)$$

$$H_{C,N}\left(e^{j\omega},\gamma\right) = \left[\sum_{n=0}^{N} c_n \cdot \left[\gamma \cos\left(\frac{\omega}{2}\right)\right]^n\right] e^{-j\omega N/2}, \quad (7)$$

with

$$\gamma \leq \frac{1}{\cos(\pi/2 - \pi/4R)}, \quad (8)$$

where c_n are the coefficients of the Chebyshev polynomial of first kind, represented in (5), and $H(z)$ is given in (1). To obtain a low-complexity multiplierless implementation, the constant γ must be expressible as a Sum of Powers of Two (SOPOT). To this end, we set

$$\gamma = f\left(2^{-B}\left\lfloor \frac{2^B}{\cos(\pi/2 - \pi/4R)} \right\rfloor, 1\right), \quad (9)$$

where $f(a,b)$ denotes "the closest value less than or equal to a that can be realized with at most b adders" and $\lfloor x \rfloor$ denotes rounding x to the closest integer less than or equal to x. To provide an improved attenuation around the zero of the cosine filter, γ must be as close as possible to its upper limit [8]. This is achieved by increasing the integer B. The value R in (8)-(9) is usually set as an integer equal to or greater than 2 for applications in decimation processes [8]. However, we will allow for more flexibility to the parameter R in this paper, as will be explained in the next section.

The transfer function and frequency response of a cascaded expanded CSCF are, respectively, defined as

$$G(z) = \prod_{m=1}^{M} \left[H_{C,N_m}(z^m, \gamma_m)\right]^{K_m}, \tag{10}$$

$$G(e^{j\omega}) = \left\{ \prod_{m=1}^{M} \left[\sum_{n=0}^{N_m} c_n \cdot \left[\gamma_m \cos\left(m \cdot \frac{\omega}{2}\right)\right]^n \right]^{K_m} \right\} \cdot e^{-j\omega \sum_{m=1}^{M} m \cdot K_m \cdot N_m/2}, \tag{11}$$

where the integer M indicates the number of cascaded CSCF blocks, each of them repeated K_m times, with $m = 1, 2, \ldots, M$. Every value of m is a distinct factor that expands a different CSCF whose corresponding order is N_m. These CSCFs have different factors γ_m, which can be obtained using (9), just replacing B by B_m and R by R_m, where B_m and R_m are integer parameters that correspond to the mth CSCF in the cascade. Figure 1(a) shows the structure of the CSCF, where we have that $d_i = c_{2i+\nu}$, with $i = 0, 1, 2, \ldots, D = (N - \nu)/2$ and with $\nu = 1$ if N is odd or $\nu = 0$ if N is even. Dashed blocks in Figure 1(a) appear only if N is odd. Figure 1(b) presents the structure of the cascaded expanded CSCF whose transfer function is given in (10).

3. Proof of Minimum Phase Property in CSCFs

The proof starts with the expression of the Chebyshev polynomial from (5) in the form of a product of first-order terms as [9]

$$P(x) = \sum_{n=0}^{N} c_n \cdot x^n = \prod_{n=1}^{N} (x - \sigma_n), \tag{12}$$

$$\sigma_n = \cos\left(\frac{\pi}{2} \cdot \frac{2n-1}{N}\right). \tag{13}$$

On the other hand, we rewrite the transfer function of the CSCF from (6) as

$$H_{C,N}(z, \gamma) = z^{-N/2} \sum_{n=0}^{N} c_n \cdot \left[z^{1/2} \gamma H(z)\right]^n. \tag{14}$$

Using (12), and after simple rearrangement of terms, we express $H_{C,N}(z, \gamma)$ as follows:

$$H_{C,N}(z, \gamma) = \prod_{n=1}^{N} \left[\gamma H(z) - z^{-1/2} \sigma_n\right], \tag{15}$$

which can be rewritten as

$$H_{C,N}(z, \gamma) = \begin{cases} \prod_{n=1}^{N/2} \left\{\left[\gamma H(z) - z^{-1/2}\sigma_n\right]\left[\gamma H(z) - z^{-1/2}\sigma_{N-(n-1)}\right]\right\}; & N \text{ even}, \\ \left[\gamma H(z) - z^{-1/2}\sigma_{\lceil N/2 \rceil}\right] \prod_{n=1}^{\lceil N/2 \rceil - 1} \left\{\left[\gamma H(z) - z^{-1/2}\sigma_n\right]\left[\gamma H(z) - z^{-1/2}\sigma_{N-(n-1)}\right]\right\}; & N \text{ odd}, \end{cases} \tag{16}$$

where $\lceil x \rceil$ denotes rounding x to the closest integer greater than or equal to x.

At this point, it is worth highlighting that the antisymmetry relations

$$\sigma_n = -\sigma_{N-(n-1)}, \quad n = 1, 2, \ldots, \left\lceil \frac{N}{2} \right\rceil, \qquad \sigma_{\lceil N/2 \rceil} = 0 \quad \text{for } N \text{ odd} \tag{17}$$

hold [9]. Thus, replacing (17) in (16), and after simple manipulation of terms, we have

$$H_{C,N}(z, \gamma) = \begin{cases} \prod_{n=1}^{N/2} Q_n(z); & N \text{ even}, \\ \gamma H(z) \prod_{n=1}^{\lceil N/2 \rceil - 1} Q_n(z); & N \text{ odd}, \end{cases} \tag{18}$$

$$Q_n(z) = \gamma^2 H^2(z) - \sigma_n^2 z^{-1}. \tag{19}$$

From (18) we have that $H_{C,N}(z, \gamma)$ consists of a product of either several terms $Q_n(z)$ if N is even or several terms $Q_n(z)$ and a term $\gamma H(z)$ if N is odd, with $n = 1, 2, \ldots, N$. Thus, to prove the MP property of the CSCF, it is only necessary to ensure that $Q_n(z)$ and $\gamma H(z)$ have MP characteristic for all values n.

Using (1), it is easy to see that the term $\gamma H(z)$ has a root on the unit circle and thus it corresponds to a MP filter. On the other hand, replacing (1) into (19) and after simple rearrangement of terms, we get

$$Q_n(z) = \frac{\gamma^2}{4}\left[1 - \left(\frac{4\sigma_n^2}{\gamma^2} - 2\right) z^{-1} + z^{-2}\right]. \tag{20}$$

From (20) it is easy to show that the roots of $Q_n(z)$ are placed on the unit circle; that is,

$$Q_n(z) = \left(1 - e^{j2\varphi_n} z^{-1}\right)\left(1 - e^{-j2\varphi_n} z^{-1}\right), \tag{21}$$

$$\varphi_n = \arccos\left(\sigma_n \cdot \gamma^{-1}\right), \tag{22}$$

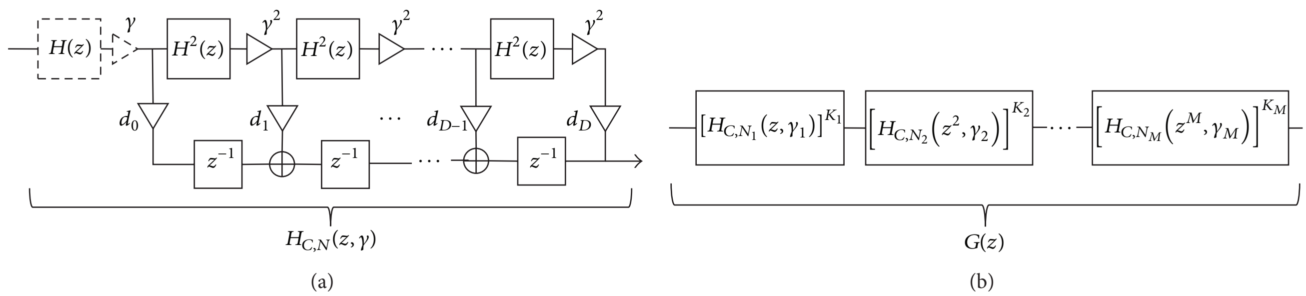

FIGURE 1: General structure of the filters: (a) Chebyshev-Sharpened Cosine Filter (CSCF); (b) cascaded expanded CSCF.

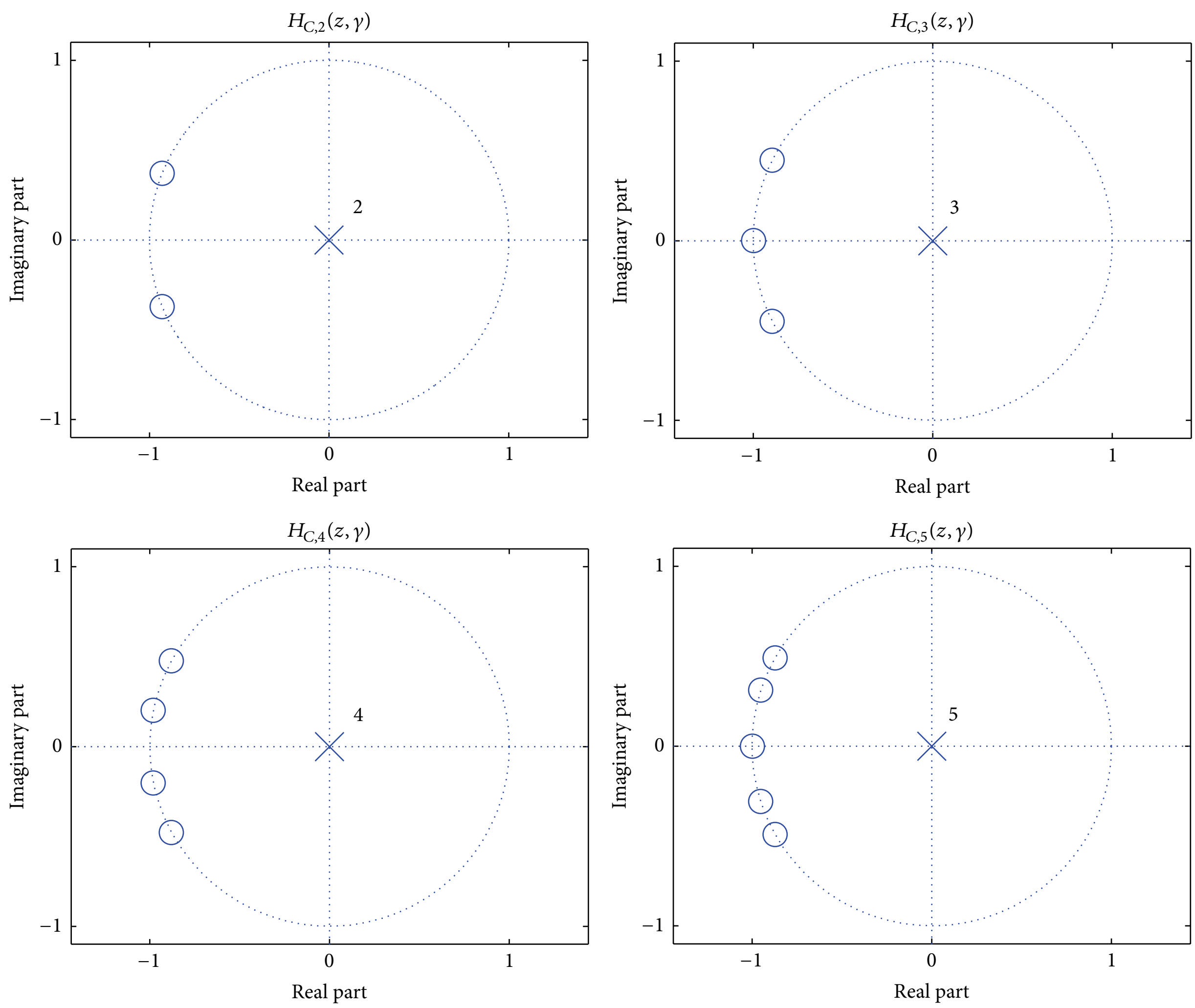

FIGURE 2: Pole-zero plots for CSCFs $H_{C,2}(z,\gamma)$, $H_{C,3}(z,\gamma)$, $H_{C,4}(z,\gamma)$, and $H_{C,5}(z,\gamma)$, where $\gamma = 2^{-3} \times 15$.

if the argument $\sigma_n \cdot \gamma^{-1}$ in (22) is preserved into the range $[-1, 1]$. From (13) we have that $-1 \leq \sigma_n \leq 1$ holds. Additionally, by setting

$$R \geq 0.5 \tag{23}$$

in (8)-(9), we ensure $\gamma \geq 1$. Under this condition for R, we have that $-1 \leq \gamma^{-1} \leq 1$ holds. In this case, $Q_n(z)$ has its roots on the unit circle for all the valid values n and, as a consequence, the filter $H_{C,N}(z,\gamma)$ has a MP characteristic.

Figure 2 shows the pole-zero plots for the filters $H_{C,2}(z,\gamma)$, $H_{C,3}(z,\gamma)$, $H_{C,4}(z,\gamma)$, and $H_{C,5}(z,\gamma)$. For all of these filters, we have $\gamma = 2^{-3} \times 15$, which is implemented with just one subtraction.

4. Proof of Minimum Phase Property in Cascaded Expanded CSCFs

The proof starts with the expression of every CSCF of the cascaded expanded CSCFs from (10) in the form of a product of second-order expanded transfer functions using (18) and (20); that is,

$$H_{C,N_m}\left(z^m,\gamma_m\right) = \begin{cases} \prod_{n=1}^{N_m/2} Q_n\left(z^m\right); & N_m \text{ even}, \\ \gamma_m H\left(z^m\right) \prod_{n=1}^{\lceil N_m/2\rceil-1} Q_n\left(z^m\right); & N_m \text{ odd}, \end{cases} \quad (24)$$

$$Q_n\left(z^m\right) = \frac{{\gamma_m}^2}{4}\left[1-\left(\frac{4\sigma_n^2}{\gamma_m^2}-2\right)z^{-m}+z^{-2m}\right], \quad (25)$$

where $m = 1,2,\ldots,M$ and $n = 1,2,\ldots,N_m$. Since the transfer function of the cascaded expanded CSCF from (10) consists of a product of several terms $[H_{C,N_m}(z^m,\gamma_m)]^{K_m}$ with different values m, it is only necessary to ensure that $H_{C,N_m}(z^m,\gamma_m)$ has a MP characteristic for all values m. Moreover, from (24) we see that $H_{C,N_m}(z^m,\gamma_m)$ is expressed as a product of either several terms $Q_n(z^m)$ if N_m is even or several terms $Q_n(z^m)$ and $\gamma_m H(z^m)$ if N_m is odd. Thus, to prove the MP property in cascaded expanded CSCFs we only need to ensure that $Q_n(z^m)$ and $\gamma_m H(z^m)$ have MP characteristic for all values n and m.

By replacing (1) in the term $\gamma_m H(z^m)$ and then making the resulting expression equal to zero, we can find the m roots of $\gamma_m H(z^m)$. These roots turn out to be the m complex roots of -1, which have unitary magnitude. Thus, $\gamma_m H(z^m)$ has MP characteristic, since its roots are placed on the unit circle. On the other hand, using (21) we can express (25) as follows:

$$Q_n\left(z^m\right) = \left(1-e^{j2\varphi_n}z^{-m}\right)\left(1-e^{-j2\varphi_n}z^{-m}\right), \quad (26)$$

$$\varphi_n = \arccos\left(\sigma_n \cdot \gamma_m^{-1}\right). \quad (27)$$

To preserve the argument $\sigma_n \cdot \gamma_m^{-1}$ in (27) into the range $[-1,1]$, we set

$$R_m \geq 0.5, \quad m = 1,2,\ldots,M. \quad (28)$$

Under this condition for R_m, we have that $-1 \leq \gamma_m^{-1} \leq 1$ holds. In this case, the respective m roots of factors $(1-e^{j2\varphi_n}z^{-m})$ and $(1-e^{-j2\varphi_n}z^{-m})$ in (25) are the m roots of the complex numbers $e^{j2\varphi_n}$ and $e^{-j2\varphi_n}$, which have unitary magnitude for all the valid values n. Therefore, $Q_n(z^m)$ has MP characteristic, since its roots are placed on the unit circle. Finally, since $Q_n(z^m)$ and $\gamma_m H(z^m)$ have MP characteristic, the overall cascaded expanded CSCF from (9), $G(z)$, also has MP characteristic.

Figure 3 shows the pole-zero plots for the filters $H_{C,2}(z^5,\gamma)$, $H_{C,3}(z^4,\gamma)$, $H_{C,4}(z^3,\gamma)$, and $H_{C,5}(z^2,\gamma)$. For all these filters, we have $\gamma = 2^{-3} \times 15$, which is implemented with just one subtraction.

5. Characteristics and Applications of Cascaded Expanded CSCFs

A cascaded expanded CSCF has both MP and LP characteristics. The former was proven in Section 4, whereas the latter is easily seen from the frequency response $G(e^{j\omega})$ given in (11). A consequence of this is that the cascaded expanded CSCF has a passband droop in its magnitude response. Due to this passband droop, the cascaded expanded CSCF should be employed only to provide a given attenuation requirement of an overall MP FIR filter over a prescribed stopband region (depending on the application). Thus, the resulting structure to design an overall MP FIR filter can be associated with the prefilter-equalizer scheme of [10], shown in Figure 4, where the prefilter provides the required attenuation whereas the equalizer corrects the passband droop of the prefilter. The cascaded expanded CSCF, with transfer function $G(z)$ defined in (10), can be used as prefilter. Note that since a cascaded expanded cosine filter (whose transfer function $H_A(z)$ is defined in (3)) also has both LP and MP properties, it is used as prefilter in [6].

Even though this paper is not focused on the design of the equalizer, it is worthwhile to spend some words on how this filter could be designed. An Infinite Impulse Response (IIR) filter with optimally located poles based, for example, in the Least Squares criterion as shown in [11] can form a proper equalizer. However, FIR filters are usually preferred over their IIR counterparts because they have guaranteed stability, they are free of limit-cycle oscillations, and their polyphase decomposition in multirate schemes allows them to reduce the computational load, among other characteristics [12]. Thus, we are more concerned here with FIR equalizers. Since a FIR equalizer with LP characteristic has its zeros placed in quadruplets around the unit circle [1], it does not accomplish the MP characteristic. Therefore, a MP FIR equalizer (i.e., that filter whose zeros appear inside the unit circle) does not have a Linear Phase.

When a LP FIR filter is designed by sharpening cascaded expanded cosine filters with the traditional sharpening polynomial $3x^2 - 2x^3$ from [7], the resulting filter has a prefilter given by $[H_A(z)]^2$ and a LP equalizer given by $[3z^{-D} - 2H_A(z)]$, where D is the group delay of $H_A(z)$ used to preserve the Linear Phase characteristic. In method [6] the delay D has been removed to obtain a MP FIR equalizer. Thus, a first option would be to use the same approach of [6] to design a FIR equalizer. However, it is worth highlighting that the recent LP droop compensators proposed in literature (see, e.g., [13, 14]) are novel low-complexity alternatives to the aforementioned LP equalizer based on the traditional sharpening. Inspired by these alternatives, a more convenient approach would be to design MP droop compensators as counterparts of the MP equalizer based on sharpening, proposed in [6].

Besides method [6], other design methods for MP FIR filters have been introduced, for example, in [15–22]. They can be classified in general terms as methods based on cepstrum [15–17] and methods based on the design of a LP FIR filter [18–22]. However, in general, these methods have the inconvenience of producing filtering solutions that require

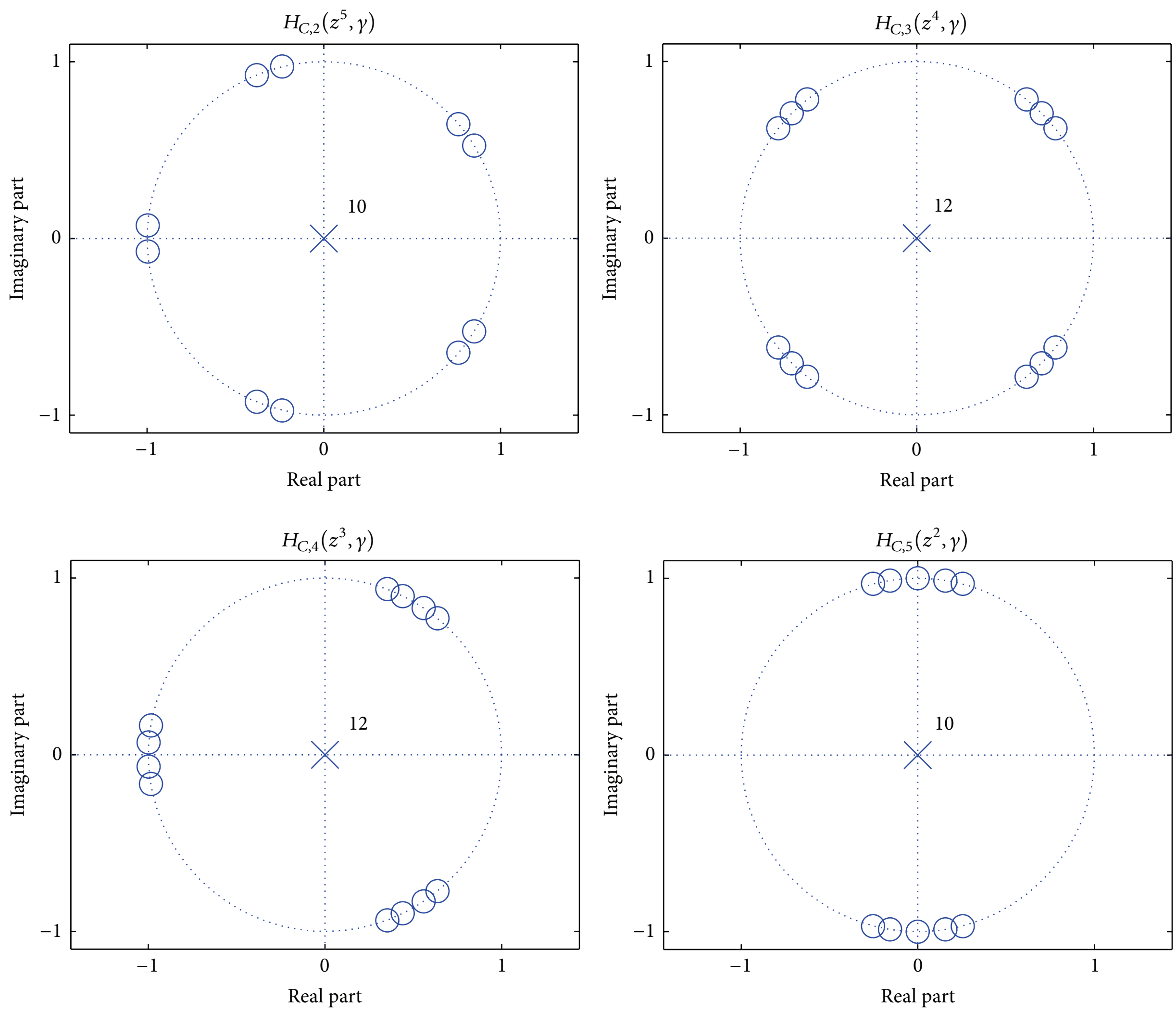

Figure 3: Pole-zero plots for cascaded expanded CSCFs $H_{C,2}(z^5, \gamma)$, $H_{C,3}(z^4, \gamma)$, $H_{C,4}(z^3, \gamma)$, and $H_{C,5}(z^2, \gamma)$, where $\gamma = 2^{-3} \times 15$.

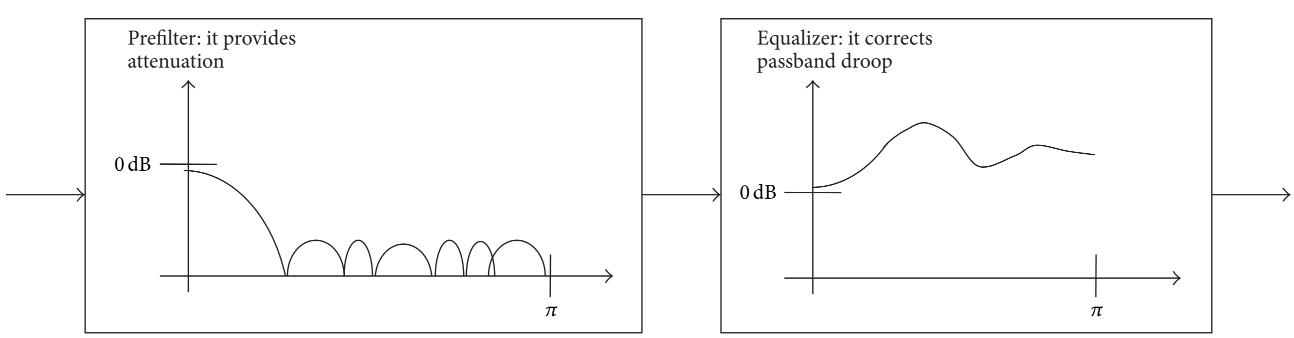

Figure 4: Prefilter-equalizer scheme to design an overall FIR filter. For a MP FIR filter design, the prefilter and the equalizer must be MP filters.

multipliers, which are the most costly elements in a digital filter [5]. To solve this problem, the cascaded expanded CSCF can be used as a prefilter to implement an overall MP FIR filter using several multiplierless CSCFs. A similar approach can be followed with the use of a cascaded expanded cosine filter from [6]. Nevertheless, as we mentioned earlier, the problem with method [6] is that the resulting filter requires a large cascade of expanded cosine filters, increasing the group delay of the resulting filter.

Finally, it is worth highlighting that, in comparison to the filter $H_A(z)$, the filter $G(z)$ has many more parameters, namely, N_m, K_m, B_m, and R_m, with $m = 1, 2, \ldots, M$,

Table 1: First half of the symmetric impulse response of the filter designed with method [6] in Example 1.

n	$h_A(n)$	n	$h_A(n)$	n	$h_A(n)$
1	0.000030517578125	9	0.004943847656250	17	0.037902832031250
2	0.000091552734375	10	0.007110595703125	18	0.043304443359375
3	0.000183105468750	11	0.009887695312500	19	0.048431396484375
4	0.000396728515625	12	0.013275146484375	20	0.052825927734375
5	0.000732421875000	13	0.017272949218750	21	0.056488037109375
6	0.001281738281250	14	0.021881103515625	22	0.058959960937500
7	0.002136230468750	15	0.026916503906250	23	0.060241699218750
8	0.003295898437500	16	0.032409667968750		

Table 2: First half of the symmetric impulse response of the proposed filter in Example 1.

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	0.000365884150812	7	0.014876445801089	13	0.055883940227800
2	0.001024551768820	8	0.020473561194634	14	0.061849547506437
3	0.002122204221257	9	0.027011012207314	15	0.066392852921837
4	0.003834694340150	10	0.034112182959393	16	0.069389704077153
5	0.006627799290290	11	0.041585018019638	17	0.070269686319927
6	0.010243501009105	12	0.049072257144308		

to be tuned in order to find a desired attenuation. This characteristic provides more flexibility for the design of MP FIR filters in comparison to $H_A(z)$. Moreover, by setting $M = K$, $N_m = 1$, $R_m = 0.5$, and $K_m = L$ for all m in (10), we obtain the same expression as (3). Thus, the cascaded expanded CSCF from (10) is a generalized case of (3).

6. Examples and Discussion

This section presents a couple of examples (Examples 1 and 2) that compare the cascaded expanded CSCFs $G(z)$ with cascaded expanded cosine filters $H_A(z)$ from [6]. This comparison is made in terms of

(a) group delay, measured in samples and defined as follows [1]:

$$\tau(\omega) = -\frac{d}{d\omega}\left\{\arg\left[F\left(e^{j\omega}\right)\right]\right\}, \tag{29}$$

where $F(e^{j\omega})$ is the frequency response of the corresponding filter;

(b) implementation complexity, measured in the required number of adders and delays for a given attenuation over a prescribed stopband region.

Additionally, an engineering application is provided in Example 3, namely, the antialiasing filtering process used in the first stage of a two-stage decimation structure applied in a low-delay Sigma-Delta ADC for audio systems, detailed in [3]. In this case, comparisons are made in terms of group delay referred to high rate, computational complexity counted in Additions per Output Sample (APOS), and number of hardware elements assuming that both filters, the one used in method [3] and the proposed filter, are implemented in recursive form.

Example 1. Design a MP FIR filter with minimum attenuation equal to 60 dB over the range from $\omega = 0.17\pi$ to $\omega = \pi$ (see Figure 1 of [6]).

In [6], the filter employed to accomplish such characteristic is obtained from (3) using $K = 5$ and $L = 3$. The group delay is obtained by replacing these values in (4) and then using (4) in (29). This filter requires 15 adders and 45 delays, but it has a group delay of 22.5 samples.

If we use $M = 4$, $N_1 = N_3 = N_4 = 3$, $N_2 = 4$, $R_1 = 3$, $R_2 = 1.5$, $R_3 = 0.9$, and $R_4 = 2$, with $B_m = 4$ and $K_m = 1$ for all m in (10), we get a filter whose group delay, obtained by replacing the aforementioned parameters in (11) and then using (11) in (29), is 16 samples, that is, nearly 30% less delay than that of [6]. Since this filter uses 30 adders and 44 delays, the price to pay is $100 \times \{[(30 + 44)/(15 + 45)] - 1\} \approx 23\%$ of additional implementation complexity. Figure 5 shows the magnitude responses and group delays of both filters. Moreover, Tables 1 and 2 present, respectively, the first half of the symmetric impulse response of the filter designed with method [6] and the proposed filter.

Example 2. Design a MP FIR filter with minimum attenuation equal to 100 dB over the range from $\omega = 0.15\pi$ to $\omega = \pi$ (see Example 1 of [6]).

In [6], the filter employed to accomplish such characteristic is obtained from (3) using $K = 7$ and $L = 4$. The group delay is obtained by replacing these values in (4) and then using (4) in (29). This filter requires 24 adders and 120 delays. However, its group delay is 56 samples.

By using $M = 4$, $N_1 = 4$, $N_2 = 6$, $N_3 = 4$, $N_4 = 8$, $R_1 = 6$, $R_2 = 2$, $R_3 = 0.8$, $R_4 = 1.2$, $B_1 = 2$, $B_2 = B_4 = 3$, $B_3 = 4$, and $K_m = 1$ for all m in (10), we get a filter whose group delay, obtained by replacing the aforementioned parameters in (11) and then

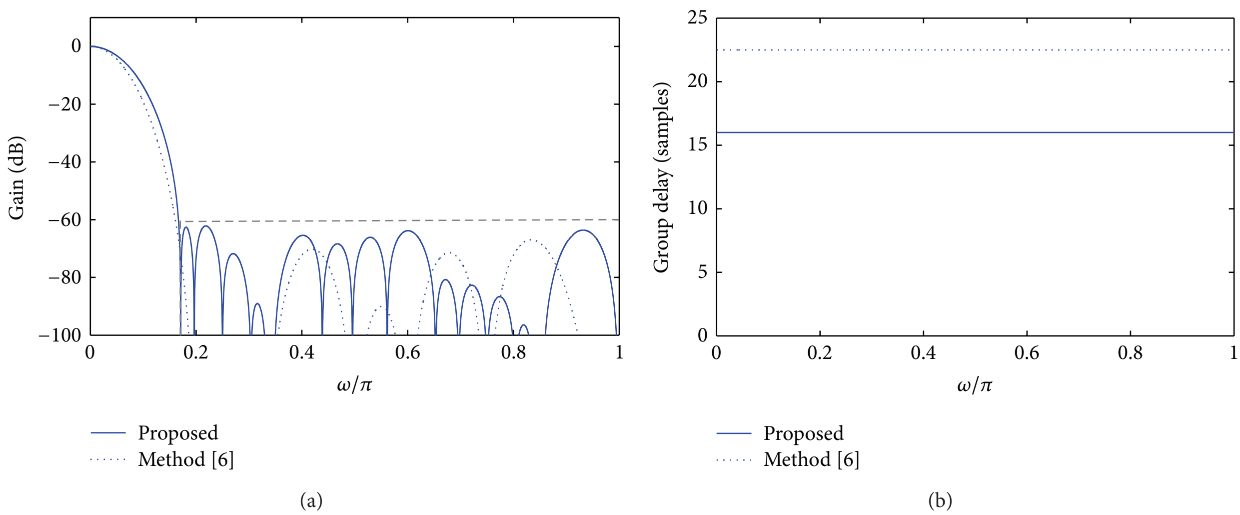

Figure 5: Magnitude responses (a) and group delays (b) of the cascaded expanded CSCF (10) and the cascaded expanded cosine filter from [6] (3), accomplishing the attenuation required in Example 1.

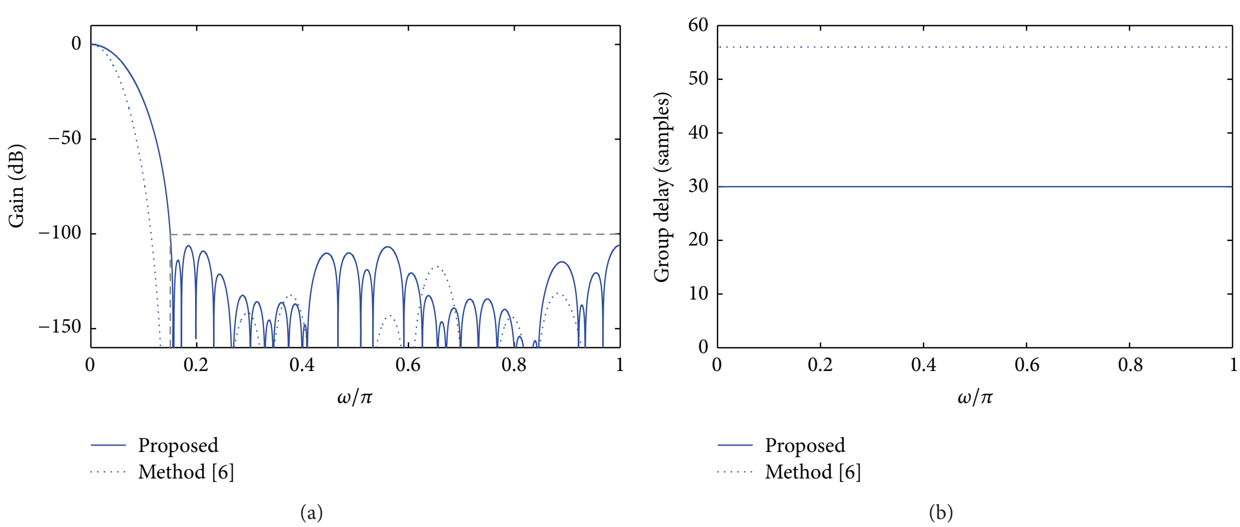

Figure 6: Magnitude responses (a) and group delays (b) of the cascaded expanded CSCF (10) and the cascaded expanded cosine filter from [6] (3), accomplishing the attenuation required in Example 2.

using (11) in (29), is 30 samples, that is, approximately 47% less delay than that of [6]. This filter uses 58 adders and 90 delays; thus the price to pay is a $100 \times \{[(58+90)/(24+120)] - 1\} \approx 3\%$ of additional implementation complexity. Figure 6 shows the magnitude responses and group delays of both filters. Tables 3 and 4, respectively, show the first half of the symmetric impulse response of the filter of method [6] and the proposed filter in Example 2.

Table 5 summarizes the results from the previous examples. From them we observe that the cascaded expanded CSCFs achieve a lower group delay in comparison to the cascaded expanded cosine filters from [6]. This characteristic, desirable for MP filters, occurs because CSCFs take advantage of the zero-rotation effect (see Figures 2 and 3), provided by Chebyshev sharpening, to achieve a given attenuation using less cascaded filters. Nevertheless, since every CSCF needs in general a few more hardware resources (adders and delays) than their cascaded cosine counterparts, the price to pay is an increase in the implementation complexity. Note, however, that the resulting filters still are low-complexity multiplierless solutions.

In the following example we show the design of the anti-aliasing filter used in the first stage of a two-stage decimation structure applied in a low-delay Sigma-Delta ADC for audio systems, where the second stage is a FIR filter with droop compensation characteristic followed by a downsampler by 2 [3].

Example 3. Design a MP FIR filter for a decimation factor $M = 32$ and residual decimation factor $R = 2$, with minimum

Table 3: First half of the symmetric impulse response of the filter of method [6] in Example 2.

n	$h_A(n)$	n	$h_A(n)$	n	$h_A(n)$
1	0.000000003725290	20	0.000197932124138	39	0.010904885828495
2	0.000000014901161	21	0.000268835574389	40	0.012328892946243
3	0.000000037252903	22	0.000360459089279	41	0.013833809643984
4	0.000000089406967	23	0.000477448105812	42	0.015406861901283
5	0.000000189989805	24	0.000624939799309	43	0.017032675445080
6	0.000000372529030	25	0.000808782875538	44	0.018693551421165
7	0.000000707805157	26	0.001035392284393	45	0.020369183272123
8	0.000001281499863	27	0.001311570405960	46	0.022037297487259
9	0.000002223998308	28	0.001644581556320	47	0.023674391210079
10	0.000003740191460	29	0.002041984349489	48	0.025255739688873
11	0.000006116926670	30	0.002511218190193	49	0.026756063103676
12	0.000009730458260	31	0.003059625625610	50	0.028150811791420
13	0.000015132129192	32	0.003694280982018	51	0.029415860772133
14	0.000023052096367	33	0.004421260207891	52	0.030528664588928
15	0.000034391880035	34	0.005245819687843	53	0.031469188630581
16	0.000050395727158	35	0.006171941757202	54	0.032219946384430
17	0.000072613358498	36	0.007201746106148	55	0.032766595482826
18	0.000102937221527	37	0.008335486054420	56	0.033098891377449
19	0.000143751502037	38	0.009571358561516	57	0.033210486173630

Table 4: First half of the symmetric impulse response of the proposed filter in Example 2.

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	0.000001616556925	12	0.002625166590998	23	0.030805600150354
2	0.000006351272540	13	0.003671733802111	24	0.034481874096793
3	0.000017617899954	14	0.004998834324957	25	0.037993214348607
4	0.000040689528112	15	0.006638831718593	26	0.041220432511362
5	0.000085616315587	16	0.008615391803219	27	0.044049047537715
6	0.000163942825303	17	0.010940958165390	28	0.046372196978920
7	0.000292679744504	18	0.013612477204781	29	0.048100695572177
8	0.00049325324069	19	0.016611117272267	30	0.049166485033492
9	0.000793906372572	20	0.019897013838157	31	0.049526751322073
10	0.001225195293310	21	0.023412420954797		
11	0.001822819899364	22	0.027079443485406		

Table 5: Comparison of results in Examples 1 and 2.

	Example 1		Example 2	
	Proposed	Method [6]	Proposed	Method [6]
Group delay (samples)	16	22.5	30	56
Complexity of implementation (number of adders/number of delays)	30/44	15/45	58/90	24/120
% improvement in group delay (compared with method [6])	≈30%	—	≈47%	—
% increase in complexity of implementation (compared with method [6])	≈23%	—	≈3%	—

attenuation equal to 95 dB over the range of frequencies from $\omega_{1,k}$ to $\omega_{2,k}$, where these frequencies are given by

$$\omega_{1,k} = \frac{\pi}{16}k - \frac{\pi}{64}, \quad \text{for } k = 1, 2, \ldots, 16,$$
$$\omega_{2,k} = \begin{cases} \dfrac{\pi}{16}k + \dfrac{\pi}{64}; & \text{for } k = 1, 2, \ldots, 15, \\ \pi; & \text{for } k = 16. \end{cases} \tag{30}$$

In [3], the filter employed to accomplish such characteristic is obtained from method [23]. Its transfer function $H_{\text{comb}}(z)$ and frequency response $H_{\text{comb}}(e^{j\omega})$ are, respectively, given as

$$H_{\text{comb}}(z) = \left[\frac{1 - z^{-32}}{1 - z^{-1}}\right]^{10} = \left[\sum_{k=0}^{31} z^{-k}\right]^{10}, \tag{31}$$

$$H_{\text{comb}}\left(e^{j\omega}\right) = \left[\frac{\sin(16\omega)}{\sin(\omega/2)}\right]^{10} e^{-j\omega(10\times 31)/2}. \tag{32}$$

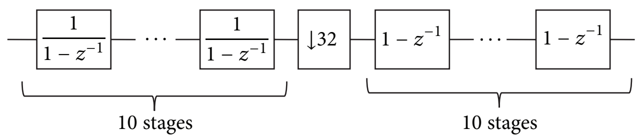

FIGURE 7: CIC structure for the first-stage decimation filter of Example 3, used in method [3].

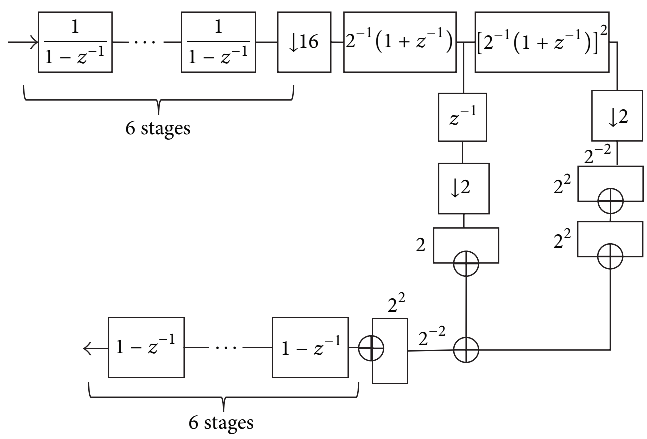

FIGURE 8: Proposed CIC-based structure for the first-stage decimation filter of Example 3.

This filter has a group delay of $(10 \times 31)/2 = 155$ samples. Moreover, its Cascaded Integrator-Comb (CIC) structure, shown in Figure 7, performs $10 \times 32 + 10 = 330$ Additions per Output Sample (APOS) and uses 20 adders and 20 delays.

Now consider the proposed filter, whose transfer function $H_{\text{prop}}(z)$ is given by

$$H_{\text{prop}}(z) = \left[\frac{1 - z^{-32}}{1 - z^{-1}}\right]^6 G(z) = \left[\sum_{k=0}^{31} z^{-k}\right]^6 G(z), \quad (33)$$

where $G(z)$ is an expanded CSCF given by

$$G(z) = \left[H_{C,3}\left(z^{16}, \gamma\right)\right], \quad \gamma = 2^{-2} \times 5. \quad (34)$$

The frequency response $H_{\text{prop}}(e^{j\omega})$ is given by

$$H_{\text{prop}}\left(e^{j\omega}\right) = \left[\frac{\sin(16\omega)}{\sin(\omega/2)}\right]^6 \cdot \left\{\sum_{n=0}^{3} c_n \cdot \left[\gamma \cos\left(16 \cdot \frac{\omega}{2}\right)\right]^n\right\} \cdot e^{-j\omega[(6\times31)/2+8\times3]}. \quad (35)$$

This filter has a group delay of $[(6 \times 31)/2 + 8 \times 3] = 117$ samples, which is approximately 24% less delay than that of [3]. The filter $G(z)$ can be moved after a downsampling by 16 because it is actually a CSCF expanded by 16. Thus, the resulting CIC-based structure, shown in Figure 8, performs $(6 \times 32 + 6) + (3 \times 2 + 5 + 6) = 215$ Additions per Output Sample (APOS), that is, nearly 35% less computational complexity with regard to the filter used in [3]. Moreover, this filter uses 20 adders and 16 delays, which represents 10% lower usage of hardware resources compared with method [3].

Figure 9 shows the magnitude responses and group delays referred to high rate of both the filter used in method [3] and the proposed filter. Tables 6 and 7 show the first half of the symmetric impulse response referred to high rate of the filters obtained with method [3] and the proposed method, respectively, whereas the summary of results is given in Table 8.

It is worth highlighting that the implementation of the comb decimation filter in a CIC structure has been employed in method [3] due to its regularity and simplicity, which has a low usage of hardware resources (see Figure 7). However, the price to pay for such desirable characteristics is a high computational complexity. Our proposed solution has taken advantage of the possibility of factorize the decimation factor $M = 32$ as $M = 16 \times 2$. With this we have used an expanded-by-16 CSCF as an additional filter that contributes to improving the attenuation in the first stopband, where the comb filter has the worst attenuation. The first advantage of doing so is observed in the reduction of the number of Integrator-Comb stages from 10 to 6. Moreover, since the CSCF can operate at a sampling rate reduced by 16, the computational complexity of the decimation process is reduced and the number of hardware elements is also reduced. Of course one can resort to other types of architectures, such as the nonrecursive form of the comb filter and its subsequent polyphase decomposition. However, this decreases the computational complexity at the cost of a considerable increase of the number of hardware elements.

Table 6: First half of the symmetric impulse response of the filter of method [3], referred to high rate, in Example 3.

n	$h_{\text{comb}}(n)$	n	$h_{\text{comb}}(n)$	n	$h_{\text{comb}}(n)$
1	0.000000000000001	53	0.000015313628992	105	0.003067434040211
2	0.000000000000009	54	0.000017891039557	106	0.003254277926132
3	0.000000000000049	55	0.000020842073902	107	0.003447970844412
4	0.000000000000195	56	0.000024211829441	108	0.003648431922869
5	0.000000000000635	57	0.000028049632732	109	0.003855552947449
6	0.000000000001778	58	0.000032409292032	110	0.004069197518604
7	0.000000000004445	59	0.000037349349356	111	0.004289200321715
8	0.000000000010161	60	0.000042933330064	112	0.004515366520934
9	0.000000000021592	61	0.000049229987809	113	0.004747471285302
10	0.000000000043183	62	0.000056313542416	114	0.004985259455488
11	0.000000000082048	63	0.000064263908115	115	0.005228445358849
12	0.000000000149178	64	0.000073166909260	116	0.005476712779839
13	0.000000000261062	65	0.000083114480507	117	0.005729715091984
14	0.000000000441798	66	0.000094204848246	118	0.005987075556853
15	0.000000000725811	67	0.000106542689959	119	0.006248387794461
16	0.000000001161297	68	0.000120239268071	120	0.006513216428630
17	0.000000001814526	69	0.000135412534785	121	0.006781097909681
18	0.000000002775158	70	0.000152187204385	122	0.007051541515750
19	0.000000004162737	71	0.000170694789450	123	0.007324030532772
20	0.000000006134560	72	0.000191073597477	124	0.007598023611877
21	0.000000008895111	73	0.000213468684471	125	0.007872956301611
22	0.000000012707302	74	0.000238031762123	126	0.008148242750906
23	0.000000017905744	75	0.000264921055368	127	0.008423277577236
24	0.000000024912339	76	0.000294301107232	128	0.008697437892806
25	0.000000034254466	77	0.000326342528076	129	0.008970085480113
26	0.000000046586074	78	0.000361221686589	130	0.009240569106890
27	0.000000062712022	79	0.000399120340084	131	0.009508226969148
28	0.000000083616030	80	0.000440225201999	132	0.009772389249925
29	0.000000110492611	81	0.000484727444753	133	0.010032380780283
30	0.000000144783421	82	0.000532822136527	134	0.010287523788204
31	0.000000188218447	83	0.000584707610845	135	0.010537140720208
32	0.000000242862512	84	0.000640584768306	136	0.010780557119830
33	0.000000311167585	85	0.000700656310233	137	0.011017104546500
34	0.000000396031394	86	0.000765125904472	138	0.011246123517914
35	0.000000500862858	87	0.000834197284103	139	0.011466966458595
36	0.000000629654824	88	0.000908073280357	140	0.011679000637130
37	0.000000787064622	89	0.000986954791600	141	0.011881611074397
38	0.000000978502888	90	0.001071039690856	142	0.012074203405099
39	0.000001210231111	91	0.001160521674976	143	0.012256206674994
40	0.000001489468318	92	0.001255589059213	144	0.012427076056412
41	0.000001824507242	93	0.001356423521699	145	0.012586295464958
42	0.000002224840312	94	0.001463198803009	146	0.012733380060741
43	0.000002701295694	95	0.001576079366785	147	0.012867878617966
44	0.000003266183590	96	0.001695219028193	148	0.012989375747416
45	0.000003933452897	97	0.001820759557685	149	0.013097493957082
46	0.000004718858246	98	0.001952829268212	150	0.013191895537066
47	0.000005640137379	99	0.002091541594584	151	0.013272284255899
48	0.000006717198673	100	0.002236993674174	152	0.013338406856462
49	0.000007972318561	101	0.002389264938603	153	0.013390054340938
50	0.000009430348458	102	0.002548415726373	154	0.013427063035536
51	0.000011118930675	103	0.002714485926685	155	0.013449315427145
52	0.000013068722691	104	0.002887493664900	156	0.013456740765633

Table 7: First half of the symmetric impulse response of the proposed filter, referred to high rate, in Example 3.

n	$h_{\text{prop}}(n)$	n	$h_{\text{prop}}(n)$	n	$h_{\text{prop}}(n)$
1	0.000000000132290	41	0.000200623662253	81	0.005950250387700
2	0.000000000793741	42	0.000227922965264	82	0.006248059555550
3	0.000000002778093	43	0.000258185937789	83	0.006551124134355
4	0.000000007408248	44	0.000291642986915	84	0.006858858787878
5	0.000000016668557	45	0.000328534155745	85	0.007170639614659
6	0.000000033337115	46	0.000369109128687	86	0.007485806476325
7	0.000000061118044	47	0.000413627236743	87	0.007803665325892
8	0.000000104773790	48	0.000462357462807	88	0.008123490536077
9	0.000000170257408	49	0.000515576579015	89	0.008444527227601
10	0.000000264844857	50	0.000573567284102	90	0.008765993597494
11	0.000000397267286	51	0.000636616340754	91	0.009087083247406
12	0.000000577843325	52	0.000705012712966	92	0.009406967511909
13	0.000000818611377	53	0.000779045703398	93	0.009724797786806
14	0.000001133461906	54	0.000859003090723	94	0.010039707857438
15	0.000001538269730	55	0.000945169266991	95	0.010350816226988
16	0.000002051026306	56	0.001037823374976	96	0.010657228444788
17	0.000002692305398	57	0.001137237445536	97	0.010958041789391
18	0.000003485728733	58	0.001243674534966	98	0.011252349951643
19	0.000004458431662	59	0.001357386862351	99	0.011539247717751
20	0.000005641528829	60	0.001478613946926	100	0.011817835652354
21	0.000007070579820	61	0.001607580745423	101	0.012087224781598
22	0.000008786054836	62	0.001744495789436	102	0.012346541276202
23	0.000010833800347	63	0.001889549322765	103	0.012594931134531
24	0.000013265504756	64	0.002042911438779	104	0.012831564865668
25	0.000016139164059	65	0.002204730201894	105	0.013055642172483
26	0.000019519547509	66	0.002375129769048	106	0.013266396634704
27	0.000023478663272	67	0.002554208511190	107	0.013463100391990
28	0.000028096224097	68	0.002742037134753	108	0.013645068827001
29	0.000033460112966	69	0.002938656803136	109	0.013811665248464
30	0.000039666848765	70	0.003144077258185	110	0.013962305574255
31	0.000046822051941	71	0.003358274941671	111	0.014096463014456
32	0.000055040910163	72	0.003581191116775	112	0.014213672754439
33	0.000064448183614	73	0.003812729989559	113	0.014313531954857
34	0.000075178210285	74	0.004052756830457	114	0.014395699751648
35	0.000087374911263	75	0.004301096095746	115	0.014459897256033
36	0.000101191796024	76	0.004557529549030	116	0.014505907554518
37	0.000116791967726	77	0.004821794382720	117	0.014533575708893
38	0.000134348128499	78	0.005093581339514	118	0.014542808756232
39	0.000154042584737	79	0.005372532833876		
40	0.000176067252389	80	0.005658241073516		

7. Conclusion

In this paper we have presented the mathematical demonstration that the application of Chebyshev sharpening to cosine filters results in filters with zeros on the unit circle, that is, with Minimum Phase (MP) characteristic. From this, we have proven that filters composed by a cascade of Chebyshev-Sharpened Cosine Filters (CSCFs) expanded by different factors, called cascaded expanded CSCFs, also have MP property. The cascaded expanded CSCFs are useful prefilters that provide the attenuation in an overall MP FIR filter. Moreover, these filters are a general case where the cascaded expanded cosine filters are a subset. The CSCFs are low-complexity filters, since they do not need multipliers.

It has been shown with three examples that, for a desired attenuation in the magnitude response, cascaded expanded CSCFs achieve a lower group delay in comparison to cascaded expanded cosine filters. This lower group delay is desirable in the design of MP FIR filters. Since the purpose of this paper is to prove the suitability of cascaded expanded

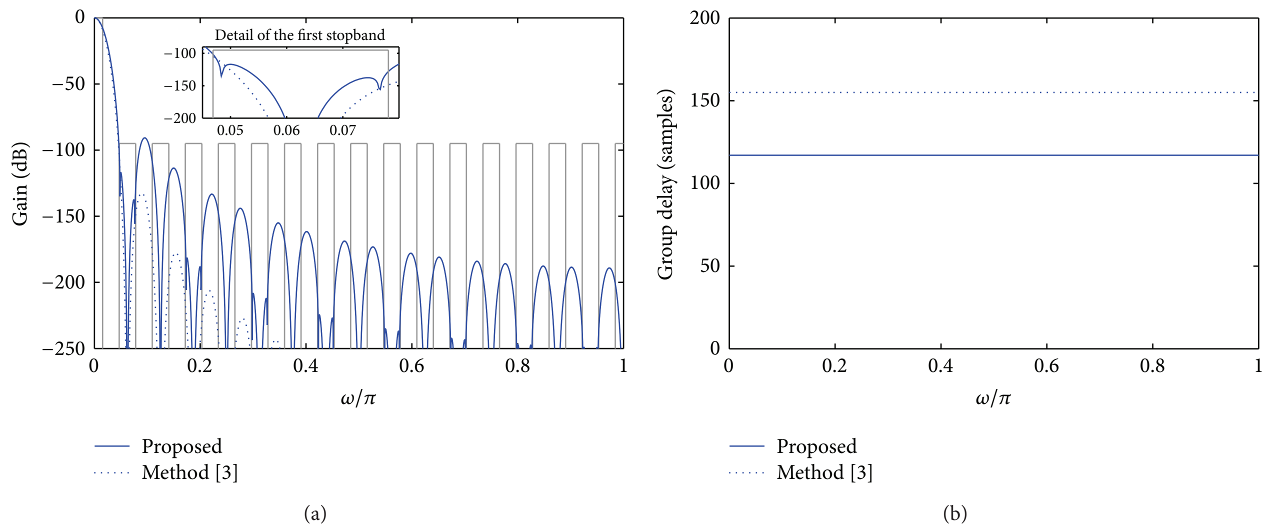

Figure 9: Magnitude responses (a) and group delays (b) of the proposed decimation filter and the filter from [3] (31), accomplishing the attenuation required in Example 3.

Table 8: Comparison of results in Example 3.

	Proposed	Method [3]
Group delay (samples) referred to high rate	117	155
Complexity of implementation (number of adders/number of delays)	20/16	20/20
Computational complexity (APOS)	215	330
% improvement in group delay (compared with method [3])	≈24%	—
% saving in APOS (compared with method [3])	≈35%	—
% saving in complexity of implementation (compared with method [3])	≈10%	—

CSCFs as MP prefilters, the CSCF-based solutions provided in the examples of this work are suboptimal.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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