Research Article

On the Asymptotic Behavior of Positive Solutions of Certain Fractional Differential Equations

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This paper deals with the asymptotic behavior of positive solutions of certain forced fractional differential equations of the form ${}^C D_c^\alpha y(t) = e(t) + f(t, x(t))$, $c > 1$, $\alpha \in (0, 1)$, where $y(t) = (a(t)x'(t))'$, $c_0 = y(c)/\Gamma(1) = y(c)$, and c_0 is a real constant. From the obtained results, we derive a technique which can be applied to some related fractional differential equations.

1. Introduction

Consider the forced fractional differential equation

$${}^C D_c^\alpha y(t) = e(t) + f(t, x(t)), \quad c > 1, \alpha \in (0, 1), \quad (1)$$

where $y(t) = (a(t)x'(t))'$, $c_0 = y(c)/\Gamma(1) = y(c)$, and c_0 is a real constant.

In the sequel we assume that

- (i) $a : [c, \infty) \rightarrow \mathbb{R}^+ = (0, \infty)$ is a continuous function;
- (ii) $f : [c, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and assume that there exists a continuous function;
 $h : [c, \infty) \rightarrow (0, \infty)$ and a real number, λ , $0 < \lambda \leq 1$ and γ is a real number such that

$$0 \leq x f(t, x) \leq t^{\gamma-1} h(t) |x|^{\lambda+1} \quad \text{for } x \neq 0, t \geq c. \quad (2)$$

We only consider those solutions of (1) that are continuous and nontrivial in any neighborhood of ∞ . Such solution is said to be oscillatory if there exists a sequence $\{t_n\} \subseteq [c, \infty)$, $t_n \rightarrow \infty$ such that $x(t_n) = 0$, and it is nonoscillatory otherwise.

In the last few decades, integral, integrodifferential, and fractional differential equations have gained considerably more attention due to their applications in many engineering and scientific disciplines as the mathematical models for systems and processes in fields of physics, mechanics, chemistry,

aerodynamics, and the electrodynamics of complex media. For more details one can refer to [1–8].

Oscillation and asymptotic behavior results for integral, integrodifferential, and fractional differential equations are scarce; some results can be found in [5, 9–13]. It seems that there are no such results for forced fractional differential equations of type (1). The main objective of this paper is to establish some new criteria on the asymptotic behavior of positive solutions of (1) which is equivalent to the Volterra type equation:

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} [e(s) + f(s, x(s))] ds, \quad (3)$$

$$c > 1, \alpha > 0.$$

From the obtained results, we derive a technique which can be applied to some related fractional differential equations.

We note that

$${}^C D_a^\alpha x(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds \quad (4)$$

is the Caputo derivative of the order $\alpha \in (n-1, n)$ of a C -scalar valued function $x(t)$ defined on the interval $[c, \infty)$, $x^{(n)}(t) = d^n x(t)/dt^n$. For the case when $\alpha \in (n-1, n)$, this definition has been given by Caputo [14]. For the definition of the Caputo derivative of order $\alpha \in (n-1, n)$, $n \geq 1$, see [1, 15, 16].

2. Main Results

To obtain our main results of this paper, we need the following two lemmas.

Lemma 1 (see [5, 7]). *Let β , γ , and p be positive constants such that $[p(\beta - 1) + 1] > 0$, $p(\beta - 1) + 1 > 0$. Then*

$$\int_0^t (t-s)^{p(\beta-1)} s^{p(\gamma-1)} ds = t^\theta B, \quad t \geq 0, \quad (5)$$

where $B := B[p(\gamma - 1) + 1, p(\beta - 1) + 1]$, $B[\xi, \eta] = \int_0^1 s^{\xi-1} (1-s)^{\eta-1} ds$, $(\xi > 0, \eta > 0)$, and $\theta = p(\beta + \gamma - 2) + 1$.

Lemma 2 (see [17]). *If X and Y are nonnegative, then*

$$X^\lambda - (1-\lambda)Y^\lambda - \lambda XY^{\lambda-1} \leq 0, \quad 0 < \lambda < 1, \quad (6)$$

where equality hold if and only if $X = Y$.

In what follows, we let

$$g(t) = \frac{((1-\lambda)\lambda^{\lambda/(1-\lambda)})}{\Gamma(\alpha)} \cdot \int_{t_1}^t (t-s)^{\alpha-1} (s^{\gamma-1} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)) ds \quad (7)$$

and $0 < \lambda < 1$, $t \geq t_1$ for some $t_1 \geq c$, where $m : [c, \infty) \rightarrow (0, \infty)$ is a continuous function.

Now we give sufficient conditions under which any solution x of (1) satisfies $x(t) = O(t^2)$ as $t \rightarrow \infty$.

Theorem 3. *Let $0 < \lambda < 1$ and suppose that $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$ and $\gamma = 2 - \alpha - 1/p$, $p(\alpha - 1) + 1 > 0$, $p(\gamma - 1) + 1 > 0$, and $t/a(t)$ is bounded on $[c, \infty)$,*

$$\int_{t_1}^\infty \frac{s}{a(s)} ds < \infty, \quad (8)$$

$$\int_{t_1}^\infty (s^2 m(s))^q ds < \infty, \quad (9)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t^2} \int_c^t \frac{1}{a(v)} \int_{t_1}^v \int_c^u (u-s)^{\alpha-1} e(s) ds du dv < \infty, \quad (10)$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t^2} \int_c^t \frac{1}{a(v)} \int_{t_1}^v \int_c^u (u-s)^{\alpha-1} e(s) ds du dv > -\infty,$$

$$\lim_{t \rightarrow \infty} \frac{1}{t^2} \int_{t_1}^t \frac{1}{a(u)} \int_{t_1}^u g(s) ds du < \infty, \quad (11)$$

where the function $g(t)$ is defined as in (7) for any $t_1 \geq c$. If x is a positive solution of (1), then $\limsup_{t \rightarrow \infty} (x(t)/t^2) < \infty$.

Proof. Let $x(t)$ be an eventually positive solution of (1). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (i) and (ii) we may then write

$$\begin{aligned} (a(t)x'(t))' &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} e(s) ds + \frac{1}{\Gamma(\alpha)} \\ &\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} [h(s)x^\lambda(s) - m(s)x(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s)x(s) ds. \end{aligned} \quad (12)$$

Applying (6) of Lemma 2 to $[h(s)x^\lambda(s) - m(s)x(s)]$ with

$$\begin{aligned} X &= (h)^{1/\lambda} x, \\ Y &= \left(\frac{1}{\lambda} m(h^{-1/\lambda}) \right)^{1/(\lambda-1)} \end{aligned} \quad (13)$$

we have

$$\begin{aligned} h(s)x^\lambda(s) - m(s)x(s) \\ \leq (1-\lambda)\lambda^{\lambda/(1-\lambda)} m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s), \end{aligned} \quad (14)$$

and hence we obtain

$$\begin{aligned} (a(t)x'(t))' &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} e(s) ds + \frac{((1-\lambda)\lambda^{\lambda/(1-\lambda)})}{\Gamma(\alpha)} \\ &\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} [m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s)x(s) ds. \end{aligned} \quad (15)$$

Integrating inequality (15) from t_1 to t and interchanging the order of integration, one can easily obtain

$$\begin{aligned} x'(t) &\leq \frac{a(t_1)x'(t_1)}{a(t)} + \frac{c_0(t-t_1)}{a(t)} + \frac{1}{a(t)} \int_{t_1}^t g(s) ds \\ &+ \frac{1}{a(t)\Gamma(\alpha)} \int_{t_1}^t \int_{t_1}^u (u-s)^{\alpha-1} s^{\gamma-1} m(s)x(s) ds du \\ &+ \frac{1}{a(t)\Gamma(\alpha)} \int_{t_1}^t \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du \\ &+ \frac{1}{a(t)\Gamma(\alpha)} \int_{t_1}^t \int_c^u (u-s)^{\alpha-1} e(s) ds du. \end{aligned} \quad (16)$$

Interchanging the order of integration in second integral, we have

$$\begin{aligned} x'(t) &\leq \frac{a(t_1)x'(t_1)}{a(t)} + \frac{c_0(t-t_1)}{a(t)} + \frac{1}{a(t)} \int_{t_1}^t g(s) ds \\ &\quad + k \int_{t_1}^t (t-s)^\alpha s^{\gamma-1} m(s) x(s) ds \\ &\quad + \frac{1}{a(t)\Gamma(\alpha)} \int_{t_1}^t \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du \\ &\quad + \frac{1}{a(t)\Gamma(\alpha)} \int_{t_1}^t \int_c^u (u-s)^{\alpha-1} e(s) ds du, \end{aligned} \quad (17)$$

where k is the upper bound of the function $1/\alpha\Gamma(\alpha)a(t)$.

Integrating (17) from t_1 to t and interchanging the order of integration in the last integral, we find

$$\begin{aligned} x(t) &\leq x(t_1) + a(t_1)x'(t_1) \left(\int_{t_1}^t \frac{1}{a(s)} ds \right) \\ &\quad + \int_{t_1}^t \frac{c_0(s-t_1)}{a(s)} ds + \int_{t_1}^t \frac{1}{a(u)} \int_{t_1}^u g(s) ds du \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{1}{a(v)} \int_{t_1}^v \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du dv \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_c^t \frac{1}{a(v)} \int_{t_1}^v \int_c^u (u-s)^{\alpha-1} e(s) ds du dv \\ &\quad + \frac{k}{\alpha+1} \left(\int_{t_1}^t (t-s)^{\alpha+1} m(s) s^{\gamma-1} x(s) ds \right). \end{aligned} \quad (18)$$

Now, one can easily see that

$$\begin{aligned} x(t) &\leq x(t_1) + a(t_1)x'(t_1) \left(\int_{t_1}^t \frac{1}{a(s)} ds \right) \\ &\quad + \int_{t_1}^t \frac{c_0(s-t_1)}{a(s)} ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{1}{a(v)} \int_{t_1}^v \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du dv \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_c^t \frac{1}{a(v)} \int_{t_1}^v \int_c^u (u-s)^{\alpha-1} e(s) ds du dv \\ &\quad + \int_{t_1}^t \frac{1}{a(u)} \int_{t_1}^u g(s) ds du \\ &\quad + k^* \left(\int_{t_1}^t (t-s)^{\alpha+1} s^{\gamma-1} m(s) x(s) ds \right), \end{aligned} \quad (19)$$

where $k^* = (k/(\alpha+1))$. From the hypotheses of the theorem, we see that

$$\begin{aligned} z(t) &:= \frac{x(t)}{t^2} \\ &\leq C + k^* \left(\int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds \right), \end{aligned} \quad (20)$$

with C as the upper bound of the function

$$\begin{aligned} &\frac{1}{t^2} \left[x(t_1) + a(t_1)x'(t_1) \left(\int_{t_1}^t \frac{1}{a(s)} ds \right) \right. \\ &\quad + \int_{t_1}^t \frac{c_0(s-t_1)}{a(s)} ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_c^t \frac{1}{a(v)} \int_{t_1}^v \int_c^u (u-s)^{\alpha-1} e(s) ds du dv \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \frac{1}{a(v)} \int_{t_1}^v \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du dv \\ &\quad \left. + \int_{t_1}^t \frac{1}{a(u)} \int_{t_1}^u g(s) ds du \right]. \end{aligned} \quad (21)$$

Applying Holder's inequality and Lemma 1, we obtain

$$\begin{aligned} &\int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds \\ &\leq \left(\int_{t_1}^t (t-s)^{p(\alpha-1)} s^{p(\gamma-1)} ds \right)^{1/p} \\ &\quad \cdot \left(\int_{t_1}^t m^q(s) x^q(s) ds \right)^{1/q} \\ &\leq \left(\int_0^t (t-s)^{p(\alpha-1)} s^{p(\gamma-1)} ds \right)^{1/p} \\ &\quad \cdot \left(\int_{t_1}^t m^q(s) x^q(s) ds \right)^{1/q} \leq (Bt^\theta)^{1/p} \\ &\quad \cdot \left(\int_{t_1}^t m^q(s) x^q(s) ds \right)^{1/q}, \end{aligned} \quad (22)$$

where $B = B[p(\gamma-1)+1, p(\alpha-1)+1]$ and $\theta = p(\alpha+\gamma-2)+1 = 0$ and so

$$\begin{aligned} &\int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds \\ &\leq B^{1/p} \left(\int_{t_1}^t m^q(s) x^q(s) ds \right)^{1/q}. \end{aligned} \quad (23)$$

Thus, inequality (20) becomes

$$\begin{aligned} z(t) &:= \frac{x(t)}{t^2} \leq C + k^* B^{1/p} \left(\int_{t_1}^t m^q(s) x^q(s) ds \right)^{1/q} \\ &= C + k^* B^{1/p} \left(\int_{t_1}^t s^{2q} m^q(s) \left(\frac{x(s)}{s^2} \right)^q ds \right)^{1/q} \\ &= C + k^* B^{1/p} \left(\int_{t_1}^t s^{2q} m^q(s) z^q(s) ds \right)^{1/q}. \end{aligned} \quad (24)$$

Using this inequality and the elementary inequality

$$(x + y)^q \leq 2^{q-1} (x^q + y^q), \quad x, y \geq 0, \quad q > 1, \quad (25)$$

we obtain from (24)

$$\begin{aligned} z^q(t) &\leq 2^{q-1} \left((1 + C)^q + k^q B^{q/p} \int_{t_1}^t s^{2q} m^q(s) z^q(s) ds \right). \end{aligned} \quad (26)$$

If we denote $u(t) = z^q(t)$, that is, $z(t) = u^{1/q}(t)$, $P = 2^{q-1}[(1 + C)^q]$, and $Q = 2^{q-1}k^q B^{q/p}$, then

$$u(t) \leq P + Q \int_{t_1}^t s^{2q} m^q(s) u(s) ds, \quad t \geq t_1 \geq c. \quad (27)$$

The conclusions follow from Gromwell's inequality and conclude that

$$\limsup_{t \rightarrow \infty} \frac{x(t)}{t^2} < \infty. \quad (28)$$

This completes the proof. $\square$

Remark 4. Condition (10) can be replaced by

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t^2} \int_c^t \frac{1}{a(v)} \int_{t_1}^v \int_c^u (u-s)^{\alpha-1} |e(s)| ds du dv \\ < \infty \end{aligned} \quad (29)$$

and the result remains valid.

The following example is illustrative.

Example 5. Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly

$$\begin{aligned} p(\alpha - 1) + 1 &= p(\gamma - 1) + 1 = p \left(1 - \frac{1}{2p} - 1 \right) + 1 \\ &= \frac{1}{2} > 0, \\ \theta &= p(\alpha + \gamma - 2) + 1 = 0. \end{aligned} \quad (30)$$

Let the functions $a(t)$ and $b(t)$ be as in (i) and (ii) with $b(t)$ being a bounded function and let $a(t) = e^t$, $e(t) = te^t \sin t$,

and $f(t, x) = t^{\gamma-1} h(t) x^\lambda$, where $0 < \lambda < 1$ and $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with $h(t) = m(t)$,

$$\begin{aligned} \int_{t_1}^\infty (s^2 h(s))^q ds &< \infty, \\ \limsup_{t \rightarrow \infty} \frac{1}{t^2} \left(\int_{t_1}^t e^{-u} \int_{t_1}^u (u-s)^{\alpha-1} s^{\gamma-1} h(s) ds du \right) &< \infty. \end{aligned} \quad (31)$$

Condition (10) is also fulfilled. Thus, all conditions of Theorem 3 are satisfied and hence every positive solution x of (1) satisfies $\limsup_{t \rightarrow \infty} (x(t)/t^2) < \infty$.

Next, we consider the fractional differential equation

$${}_C D_c^\alpha y(t) = e(t) + f(t, x(t)), \quad c > 1, \quad \alpha \in (0, 1), \quad (32)$$

where $y(t) = x'(t)$, $c_0 = y(c)/\Gamma(1) = y(c)$, and c_0 is a real constant.

Now we give sufficient conditions under which any positive solution x of (32) satisfies $x(t) = O(t)$ as $t \rightarrow \infty$.

Theorem 6. Let $0 < \lambda < 1$ and suppose that $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$ and $\gamma = 2 - \alpha - 1/p$, $p(\alpha-1) + 1 > 0$, and $p(\gamma-1) + 1 > 0$,

$$\int_{t_1}^\infty (sm(s))^q ds < \infty, \quad (33)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t \int_c^u (t-s)^{\alpha-1} e(s) ds du < \infty, \quad (34)$$

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t \int_c^u (t-s)^{\alpha-1} e(s) ds du > -\infty,$$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{t_1}^t g(s) ds < \infty, \quad (35)$$

$g(t)$ is defined as in (7),

for any $t_1 \geq c$. If x is a positive solution of (32), then $\limsup_{t \rightarrow \infty} (x(t)/t) < \infty$.

Proof. Let $x(t)$ be an eventually positive solution of (32). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (ii) we may then write

$$\begin{aligned} x'(t) &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds + \frac{1}{\Gamma(\alpha)} \\ &\cdot \int_c^t (t-s)^{\alpha-1} e(s) ds + \frac{1}{\Gamma(\alpha)} \\ &\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} [h(s) x^\lambda(s) - m(s) x(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds. \end{aligned} \quad (36)$$

Proceeding as in the proof of Theorem 3, we obtain

$$\begin{aligned}
 x(t)' &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds + \frac{1}{\Gamma(\alpha)} \\
 &\cdot \int_c^t (t-s)^{\alpha-1} e(s) ds + \frac{((1-\lambda)\lambda^{\lambda/(1-\lambda)})}{\Gamma(\alpha)} \\
 &\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} [m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)] ds \\
 &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds.
 \end{aligned} \quad (37)$$

Integrating inequality (37) from t_1 to t and interchanging the order of integration, one can easily obtain

$$\begin{aligned}
 x(t) &\leq x(t_1) + c_0(t-t_1) + \int_{t_1}^t g(s) ds + \frac{1}{\Gamma(\alpha)} \\
 &\cdot \int_{t_1}^t \int_{t_1}^u (u-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds du \frac{1}{\Gamma(\alpha)} \\
 &\cdot \int_{t_1}^t \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du + \frac{1}{\Gamma(\alpha)} \\
 &\cdot \int_{t_1}^t \int_c^u (t-s)^{\alpha-1} e(s) ds du.
 \end{aligned} \quad (38)$$

Interchanging the order of integration in second integral we have

$$\begin{aligned}
 x(t) &\leq x(t_1) + c_0(t-t_1) + \int_{t_1}^t g(s) ds \\
 &+ \frac{t}{\alpha\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds \\
 &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \int_c^{t_1} (t_1-s)^{\alpha-1} |F(s)| ds du \\
 &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \int_c^u (t-s)^{\alpha-1} e(s) ds du.
 \end{aligned} \quad (39)$$

The rest of the proof is similar to that of Theorem 3 and hence is omitted. $\square$

Example 7. Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly

$$\begin{aligned}
 p(\alpha-1)+1 &= p(\gamma-1)+1 = p\left(1-\frac{1}{2p}-1\right)+1 \\
 &= \frac{1}{2} > 0, \\
 \theta &= p(\alpha+\gamma-2)+1 = 0.
 \end{aligned} \quad (40)$$

Let the functions $e(t) = t \sin t$ and $f(t, x) = t^{\gamma-1} h(t) x^\lambda$, where $0 < \lambda < 1$ and $h: R^+ \rightarrow R^+$ is a continuous function with $h(t) = m(t)$,

$$\begin{aligned}
 &\int_c^\infty (sh(s))^q ds < \infty, \\
 &\limsup_{t \rightarrow \infty} \frac{1}{t} \left(\int_{t_1}^u (u-s)^{\alpha-1} s^{\gamma-1} h(s) ds \right) < \infty.
 \end{aligned} \quad (41)$$

Condition (34) is also fulfilled. Thus, all conditions of Theorem 8 are satisfied and hence every positive solution x of (32) satisfies $\limsup_{t \rightarrow \infty} (x(t)/t) < \infty$.

Finally, we consider the fractional differential equation

$${}_C D_c^\alpha x(t) = e(t) + f(t, x(t)), \quad c > 1, \alpha \in (0, 1), \quad (42)$$

where $x(c) = c_0$ and c_0 is a real constant.

Now, we give sufficient conditions for the boundedness of any positive solution of (42).

Theorem 8. Let $0 < \lambda < 1$ and suppose that $p > 1$, $q = p/(p-1)$, $0 < \alpha < 1$ and $\gamma = 2 - \alpha - 1/p$, $p(\alpha-1)+1 > 0$, and $p(\gamma-1)+1 > 0$,

$$\begin{aligned}
 &\int_{t_1}^\infty m^q(s) ds < \infty, \\
 &\limsup_{t \rightarrow \infty} \int_{t_1}^t (t-s)^{\alpha-1} e(s) ds < \infty, \\
 &\liminf_{t \rightarrow \infty} \int_{t_1}^t (t-s)^{\alpha-1} e(s) ds > -\infty, \\
 &\lim_{t \rightarrow \infty} g(t) < \infty, \\
 &g(t) \text{ is defined as in (7),}
 \end{aligned} \quad (43)$$

for any $t_1 \geq c$. If x is a positive solution of (42), then $x(t)$ is bounded.

Proof. Let $x(t)$ be an eventually positive solution of (42). We may assume that $x(t) > 0$ for $t \geq t_1$ for some $t_1 \geq c$. We let $F(t) = f(t, x(t))$. In view of (ii) we may then write

$$\begin{aligned}
 x(t) &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds + \frac{1}{\Gamma(\alpha)} \\
 &\cdot \int_c^t (t-s)^{\alpha-1} e(s) ds + \frac{1}{\Gamma(\alpha)} \\
 &\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} [h(s) x^\lambda(s) - m(s) x(s)] ds \\
 &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds.
 \end{aligned} \quad (44)$$

Proceeding as in the proof of Theorem 3, we obtain

$$\begin{aligned} x(t) &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds + \frac{1}{\Gamma(\alpha)} \\ &\cdot \int_c^t (t-s)^{\alpha-1} e(s) ds + \frac{((1-\lambda)\lambda^{\lambda/(1-\lambda)})}{\Gamma(\alpha)} \\ &\cdot \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} [m^{\lambda/(\lambda-1)}(s) h^{1/(1-\lambda)}(s)] ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds \end{aligned} \quad (45)$$

or

$$\begin{aligned} x(t) &\leq c_0 + \frac{1}{\Gamma(\alpha)} \int_c^{t_1} (t-s)^{\alpha-1} |F(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_c^t (t-s)^{\alpha-1} e(s) ds + g(t) \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} m(s) x(s) ds. \end{aligned} \quad (46)$$

The rest of the proof is similar to that of Theorem 3 and hence is omitted. $\square$

Example 9. Let $p > 1$, $0 < \alpha = 1 - 1/2p < 1$, $\alpha = \gamma$, and $q = p/(p-1)$. Clearly

$$\begin{aligned} p(\alpha-1) + 1 &= p(\gamma-1) + 1 = p\left(1 - \frac{1}{2p} - 1\right) + 1 \\ &= \frac{1}{2} > 0, \\ \theta &= p(\alpha + \gamma - 2) + 1 = 0. \end{aligned} \quad (47)$$

Let the functions $e(t) = \sin t$ and $f(t, x) = t^{\gamma-1} h(t) x^\lambda$, where $0 < \lambda < 1$ and $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with $h(t) = m(t)$,

$$\begin{aligned} \int_0^\infty h^q(s) ds &< \infty, \\ \limsup_{t \rightarrow \infty} \left(\int_{t_1}^t (t-s)^{\alpha-1} s^{\gamma-1} h(s) ds \right) &< \infty. \end{aligned} \quad (48)$$

Condition (34) is also fulfilled. Thus, all conditions of Theorem 8 are satisfied and hence every positive solution of (42) is bounded.

Similar reasoning to that in the sublinear case guarantees the following theorems for the integrodifferential equation (1) when $\lambda = 1$.

Theorem 10. Let $\lambda = 1$ and the hypotheses of Theorems 3–8 hold with $m(t) = h(t)$.

Then the conclusion of Theorems 3–8 holds.

General Remarks

- (1) The results of this paper are presented in a form which is essentially new and it can also be obtained for higher fractional differential equations (1) of order $\alpha \in (n-1, n)$, $n \geq 1$.
- (2) It would be of interest to study (1) when f satisfies condition (ii) with $\lambda > 1$.

Conflict of Interests

The author declares that they have no competing interests.

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