Research Article

Multidromion Soliton and Rouge Wave for the $(2 + 1)$ -Dimensional Broer-Kaup System with Variable Coefficients

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Broad new families of rational form variable separation solutions with two arbitrary lower-dimensional functions of the $(2 + 1)$ -dimensional Broer-Kaup system with variable coefficients are derived by means of an improved mapping approach and a variable separation hypothesis. Based on the derived variable separation excitation, some new special types of localized solutions such as rouge wave, multidromion soliton, and soliton vanish phenomenon are revealed by selecting appropriate functions of the general variable separation solution.

1. Introduction

Searching for explicit solutions of nonlinear evolution equations by using various different methods is useful and meaningful in physical science and nonlinear science. Many powerful methods have been presented, such as inverse scattering transform [1], Hirota's bilinear form method [2], two-soliton method [3], homoclinic test technique [4–6], the Bäcklund transformation method [7], three-wave type of ansatz approach [8, 9], projective equation method [10], and multilinear variable separation method [11].

In the line with the development of symbolic computation, much work has been focused on the various extensions and application of the known algebraic methods to construct solutions of nonlinear evolution equations [12–14].

In this paper, we will apply a projective equation method [15] with a variable separation hypothesis to look for new families of variable separation solutions to the $(2 + 1)$ -dimensional Broer-Kaup system with variable coefficients (vcBK) [16]:

$$\begin{aligned} u_{yt} - B(t) [u_{xxy} - 2(uu_x)_y - 2v_{xx}] &= 0, \\ v_t + B(t) [v_{xx} + 2(uv)_x] &= 0, \end{aligned} \quad (1)$$

where $B(t)$ is an arbitrary function of time t and $B(t) \neq 0$. When $B(t) = 1$, the system reduces to the celebrated $(2 + 1)$ -dimensional Broer-Kaup system.

The vcBK system (1) is an important mathematical model in nonlinear optics, plasma physics, and statistical physics [16–18]. In our work, we apply a projective equation method and a variable separation hypothesis to the BK equations (1) and obtain its exact excitations. We give some selected oscillating solitons, multidromion soliton, and cross-like fractal structures by selecting appropriate functions in the general variable separation solution of the vcBK equations to demonstrate some interesting outcomes most of which are new by comparing with the solutions of the references.

2. The Nontraveling Wave Solutions of the vcBK Equations

By letting $v = u_y$, (1) can be converted into a new equation

$$u_{yt} + B(t) [u_{xxy} + 2(uu_x)_y] = 0. \quad (2)$$

Suppose that the solution of (2) can be expressed as follows:

$$u = a_0(X) + \sum_{i=1}^n \frac{a_i(X)}{F^i(\phi(X))} + \sum_{i=1}^n \frac{b_i(X) F^i(\phi(X))}{(1 + r(x, t) F(\phi(X)))^i}, \quad (3)$$

where $a_0(X)$, $a_i(X)$, $b_i(X)$ ($i = 1, 2, \dots, n$), and $\phi(X)$ are all functions of X to be determined, $r(x, t)$ is arbitrary function of $\{x, t\}$, and $F(\phi(X))$ satisfies the Riccati equation

$$F'(\phi(X)) = \delta + F^2(\phi(X)) \quad (4)$$

with δ an arbitrary constant. Notice that the Riccati equation (4) possesses the following solutions:

- (i) when $\delta = 0$, $F(\phi(X)) = -1/\phi(X)$;
- (ii) when $\delta < 0$, $F(\phi(X)) = -\sqrt{-\delta} \tanh(\sqrt{-\delta}\phi(X))$;
- (iii) when $\delta > 0$, $F(\phi(X)) = \sqrt{\delta} \tan(\sqrt{\delta}\phi(X))$.

By balancing the linear term of the highest order with the nonlinear term in (2), we get $n = 1$; then according to the ansatz (3), the solution of (2) reads

$$u(x, y, t) = a_0(x, y, t) + \frac{a_1(x, y, t)}{F(\phi(x, y, t))} + \frac{a_2(x, y, t) F(\phi(x, y, t))}{1 + r(x, t) F(\phi(x, y, t))}, \quad (5)$$

where $a_0(x, y, t)$, $a_1(x, y, t)$, $a_2(x, y, t)$, and $\phi(x, y, t)$ are arbitrary functions of (x, y, t) to be determined, $r(x, t)$ is arbitrary functions of $\{x, t\}$ to be determined, and F satisfies (4). Inserting (5) with (4) into (2), selecting the variable separation ansatz

$$\phi = p(x, t) + q(y) \quad (6)$$

with $r(x, t) = 0$ (for simplicity), and eliminating all the coefficients of polynomials of F , one gets a set of partial differential equations as follows:

$$6B\delta^2 a_1^2 p_x q_y - 6B\delta^3 a_1 p_x^2 q_y = 0,$$

$$6Ba_2^2 p_x q_y + 6Ba_2 p_x^2 q_y = 0,$$

$$2\delta^2 a_1 p_t q_y + 4B\delta^2 a_0 a_1 p_x q_y + 2B\delta^2 a_1 p_{xx} q_y - 4B\delta a_1 a_{1y} p_x + 2B\delta^2 a_{1y} p_x^2 - 4B\delta a_1 a_{1x} q_y + 4B\delta^2 a_{1x} p_x q_y = 0,$$

$$2a_2 p_t q_y + 4Ba_0 a_2 p_x q_y + 2Ba_2 p_{xx} q_y + 4Ba_2 a_{2y} p_x + 2Ba_{2y} p_x^2 + 4Ba_2 a_{2x} q_y + 4Ba_{2x} p_x q_y = 0,$$

$$8B\delta a_1^2 p_x q_y - 8B\delta^2 a_1 p_x^2 q_y - \delta a_{1t} q_y - 2B\delta a_1 a_{0y} p_x - \delta a_{1y} p_t - 2B\delta a_0 a_{1y} p_x - B\delta a_{1y} p_{xx} - 2B\delta a_1 a_{0x} q_y - 2B\delta a_0 a_{1x} q_y + 2Ba_{1x} a_{1y} + 2Ba_1 a_{1xy} - 2B\delta a_{1xy} p_x - B\delta a_{1xx} q_y = 0,$$

$$8B\delta a_2^2 p_x q_y + 8B\delta a_2 p_x^2 q_y + a_{2t} q_y + 2Ba_2 a_{0y} p_x + a_{2y} p_t + 2Ba_0 a_{2y} p_x + Ba_{2y} p_{xx} + 2Ba_2 a_{0x} q_y + 2Ba_0 a_{2x} q_y + 2Ba_{2x} a_{2y} + 2Ba_2 a_{2xy} + 2Ba_{2xy} p_x + Ba_{2xx} q_y = 0,$$

$$2\delta a_1 p_t q_y + 4B\delta a_0 a_1 p_x q_y + 2B\delta a_1 p_{xx} q_y - 4Ba_1 a_{1y} p_x + 2B\delta a_{1y} p_x^2 + a_{1yt} + 2Ba_{0x} a_{1y} - 4Ba_1 a_{1x} q_y + 4B\delta a_{1x} p_x q_y + 2Ba_{0y} a_{1x} + 2Ba_{0xy} a_1 + 2Ba_0 a_{1xy} + Ba_{1xy} = 0,$$

$$2\delta a_2 p_t q_y + 4B\delta a_0 a_2 p_x q_y + 2B\delta a_2 p_{xx} q_y + 4B\delta a_2 a_{2y} p_x + 2B\delta a_{2y} p_x^2 + a_{2yt} + 2Ba_{0x} a_{2y} + 4B\delta a_2 a_{2x} q_y + 4B\delta a_{2x} p_x q_y + 2Ba_{0y} a_{2x} + 2Ba_{0xy} a_2 + 2Ba_0 a_{2xy} + Ba_{2xy} = 0,$$

$$2Ba_1^2 p_x q_y + 2B\delta^2 a_2^2 p_x q_y - 2B\delta a_1 p_x^2 q_y + 2B\delta^2 a_2 p_x^2 q_y - a_{1t} q_y + \delta a_{2t} q_y - 2Ba_1 a_{0y} p_x + 2B\delta a_2 a_{0y} p_x - a_{1y} p_t - 2Ba_0 a_{1y} p_x - Ba_{1y} p_{xx} + \delta a_{2y} p_t + 2B\delta a_0 a_{2y} p_x + B\delta a_{2y} p_{xx} + a_{0yt} - 2Ba_{0x} a_1 q_y + 2B\delta a_0 a_{2x} q_y + 2Ba_{1y} a_{2x} + 2Ba_0 a_{0xy} + 2Ba_{1xy} a_2 - 2Ba_{1xy} p_x + 2Ba_1 a_{2xy} + 2B\delta a_{2xy} p_x - Ba_{1xx} q_y + B\delta a_{2xx} q_y + Ba_{0xy} = 0. \quad (7)$$

Solving the set of differential equations simultaneously, we obtain the following results.

Case 1. Consider

$$a_0 = -\frac{p_t + B(t) p_{xx}}{2B(t) p_x}, \quad a_1 = \delta p_x, \quad a_2 = 0, \quad r = 0. \quad (8)$$

Case 2. Consider

$$a_0 = -\frac{p_t + B(t) p_{xx}}{2B(t) p_x}, \quad a_1 = 0, \quad a_2 = -p_x, \quad r = 0. \quad (9)$$

Case 3. Consider

$$a_0 = -\frac{p_t + B(t) p_{xx}}{2B(t) p_x}, \quad a_1 = \delta p_x, \quad a_2 = -p_x, \quad r = 0, \quad (10)$$

where $p = p(x, t)$, $q = q(y)$ and $r = r(x, t)$ are three arbitrary functions of $\{x, t\}$, $\{y\}$, $\{x, t\}$, respectively, and δ is an arbitrary constant.

Next we only list some new rational variable separation solutions by using Case 3; however, Case 2 is the results in [19, 20]; we omit them for simplification. Therefore, we can obtain exact solutions for (1) as follows.

(I) When $\delta < 0$, we can derive the following solitary wave solution:

$$u_1(x, y, t) = -\frac{p_t + B(t) p_{xx}}{2B(t) p_x} + \frac{\sqrt{-\delta} p_x \left(1 + \tanh^2 \left[\sqrt{-\delta} (p(x, t) + q(y)) \right] \right)}{\tanh \left[\sqrt{-\delta} (p(x, t) + q(y)) \right]}, \quad (11)$$

$$v_1(x, y, t) = 4\delta p_x q_y \operatorname{csch}^2 \left[2\sqrt{-\delta} (p(x, t) + q(y)) \right], \quad (12)$$

with two arbitrary functions being $p(x, t)$ and $q(y)$.

(II) When $\delta > 0$, we can obtain the following periodic wave solution:

$$u_2(x, y, t) = -\frac{p_t + B(t) p_{xx}}{2B(t) p_x} + \frac{\sqrt{\delta} p_x \left(1 - \tan^2 \left[\sqrt{\delta} (p(x, t) + q(y)) \right] \right)}{\tan \left[\sqrt{\delta} (p(x, t) + q(y)) \right]},$$

$$v_2(x, y, t) = -4\delta p_x q_y \operatorname{csc}^2 \left[2\sqrt{\delta} (p(x, t) + q(y)) \right], \quad (13)$$

with two arbitrary functions being $p(x, t)$, $q(y)$.

(III) When $\delta = 0$, then we can derive the following variable separation solution:

$$u_3(x, y, t) = -\frac{p_t + B(t) p_{xx}}{2B(t) p_x} - \frac{p_x}{p(x, t) + q(y)},$$

$$v_3(x, y, t) = -\frac{p_x q_y}{(p + q)^2}, \quad (14)$$

with two arbitrary functions being $p(x, t)$ and $q(y)$.

By using Mathematica, we verify that all the solutions listed above satisfy the original equation (2).

3. New Localized Excitations

Owing to the arbitrary functions $p(x, t)$ and $q(y)$ involved in the solutions (11)–(14), it is convenient to excite soliton structure. After some calculations, we construct a new class of structures, such as the multidromion soliton, rouge wave, and the cross-like fractal solitons. Here, we take the solution (12) of the vcBK equations (1) as an example to study the soliton structure excitation.

Case 1. When $p(x, t)$ and $q(y)$ possess the following forms:

$$p(x, t) = 1 + 0.25 \operatorname{sech}(0.5x + t) + 0.1 \operatorname{sech}(0.2x + 2t), \quad (15)$$

$$q(y) = 1 + \operatorname{sech}(4 + y) + 0.5 \operatorname{sech}(4 - y),$$

respectively, we can obtain the multidromion soliton structures of the $(2 + 1)$ -dimensional vcBK equation. Figure 1(a) clearly indicates that the multidromion moves backwards and forwards over the same path in $x - y$ plane, oscillating in x -direction.

Case 2. If $p(x, t)$ and $q(y)$ are taken as

$$p(x, t) = \frac{1}{4}(x + t)^2, \quad q(y) = \frac{1}{8} + \frac{1}{2}y^2, \quad (16)$$

respectively, then we obtain a peakon-type rouge wave structure which is exhibited in Figure 1(b).

Case 3. When $p(x, t)$ and $q(y)$ are choosing as

$$p(x, t) = 1 + \tanh(x + t), \quad q(y) = 1 - \tanh(y), \quad (17)$$

respectively, the physical quantity v shows the soliton vanish phenomenon. Figure 2(a) shows that when $T = x + t < 0$, there is a soliton, and as $T = x + t$ increases over 0, it vanishes very soon.

Case 4. When $p(x, t)$ and $q(y)$ possess the following forms:

$$p(x, t) = \exp\left(-\frac{1}{4}x^2 + 2t - 0.5\right),$$

$$q(y) = \exp\left(-\frac{1}{2}y^2\right), \quad (18)$$

respectively, we have oscillating soliton-type structure. Their plots are presented in Figure 2(b).

FIGURE 1: (a) Multidromion soliton ($\delta = -1$, $t = 0$). (b) Rouge wave ($\delta = -4$, $t = 0$).

FIGURE 2: (a) Soliton vanish phenomenon ($\delta = -1$, $t = 0$). (b) Oscillating soliton-type structure ($\delta = -1$, $t = 0$).

Case 5. If $p(x, t)$ and $q(y)$ are selecting as

$$\begin{aligned} p(x, t) &= 4 \exp[(2x + t) \cos(x + t) - 10], \\ q(y) &= 2 \exp[y - 10], \end{aligned} \quad (19)$$

respectively, we obtain a dromion of periodic oscillation structure which is shown in Figure 3(a).

Case 6. When $p(x, t)$ and $q(y)$ possess the following forms:

$$p(x, t) = \operatorname{sech}[2(x + t)^2], \quad q(y) = \exp(-2y), \quad (20)$$

respectively, we obtain the parabola-type kink solitary wave structure which is shown in Figure 3(b).

Case 7. When $p(x, t)$ and $q(y)$ possess the following forms:

$$p(x, t) = -2x \ln(x^4 + t^2), \quad q(y) = 4 - y \ln(y^4), \quad (21)$$

respectively, we obtain the cross-like fractal structures which we can see reported in [21].

Figures 4(a) and 4(b) give the figures of the solution (12) with the settings blow, but x, y in, respectively, $[-5 \times 10^{-6}, 5 \times 10^{-6}]$ and $[-5 \times 10^{-12}, 5 \times 10^{-12}]$. The essential property of the fractal structures is the similarity of the figures in different axis scales. Figure 4 demonstrates that the cross-like fractal soliton holds its similarity in different ranges of x, y .

4. Conclusion

In this paper, we applied an improved mapping method and a variable separation hypothesis to the $(2 + 1)$ -dimensional vcBK equations and obtained a general variable separation with two arbitrary functions. Based on the general variable separation solution, abundant novel localized excitations, such as oscillating soliton, multidromion soliton, rouge wave, and cross-like fractal structures, have been constructed. The arbitrary functions in the obtained solutions imply that these solutions have rich spatial structures. And it may be helpful in future studies for the intricate nature world. This method can be also extended to the other higher dimensional nonlinear equations.

FIGURE 3: (a) Dromion of periodic oscillation structure ($\delta = -1$, $t = 0$). (b) Parabola-type kink solitary wave structure ($\delta = -4$, $t = 0$).

FIGURE 4: Cross-like fractal structures ($\delta = -1$, $t = 0$): (a) $x \in (-5 \times 10^{-6}, 5 \times 10^{-6})$, $y \in (-5 \times 10^{-6}, 5 \times 10^{-6})$ and (b) $x \in (-5 \times 10^{-12}, 5 \times 10^{-12})$, $y \in (-5 \times 10^{-12}, 5 \times 10^{-12})$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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