Research Article

A Fourth-Order Conservative Compact Finite Difference Scheme for the Generalized RLW Equation

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The generalized regularized long-wave (GRLW) equation is studied by finite difference method. A new fourth-order energy conservative compact finite difference scheme was proposed. It is proved by the discrete energy method that the compact scheme is solvable, the convergence and stability of the difference schemes are obtained, and its numerical convergence order is $O(\tau^2 + h^4)$ in the L^∞ -norm. Further, the compact schemes are conservative. Numerical experiment results show that the theory is accurate and the method is efficient and reliable.

1. Introduction

In this paper, we consider the following generalized regularized long-wave equation:

$$u_t + u_x + \alpha(u^p)_x - \beta u_{xxt} = 0, \quad (x, t) \in (x_l, x_r) \times (0, T], \quad (1)$$

with the boundary conditions

$$u(x_l, t) = u(x_r, t) = 0, \quad t \in (0, T], \quad (2)$$

and the initial condition

$$u(x, 0) = u_0(x), \quad x \in (x_l, x_r), \quad (3)$$

where $u = u(x, t)$ is a real-valued function defined on $(x_l, x_r) \times (0, T]$, $\alpha, \beta > 0$, $p \geq 2$ is a positive integer, and u_0 is a given function with Dirichlet boundary condition.

The GRLW equation was first put forward as a model for small-amplitude long waves on the surface of water in a channel by Peregrine [1, 2]. A special case of (1), that is,

$$u_t + u_x + \alpha u u_x - \beta u_{xxt} = 0, \quad (4)$$

is usually called the regularized long-wave (RLW) equation. The RLW equation is a representation form of nonlinear long wave and can describe a lot of important physical

phenomena, such as shallow waves and ionic waves. The GRLW equation can also describe that wave motion to the same order of approximation as the KDV equation, so it plays a major role in the study of nonlinear dispersive waves [3]. It is difficult to find the analytical solution for (1), which has been studied by many researchers. The finite difference method for the initial-boundary value problem of the GRLW equation had been studied in [4–8]. Other mathematical theory and numerical methods for GRLW equation were considered in [9–11]. Reference [12] solved the GRLW equation by the Petrov-Galerkin method. Numerical solution of GRLW equation used Sinc-collocation method in [13]. In [14], a time-linearization method that uses a Crank-Nicolson procedure in time and three-point, fourth-order accurate in space, compact difference equations, is presented and used to determine the solutions of the generalized regularized-long wave (GRLW) equation. Recently, there has been growing interest in high-order compact methods for solving partial differential equations [15–17].

In this paper, we consider problem (1)–(3); it has the following conservation law:

$$E(t) = \|u\|_{L^2}^2 + \beta \|u_x\|_{L^2}^2 = \text{const}. \quad (5)$$

Using a customary designation, we will refer to the functional $E(t)$ as the energy integral, although it is not necessarily identifiable with energy in the original physical problem.

We aim to present a conservative finite difference scheme for problem (1)–(3), which simulates conservation law (5) that the differential equation (1) possesses, and prove convergence and stability of the scheme. This paper is organized as follows. In Section 2, some notations are given and some useful lemmas are proposed. In Section 3, we present a nonlinear compact conservative difference scheme, discuss its discrete conservative law, prove the existence of difference solution by Brouwer fixed point theorem, give some a priori estimates, and then prove by discrete energy method that the difference scheme is uniquely solvable, unconditionally stable and that convergence of the difference solutions with $O(\tau^2 + h^4)$ order is based on some a priori estimates. In Section 4, numerical results are provided to test the theoretical results.

2. Notations and Lemmas

Let $h = (x_r - x_l)/J$ and $\tau = T/N$ be the spatial and temporal step sizes, respectively. Denote $x_j = x_l + jh$, ($0 \leq j \leq J$), $t^n = n\tau$, ($0 \leq n \leq N$). Let u_j^n denote the approximation of $u(x_j, t_n)$, and let

$$\mathbf{R}_0^J = \left\{ u_j = (u_j)_{j \in \mathbb{Z}} \mid u_0 = u_J = 0 \right\}. \quad (6)$$

As usual, the following notations will be used:

$$\begin{aligned} \delta_x^+ u_j^n &= \frac{u_{j+1}^n - u_j^n}{h}, & \delta_x^- u_j^n &= \frac{u_j^n - u_{j-1}^n}{h}, \\ \delta_x u_j^n &= \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \\ \bar{u}_j^n &= \frac{u_j^{n+1} + u_j^{n-1}}{2}, & \delta_t u_j^n &= \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \\ \mathcal{A}_1 u_j^n &= \left(1 + \frac{h^2}{12} \delta_x^+ \delta_x^- \right) u_j^n, \\ \mathcal{A}_2 u_j^n &= \left(1 + \frac{h^2}{6} \delta_x^+ \delta_x^- \right) u_j^n. \end{aligned} \quad (7)$$

We now introduce the discrete L^2 -inner product and the associated norm

$$(u^n, v^n)_h = h \sum_{j=1}^{J-1} u_j^n v_j^n, \quad \|u^n\|_h = (u^n, u^n)_h^{1/2}. \quad (8)$$

The discrete L^∞ -norm $\|\cdot\|_{\infty, h}$ is defined as

$$\|u\|_{\infty, h} = \max_j |u_j|. \quad (9)$$

Let $\Omega_h = \{x_j = x_l + jh \mid 0 \leq j \leq J\}$. It is convenient to let $L_h^2(\Omega_h)$ denote the normed vector space as $\{\mathbf{R}_0^J, \|\cdot\|_h\}$. The corresponding matrices are defined, respectively, as

$$\begin{aligned} u^n &= (u_1^n, u_2^n, \dots, u_{J-1}^n)^T, \\ (u^n)^p &= \text{diag} \left((u_1^n)^p, (u_2^n)^p, \dots, (u_{J-1}^n)^p \right), \\ A_1 &= \frac{1}{12} \begin{pmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & 10 & 1 \\ 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{J-1}, \\ A_2 &= \frac{1}{6} \begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & 4 & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{pmatrix}_{J-1}. \end{aligned} \quad (10)$$

For a simple notation, the discrete function φ is defined by

$$\varphi(u, v) = \frac{p\alpha}{p+1} \left[(u)^{p-1} H_2 \delta_x v + H_2 \delta_x (u^{p-1} v) \right], \quad (11)$$

where $H_1 = A_1^{-1}$, $H_2 = A_2^{-1}$. Obviously, A_1, A_2, H_1, H_2 are symmetric positive definite matrices. To obtain some important results, we introduce the following lemmas.

Lemma 1 (see [18]). For $u, v \in \mathbf{R}_0^J$, one has

$$(\delta_x^+ u, v)_h = -(u, \delta_x^- v)_h, \quad (\delta_x u, v)_h = -(u, \delta_x v)_h. \quad (12)$$

Lemma 2. For any real symmetric positive definite matrices H and for $u, v \in \mathbf{R}_0^J$, one can get

$$\begin{aligned} (H \delta_x^+ \delta_x^- u, v)_h &= -(H \delta_x^+ u, \delta_x^+ v)_h = -(R \delta_x^+ u, R \delta_x^+ v)_h, \\ (H \delta_x u, v)_h &= -(Hu, \delta_x v)_h = -(u, H \delta_x v)_h, \end{aligned} \quad (13)$$

where R is obtained by Cholesky decomposition of H , denoted as $H = R^T R$.

Proof. For $u, v \in \mathbf{R}_0^J$, we have

$$\begin{aligned} (H \delta_x^+ \delta_x^- u, v)_h &= (\delta_x^+ \delta_x^- Hu, v)_h = -(\delta_x^+ Hu, \delta_x^+ v)_h \\ &= -(H \delta_x^+ u, \delta_x^+ v)_h = -(R^T R \delta_x^+ u, \delta_x^+ v)_h \\ &= -(R \delta_x^+ u, R \delta_x^+ v)_h, \\ (H \delta_x u, v)_h &= (\delta_x Hu, v)_h = -(Hu, \delta_x v)_h = -(u, H \delta_x v)_h. \end{aligned} \quad (14)$$

□

Lemma 3 (see [16]). On the matrices A_1, A_2 . The eigenvalues of the matrices A_1 and A_2 are, respectively, as follows:

$$\lambda_{A_1,i} = \frac{1}{6} \left(5 + \cos \frac{2i\pi}{J} \right), \quad \lambda_{A_2,i} = \frac{1}{3} \left(2 + \cos \frac{2i\pi}{J} \right),$$

$$i = 1, 2, \dots, J-1. \quad (15)$$

Lemma 4. For $u \in \mathbf{R}_0^J$, we can get

$$\|u\|_h^2 \leq (H_1 u, u)_h = \|R_1 u\|_h^2 \leq \frac{3}{2} \|u\|_h^2, \quad (16)$$

$$\|u\|_h^2 \leq (H_2 u, u)_h = \|R_2 u\|_h^2 \leq 3 \|u\|_h^2,$$

where R_l are obtained by Cholesky decomposition of H_l , denoted as $H_l = R_l^T R_l$, ($l = 1, 2$).

Proof. It follows from Lemma 3 that the eigenvalues of H_1 and H_2 satisfy

$$1 \leq \lambda_{H_1,k} \leq \frac{3}{2}, \quad k = 1, 2, \dots, J-1, \quad (17)$$

$$1 \leq \lambda_{H_2,k} \leq 3, \quad k = 1, 2, \dots, J-1.$$

This gives the spectral radius $\rho(H_1) \leq 3/2$, $\rho(H_2) \leq 3$, and consequently

$$1 \leq \|H_1\| = \rho(H_1) \leq \frac{3}{2}, \quad 1 \leq \|H_2\| = \rho(H_2) \leq 3. \quad (18)$$

Thus

$$\|u\|_h^2 \leq (H_1 u, u)_h = (R_1 u, R_1 u)_h \leq \|H_1\| (u, u)_h \leq \frac{3}{2} \|u\|_h^2,$$

$$\|u\|_h^2 \leq (H_2 u, u)_h = (R_2 u, R_2 u)_h \leq \|H_2\| (u, u)_h \leq 3 \|u\|_h^2. \quad (19)$$

□

Lemma 5. For $u, v \in \mathbf{R}_0^J$, one has

$$(\varphi(u, v), v)_h = 0. \quad (20)$$

Proof. For $u \in \mathbf{R}_0^J$, we have

$$\begin{aligned} & (\varphi(u, v), v)_h \\ &= \frac{p\alpha}{p+1} (u^{p-1} H_2 \delta_x v + H_2 \delta_x (u^{p-1} v), v)_h \\ &= \frac{p\alpha}{p+1} [(u^{p-1} H_2 \delta_x v, v)_h + (H_2 \delta_x (u^{p-1} v), v)_h] \\ &= \frac{p\alpha}{p+1} [(H_2 \delta_x v, u^{p-1} v)_h - (u^{p-1} v, H_2 \delta_x v)_h] = 0. \end{aligned} \quad (21)$$

□

Lemma 6 (see [18]). For any discrete function $u \in \mathbf{R}_0^J$ and for any given $\varepsilon > 0$, there exists a constant $K(\varepsilon, n)$, depending only on ε and n , such that

$$\|u^n\|_{\infty,h} \leq \varepsilon \|\delta_x^+ u^n\|_h + K(\varepsilon, n) \|u^n\|_h. \quad (22)$$

Lemma 7 (see [19]). Let $(H, (\cdot, \cdot))$ be a finite-dimensional inner product space, let $\|\cdot\|$ be the associated norm, and let $g : H \rightarrow H$ be continuous. Assume, moreover, that $\exists \alpha > 0$, $\forall z \in H$, $\|z\| = \alpha$, $(g(z), z) > 0$. Then, there exists a $z^* \in H$ such that $g(z^*) = 0$ and $\|z^*\| \leq \alpha$.

Lemma 8 (see [18]). Suppose that the discrete function w_h satisfies recurrence formula

$$w_n - w_{n-1} \leq A\tau w_n + B\tau w_{n-1} + C_n\tau, \quad (23)$$

where A, B , and C_n ($n = 1, \dots, N$) are nonnegative constants. Then

$$\|w_h\| \leq \left(w_0 + \tau \sum_{k=1}^N C_k \right) e^{2(A+B)\tau}, \quad (24)$$

where τ is sufficiently small, such that $(A+B)\tau \leq (N-1)/2N$ ($N > 1$).

3. A Nonlinear-Implicit Conservative Scheme

In this section, we propose a nonlinear-implicit conservative scheme for the initial-boundary value problem (1)–(3) and give its numerical analysis.

3.1. The Nonlinear-Implicit Scheme and Its Conservative Law. Next we consider the compact finite difference scheme for problem (1)–(3) as follows:

$$\begin{aligned} & \partial_t u_j^n + \theta \mathcal{A}_2^{-1} \delta_x u_j^n + (1-\theta) \mathcal{A}_2^{-1} \delta_x \bar{u}_j^n \\ &+ \frac{p\alpha}{p+1} \left[(\bar{u}_j^n)^{p-1} \mathcal{A}_2^{-1} \delta_x \bar{u}_j^n + \mathcal{A}_2^{-1} \delta_x (\bar{u}_j^n)^p \right] \\ &- \beta \mathcal{A}_1^{-1} \delta_x^+ \delta_x^- (\partial_t u_j^n) = 0, \quad 1 \leq j \leq J-1, \quad 1 \leq n \leq N-1, \\ &u_0^n = u_j^n = 0, \quad 1 \leq n \leq N, \\ &u_j^0 = u_0(x_j), \quad 0 \leq j \leq J, \end{aligned} \quad (25)$$

where weight coefficient $\theta \in [0, 1]$. Note that we need another finite difference scheme to calculate u^1 , so the following scheme will be used:

$$\begin{aligned} u_j^1 - \beta \mathcal{A}_1^{-1} \delta_x^+ \delta_x^- u_j^1 &= u_0(x_j) - \beta \frac{\partial^2 u_0}{\partial x^2}(x_j) - \tau \frac{\partial u_0}{\partial x}(x_j) \\ &- \tau u_0(x_j) \frac{\partial u_0}{\partial x}(x_j). \end{aligned} \quad (26)$$

The matrix form of the difference scheme (25) can be written as

$$\begin{aligned} & \partial_t u^n + \theta H_2 \delta_x u^n + (1-\theta) H_2 \delta_x \bar{u}^n \\ &+ \frac{p\alpha}{p+1} \left[(\bar{u}^n)^{p-1} H_2 \delta_x \bar{u}^n + H_2 \delta_x (\bar{u}^n)^p \right] \\ &- \beta H_1 \delta_x^+ \delta_x^- (\partial_t u^n) = 0, \quad 1 \leq n \leq N-1, \end{aligned} \quad (27)$$

$$u_0^n = u_J^n = 0, \quad 1 \leq n \leq N, \quad (28)$$

$$u_j^0 = u_0(x_j), \quad 0 \leq j \leq J. \quad (29)$$

Theorem 9. Suppose that $u_0 \in H_0^1(\Omega)$; then the finite difference scheme (25) is conservative for discrete energy; that is,

$$\begin{aligned} E^n &= \frac{1}{2} \left(\|u^{n+1}\|_h^2 + \|u^n\|_h^2 \right) + \frac{\beta}{2} \left(\|R_1 \delta_x^+ u^{n+1}\|_h^2 + \|R_1 \delta_x^+ u^n\|_h^2 \right) \\ &+ \tau \theta \left(R_2 \delta_x u^n, R_2 u^{n+1} \right) = E^{n-1} = \dots = E^0, \end{aligned} \quad (30)$$

where R_l are obtained by Cholesky decomposition of H_l , denoted as $H_l = R_l^T R_l$, ($l = 1, 2$).

Proof. Taking an inner product of (27) with $u^{n+1} + u^{n-1}$, from Lemma 5, we obtain

$$\begin{aligned} &\frac{1}{2\tau} \left(\|u^{n+1}\|_h^2 - \|u^{n-1}\|_h^2 \right) + \theta (H_2 \delta_x u^n, 2\bar{u}^n)_h \\ &+ \frac{\beta}{2\tau} \left(\|R_1 \delta_x^+ u^{n+1}\|_h^2 - \|R_1 \delta_x^+ u^{n-1}\|_h^2 \right) = 0. \end{aligned} \quad (31)$$

Noting that

$$\begin{aligned} (H_2 \delta_x u^n, u^{n+1} + u^{n-1})_h &= (R_2 \delta_x u^n, R_2 u^{n+1})_h \\ &- (R_2 \delta_x u^{n-1}, R_2 u^n)_h, \end{aligned} \quad (32)$$

from (31)-(32), we obtain

$$\begin{aligned} &\frac{1}{2\tau} \left(\|u^{n+1}\|_h^2 - \|u^{n-1}\|_h^2 \right) \\ &+ \theta \left[(R_2 \delta_x u^n, R_2 u^{n+1})_h - (R_2 \delta_x u^{n-1}, R_2 u^n)_h \right] \\ &+ \frac{\beta}{2\tau} \left(\|R_1 \delta_x^+ u^{n+1}\|_h^2 - \|R_1 \delta_x^+ u^{n-1}\|_h^2 \right) = 0. \end{aligned} \quad (33)$$

Let

$$\begin{aligned} E^n &= \frac{1}{2} \left(\|u^{n+1}\|_h^2 + \|u^n\|_h^2 \right) \\ &+ \frac{\beta}{2} \left(\|R_1 \delta_x^+ u^{n+1}\|_h^2 + \|R_1 \delta_x^+ u^n\|_h^2 \right) \\ &+ \tau \theta \left(R_2 \delta_x u^n, R_2 u^{n+1} \right). \end{aligned} \quad (34)$$

Then, from (33), we get $E^n = E^{n-1}$. This completes the proof. $\square$

3.2. Existence and Prior Estimates of Difference Solution

Theorem 10. Suppose that $u_0 \in H_0^1(\Omega)$; then the finite difference scheme (25) is solvable.

Proof. Assume that there exist $u^0, u^1, \dots, u^n$ which satisfy (25) as $n \leq N-1$; now we try to prove that u^{n+1} satisfy (25). We define the mapping $g: \mathbf{R}_0^J \rightarrow \mathbf{R}_0^J$ as follows:

$$\begin{aligned} g(v) &= v - u^n - \beta (H_1 \delta_x^+ \delta_x^- v - H_1 \delta_x^+ \delta_x^- u^n) \\ &+ \tau H_2 \delta_x v + \tau \varphi(v, v) \end{aligned} \quad (35)$$

is obviously continuous. Taking in (35) the inner product with v , from Lemmas 2, 4, and 5, we obtain

$$\begin{aligned} &(g(v), v)_h \\ &= \|v\|_h^2 - (v, u^n)_h + \beta \left[\|R_1 \delta_x^+ v\|_h^2 - (R_1 \delta_x^+ v, R_1 \delta_x^+ u^n)_h \right] \\ &\geq \frac{1}{2} \left(\|v\|_h^2 - \|u^n\|_h^2 \right) + \frac{\beta}{2} \left(\|R_1 \delta_x^+ v\|_h^2 - \|R_1 \delta_x^+ u^n\|_h^2 \right) \\ &\geq \frac{1}{2} \left(\|v\|_h^2 - \|u^n\|_h^2 \right) - \frac{3\beta}{4} \|\delta_x^+ u^n\|_h^2. \end{aligned} \quad (36)$$

Thus for $\|v\|_h^2 = \|u^n\|_h^2 + (3\beta/2)\|\delta_x^+ u^n\|_h^2 + 1$, there exists $(g(v), v)_h > 0$. The existence of $\bar{u}^n$ follows from Lemma 7 and consequently the existence of $u^{n+1} = 2v - u^{n-1}$ is obtained. This completes the proof. $\square$

Next we will give some a priori estimates of difference solutions.

Lemma 11. Suppose that $u_0 \in H_0^1(\Omega)$; then there exists the estimation for the solution of problem (1)-(3):

$$\|u\|_{L^2} \leq C, \quad \|u_x\|_{L^2} \leq C, \quad \|u\|_{L^\infty} \leq K_0. \quad (37)$$

Proof. It follows from (5) that

$$\|u\|_{L^2} \leq C, \quad \|u_x\|_{L^2} \leq C. \quad (38)$$

Hence, it follows from the Sobolev inequality that

$$\|u\|_{L^\infty} \leq K_0. \quad (39)$$

$\square$

Lemma 12. Suppose that $u_0 \in H_0^1(\Omega)$; then there exists the estimation for the solution of the difference scheme (25):

$$\|u^n\|_h \leq K_1, \quad \|\delta_x^+ u^n\|_h \leq K_2, \quad \|u^n\|_{\infty, h} \leq K_3, \quad (40)$$

where $K_4 = \max\{\sqrt{6E^0/(3-5\theta\tau)}, \sqrt{2E^0}\}$, $K_2 = \max\{\sqrt{6E^0/(3\beta-5\theta\tau)}, \sqrt{2E^0/\beta}\}$, $K_3 = \varepsilon K_2 + K(\varepsilon, n)K_1$.

Proof. From Theorem 9, we obtain

$$\begin{aligned} &\frac{1}{2} \left(\|u^{n+1}\|_h^2 + \|u^n\|_h^2 \right) + \frac{\beta}{2} \left(\|R_1 \delta_x^+ u^{n+1}\|_h^2 + \|R_1 \delta_x^+ u^n\|_h^2 \right) \\ &\leq E^0 + \tau \theta \left| (R_2 \delta_x u^n, R_2 u^{n+1})_h \right| \\ &\leq E^0 + \frac{\tau \theta}{2} \left(\|R_2 \delta_x u^n\|_h^2 + \|R_2 u^{n+1}\|_h^2 \right); \end{aligned} \quad (41)$$

then from Lemma 4, we have

$$\begin{aligned} & \frac{1}{2} (\|u^{n+1}\|_h^2 + \|u^n\|_h^2) + \frac{\beta}{2} (\|\delta_x^+ u^{n+1}\|_h^2 + \|\delta_x^+ u^n\|_h^2) \\ & \leq E^0 + \frac{3\tau\theta}{2} (\|\delta_x^+ u^n\|_h^2 + \|u^{n+1}\|_h^2). \end{aligned} \quad (42)$$

That is

$$\begin{aligned} & \left(\frac{1}{2} - \frac{3\tau\theta}{2}\right) \|u^{n+1}\|_h^2 + \frac{1}{2} \|u^n\|_h^2 + \frac{\beta}{2} \|\delta_x^+ u^{n+1}\|_h^2 \\ & + \left(\frac{\beta}{2} - \frac{3\tau\theta}{2}\right) \|\delta_x^+ u^n\|_h^2 \leq E^0; \end{aligned} \quad (43)$$

let τ be small, such that $\min\{1/2 - 3\tau\theta/2, \beta/2 - 3\tau\theta/2\} > 0$; then we can get

$$\|u^n\|_h \leq K_1, \quad \|\delta_x^+ u^n\|_h \leq K_2, \quad (44)$$

where

$$\begin{aligned} K_1 &= \max \left\{ \sqrt{\frac{2E^0}{1-3\theta\tau}}, \sqrt{2E^0} \right\}, \\ K_2 &= \max \left\{ \sqrt{\frac{2E^0}{\beta-3\theta\tau}}, \sqrt{\frac{2E^0}{\beta}} \right\}. \end{aligned} \quad (45)$$

It follows from Lemma 6 that

$$\|u^n\|_{\infty, h} \leq K_3, \quad (46)$$

where $K_3 = \varepsilon K_2 + K(\varepsilon, n) K_1$.

This completes the proof. $\square$

3.3. Convergence and Stability of Difference Solution. First, we consider the truncation error of the finite difference scheme (25). Suppose that $v_j^n = u(x_j, t_n)$, which is the solution of problem (1)–(3). Then we have

$$\begin{aligned} r^n &= \partial_t v^n + \theta H_2 \delta_x v^n + (1-\theta) H_2 \delta_x \bar{v}^n - \beta H_1 \delta_x^+ \delta_x^- \partial_t v^n \\ &+ \frac{p\alpha}{p+1} [(\bar{v}^n)^{p-1} H_2 \delta_x \bar{v}^n + H_2 \delta_x (\bar{v}^n)^p]; \end{aligned} \quad (47)$$

according to Taylor's expansion, $r_j^n = O(\tau^2 + h^4)$ can be easily obtained. Next, we consider convergence and stability of the finite difference scheme (25).

Theorem 13. Suppose that $u_0 \in H_0^1(\Omega)$ and $u \in C^{(5,3)}$; then the solution of the conservative difference scheme (25) converges to the solution of problem (1)–(3) with the order $O(\tau^2 + h^4)$ by L^∞ norm.

Proof. Subtracting (27) from (47), and letting $e^n = v^n - u^n$, we have

$$\begin{aligned} r^n &= \partial_t e^n + \theta H_2 \delta_x e^n + (1-\theta) H_2 \delta_x \bar{e}^n \\ &- \beta H_1 \delta_x^+ \delta_x^- \partial_t e^n + \varphi(\bar{v}^n, \bar{v}^n) - \varphi(\bar{u}^n, \bar{u}^n); \end{aligned} \quad (48)$$

FIGURE 1: Numerical solution u^n of scheme with $p = 2$ and $\tau = h = 0.1$.

FIGURE 2: Discrete energy E^n of scheme with $p = 2$, $T = 10$, and $\tau = h = 0.1$.

taking an inner product of (48) with $e^{n+1} + e^{n-1}$, we obtain

$$\begin{aligned} & (r^n, e^{n+1} + e^{n-1})_h \\ &= \frac{1}{2\tau} (\|e^{n+1}\|_h^2 - \|e^{n-1}\|_h^2) \\ &+ \frac{\beta}{2\tau} (\|R_1 \delta_x^+ e^{n+1}\|_h^2 - \|R_1 \delta_x^+ e^{n-1}\|_h^2) \\ &+ \theta (R_2 \delta_x e^n, e^{n+1} + e^{n-1})_h \\ &+ (\varphi(\bar{v}^n, \bar{v}^n) - \varphi(\bar{u}^n, \bar{u}^n), e^{n+1} + e^{n-1})_h; \end{aligned} \quad (49)$$

TABLE 1: Errors computed by the proposed compact scheme with $\sigma = 4/3$, $p = 2$ and $h = \tau = 0.1$.

t	$\theta = 0$	$\theta = 0.25$	$\ e^n\ _{\infty,h}$ $\theta = 0.5$	$\theta = 0.75$	$\theta = 1$
0.2	2.6213×10^{-5}	1.2528×10^{-5}	1.5794×10^{-5}	3.2838×10^{-5}	5.0401×10^{-5}
0.4	5.2311×10^{-5}	2.4917×10^{-5}	3.1520×10^{-5}	6.5530×10^{-5}	1.0012×10^{-4}
0.6	7.8509×10^{-5}	3.7267×10^{-5}	4.7095×10^{-5}	9.7948×10^{-5}	1.4917×10^{-4}
0.8	1.0476×10^{-4}	4.9503×10^{-5}	6.2474×10^{-5}	1.3006×10^{-4}	1.9758×10^{-4}
1.0	1.3090×10^{-4}	6.1653×10^{-5}	7.7647×10^{-5}	1.6222×10^{-4}	2.4638×10^{-4}

TABLE 2: Errors computed by the proposed compact scheme with $\sigma = 4/3$, $p = 4$ and $h = \tau = 0.1$.

t	$\theta = 0$	$\theta = 0.25$	$\ e^n\ _{\infty,h}$ $\theta = 0.5$	$\theta = 0.75$	$\theta = 1$
0.2	1.2498×10^{-4}	1.7684×10^{-4}	2.3183×10^{-4}	2.8729×10^{-4}	3.4464×10^{-4}
0.4	2.3629×10^{-4}	3.4550×10^{-4}	4.5445×10^{-4}	5.6499×10^{-4}	6.7531×10^{-4}
0.6	3.3108×10^{-4}	5.0029×10^{-4}	6.6455×10^{-4}	8.3064×10^{-4}	9.9436×10^{-4}
0.8	4.1205×10^{-4}	6.4180×10^{-4}	8.6651×10^{-4}	1.0913×10^{-3}	1.3159×10^{-3}
1.0	4.9896×10^{-4}	7.7376×10^{-4}	1.0667×10^{-3}	1.3595×10^{-3}	1.6550×10^{-3}

from Lemma 4 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & (R_2 \delta_x e^n, e^{n+1} + e^{n-1})_h \\ & \leq 3 \|\delta_x e^n\|_h^2 + \frac{1}{2} \|e^{n+1}\|_h^2 + \frac{1}{2} \|e^{n-1}\|_h^2; \end{aligned} \quad (50)$$

according to Lemmas 2 and 4, we have

$$\begin{aligned} & (\varphi(\bar{v}^n, \bar{v}^n) - \varphi(\bar{u}^n, \bar{u}^n), e^{n+1} + e^{n-1})_h \\ & = \frac{2p\alpha}{p+1} \left\{ h \sum_{j=1}^{J-1} \left[(\bar{v}_j^n)^{p-1} \mathcal{A}_2^{-1} \delta_x \bar{v}_j^n - (\bar{u}_j^n)^{p-1} \mathcal{A}_2^{-1} \delta_x \bar{u}_j^n \right] \bar{e}_j^n \right. \\ & \quad \left. + h \sum_{j=0}^{J-1} \left[\mathcal{A}_2^{-1} \delta_x (\bar{v}_j^n)^p - \mathcal{A}_2^{-1} \delta_x (\bar{u}_j^n)^p \right] \bar{e}_j^n \right\} \\ & \leq K_4 h \sum_{j=0}^{J-1} (|\mathcal{A}_2^{-1} \delta_x \bar{e}_j^n| + |\bar{e}_j^n|) |\bar{e}_j^n| \\ & \quad + K_4 h \sum_{j=0}^{J-1} |\bar{e}_j^n| |\mathcal{A}_2^{-1} \delta_x \bar{e}_j^n| \\ & \leq K_4 (9 \|\delta_x \bar{e}^n\|_h^2 + 2 \|\bar{e}^n\|_h^2) \\ & \leq K_4 \left(\frac{9}{2} \|\delta_x e^{n+1}\|_h^2 + \frac{9}{2} \|\delta_x e^{n-1}\|_h^2 + \|e^{n+1}\|_h^2 + \|e^{n-1}\|_h^2 \right), \end{aligned} \quad (51)$$

where $K_4 = (p\alpha/2(p+1)) \max\{K_0^{p-1}, 2(p-1)K_0^{p-2}K_3, 2(p-1)K_3^{p-1}\}$.

FIGURE 3: Absolute error e^N of scheme with $p = 2$ and $\tau = h = 0.1$.

Substituting (50) and (51) into (49), we obtain

$$\begin{aligned} & \frac{1}{2\tau} (\|e^{n+1}\|_h^2 - \|e^{n-1}\|_h^2) + \frac{\beta}{2\tau} (\|R_1 \delta_x^+ e^{n+1}\|_h^2 - \|R_1 \delta_x^+ e^{n-1}\|_h^2) \\ & \leq \|r^n\|_h^2 + \frac{1}{2} (\|e^{n+1}\|_h^2 + \|e^{n-1}\|_h^2) \\ & \quad + \theta \left(3 \|\delta_x e^n\|_h^2 + \frac{1}{2} \|e^{n+1}\|_h^2 + \frac{1}{2} \|e^{n-1}\|_h^2 \right) \\ & \quad + K_4 \left(\frac{9}{2} \|\delta_x e^{n+1}\|_h^2 + \frac{9}{2} \|\delta_x e^{n-1}\|_h^2 + \|e^{n+1}\|_h^2 + \|e^{n-1}\|_h^2 \right), \end{aligned} \quad (52)$$

TABLE 3: Comparison of $\|e\|_{\infty,h}$ by the compact scheme for $h = \tau = 0.1$ with the Zhang [4] scheme for $h = 0.05, \tau = 0.1$.

t	$p = 2$		$p = 4$	
	Compact Scheme	Zhang [4]	Compact Scheme	Zhang [4]
0.2	1.2528×10^{-5}	7.756×10^{-6}	1.2498×10^{-4}	0.0049
0.4	2.4917×10^{-5}	1.575×10^{-5}	2.3629×10^{-4}	0.0098
0.6	3.7267×10^{-5}	2.357×10^{-5}	3.3108×10^{-4}	0.0148
0.8	4.9503×10^{-5}	3.129×10^{-5}	4.1205×10^{-4}	0.0198
1.0	6.1653×10^{-5}	3.963×10^{-5}	4.9896×10^{-4}	0.0239

TABLE 4: Comparison of $\|e\|_{\infty,h}$ by the compact scheme for $\sigma = 4/3, \theta = 0.25, p = 2, \tau = 0.1$ and $h = 0.05$.

t	Compact Scheme	Shao et al. [7]	$\ e^n\ _{\infty,h}$	Raslan [10]	Dogan [11]
			C-N scheme		
0.2	1.2536×10^{-5}	0.00056	0.00070	0.00190	0.00053
0.4	2.4916×10^{-5}	0.00085	0.03331	0.00283	0.00113
0.6	3.7265×10^{-5}	0.00112	0.06337	0.00403	0.00175
0.8	4.9526×10^{-5}	0.00141	0.08433	0.00481	0.00237
1.0	6.1667×10^{-5}	0.00169	0.11287	0.00563	0.00299

TABLE 5: Comparison of $\|e\|_{\infty,h}$ by the compact scheme for $\sigma = 1.03, \theta = 0.25, p = 2, \tau = h = 0.1$.

t	Compact Scheme	$\ e^n\ _{\infty,h}$	Kutluay and Esen [8]
		Bakodah and Banaja [6]	
4	7.0938×10^{-5}	1.4805×10^{-4}	1.23×10^{-4}
8	2.1660×10^{-4}	2.9961×10^{-4}	1.66×10^{-4}
12	5.1012×10^{-4}	4.5367×10^{-4}	1.79×10^{-4}

that is,

$$\begin{aligned}
& \frac{1}{2} \left(\|e^{n+1}\|_h^2 - \|e^{n-1}\|_h^2 \right) + \frac{\beta}{2} \left(\|\delta_x^+ e^{n+1}\|_h^2 - \|\delta_x^+ e^{n-1}\|_h^2 \right) \\
& \leq \tau K_5 \left(\|e^{n+1}\|_h^2 + \|e^n\|_h^2 + \|e^{n-1}\|_h^2 + \|\delta_x^+ e^{n+1}\|_h^2 \right. \\
& \quad \left. + \|\delta_x^+ e^n\|_h^2 + \|\delta_x^+ e^{n-1}\|_h^2 \right) + \tau \|r^n\|_h^2,
\end{aligned} \quad (53)$$

where $K_5 = \max\{(1 + \theta)/2 + K_4, (9/2)K_4, 3\theta\}$. Let

$$B^n = \frac{1}{2} \left(\|e^{n+1}\|_h^2 + \|e^n\|_h^2 \right) + \frac{\beta}{2} \left(\|\delta_x^+ e^{n+1}\|_h^2 + \|\delta_x^+ e^n\|_h^2 \right); \quad (54)$$

then (53) can be rewritten as

$$B^n - B^{n-1} \leq \tau \|r^n\|_h^2 + \tau K_6 (B^n + B^{n-1}), \quad (55)$$

where $K_6 = \max\{2K_5, 2K_5/\beta\}$. From Lemma 8, we have

$$B^N \leq \left(B^0 + T \sup_{1 \leq n \leq N} \|r^n\|_h^2 \right) e^{4K_6 T}. \quad (56)$$

Thus we can choose a fourth-order method to compute u^1 such that

$$B^0 \leq [O(\tau^2 + h^4)]^2; \quad (57)$$

FIGURE 4: Numerical solution u^n of scheme with $p = 4$ and $\tau = h = 0.1$.

it follows from (56) that

$$\|e^n\|_h \leq O(\tau^2 + h^4), \quad \|\delta_x^+ e^n\|_h \leq O(\tau^2 + h^4), \quad (58)$$

and then, from Lemma 6, we obtain

$$\|e\|_{\infty,h} \leq O(\tau^2 + h^4). \quad (59)$$

This completes the proof. $\square$

Below, we can similarly prove stability of the difference solution.

Theorem 14. Under the conditions of Theorem 13, the solution of conservative finite difference scheme (25) is stable by L^∞ norm.

FIGURE 5: Discrete energy E^n of scheme with $p = 4$, $T = 10$, and $\tau = h = 0.1$.

FIGURE 6: Absolute error e^n of scheme with $p = 4$ and $\tau = h = 0.1$.

4. Numerical Experiments

In this section, two examples are presented to illustrate the effectiveness of the finite difference scheme (25) in $[-50, 50]$. The single solitary wave solution of (1) is

$$u(x, t) = A \cdot \text{sech}^{2/(p-1)}[k(x + x_0 - \sigma t)], \quad (60)$$

where

$$A = \left[\frac{(p+1)(\sigma-1)}{2\alpha} \right]^{1/(p-1)}, \quad k = \frac{p-1}{2\beta} \sqrt{\frac{\sigma-1}{\sigma}}, \quad (61)$$

and σ, x_0 are arbitrary constants and $p \geq 2$.

Let $x_0 = 0$ in (60), $\alpha = 1/2$, $\beta = 1$, and $u_0(x) = A \cdot \text{sech}^{2/(p-1)}(kx)$ and consider two cases: $p = 2$ and $p = 4$. We

take $h = 0.1$ and $\tau = 0.1$, respectively. The errors $\|e^n\|_{\infty, h}$ are listed in Tables 1-2, respectively.

In Table 3, 4, and 5, the comparison of $\|e\|_{\infty, h}$ by the compact scheme for $h = \tau = 0.1$ with the Zhang [4] scheme for $h = 0.05$, $\tau = 0.1$ when $\sigma = 4/3$, $\theta = 0.25$ is shown. From Table 3, we can see that our compact scheme is acceptable.

Numerical results show that numerical precision depends on the choice of parameter θ . From Tables 1-2, $\|e^n\|_{\infty, h} \leq O(\tau^2 + h^4)$ is validated. We take different p , h , and θ values and compute the errors for the solution of problem (1)–(3). Numerical results are almost identical with the above experiment result. Hence, our schemes are efficient and reliable.

In Figures 1, 2, 3, 4, 5, and 6, we show the numerical solution and conservative discrete energy in each case.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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