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Research Article

On the Sequences Realizing Perron and Lyapunov Exponents of Discrete Linear Time-Varying Systems

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We investigate properties of partial exponents (in particular, the Lyapunov and Perron exponents) of discrete time-varying linear systems. In the set of all increasing sequences of natural numbers, we define an equivalence relation with the property that sequences in the same equivalence class have the same partial exponent. We also define certain subclass of all increasing sequences of natural numbers, including all arithmetic sequences, such that all partial exponents are achievable on a sequence from this class. Finally, we show that the Perron and Lyapunov exponents may be approximated by partial exponents achievable on sequences in certain sense similar to geometric sequences.

1. Introduction

Consider a discrete time-varying system:

$$x(n+1) = A(n)\,x(n), \quad n \geq 0, \tag{1}$$

where $A = (A(n))_{n\in\mathbb{N}}$ is a bounded sequence of invertible s-by-s real matrices such that $(A^{-1}(n))_{n\in\mathbb{N}}$ is bounded. For the coefficient matrices, denote the transition matrices

$$\Phi(m,n) = A(m-1)\cdots A(n) \quad \text{for } m > n \tag{2}$$

and $\Phi(n,n) = I$, where I is the identity matrix. For an initial condition $x_0 \in \mathbb{R}^s$, the solution of (1) is denoted by $(x(n, x_0))_{n\in\mathbb{N}}$; that is,

$$x(n, x_0) = \Phi(n,0)\,x_0. \tag{3}$$

If $a = (a(n))_{n\in\mathbb{N}}$ is a sequence of real numbers, then the Perron exponent $\pi(a)$ and the Lyapunov exponent $\lambda(a)$ of a are defined in the following ways:

$$\begin{aligned} \pi(a) &= \liminf_{n\to\infty} \frac{1}{n} \ln a(n), \\ \lambda(a) &= \limsup_{n\to\infty} \frac{1}{n} \ln a(n). \end{aligned} \tag{4}$$

By $\|\cdot\|$, denote the Euclidean norm in $\mathbb{R}^s$ and the induced operator norm. For an initial condition $x_0 \in \mathbb{R}^s$, the Perron $\pi(x_0)$ and the Lyapunov $\lambda(x_0)$ exponents of the solution $(x(n, x_0))_{n\in\mathbb{N}}$ of system (1) are defined (see [1]) as

$$\begin{aligned} \pi(x_0) &= \liminf_{n\to\infty} \frac{1}{n} \ln \|x(n, x_0)\|, \\ \lambda(x_0) &= \limsup_{n\to\infty} \frac{1}{n} \ln \|x(n, x_0)\|. \end{aligned} \tag{5}$$

It means that the Perron and Lyapunov exponents of the solution $(x(n, x_0))_{n\in\mathbb{N}}$ are the Perron and Lyapunov exponents of the sequence $(\|x(n, x_0)\|)_{n\in\mathbb{N}}$, respectively.

To characterize many properties of system (1) characteristics exponents, for example the Lyapunov, Perron, Bohl, general exponents may be used. These quantities describe the different types of stability and trajectories growth rate. For interesting summary on main properties of the Lyapunov, Perron, Bohl, general exponents of the discrete time-varying linear system, and relations between these exponents and different types of stability of the considered system see [2].

The Lyapunov [3–15] and the Perron [16–23] exponents are one of the most commonly used numerical characteristics of dynamical systems. They describe, inter alia, such

important properties like stability (exponential and Poisson). Numerical calculation of them is related to two main problems. The first one is that they are very sensitive to inaccuracies in the coefficients (they are not even continuous functions of the coefficients; see [24–29]). The second problem is that these quantities are defined by the partial limits (upper and lower one), and we do not know in advance what time sequence they are achieved on; therefore, an a priori one would need to look into all increasing sequences of natural numbers [30, 31].

This paper is linked to the second problem. In the paper, we try to describe a smaller class of all growing sequences of natural numbers with the property that the Lyapunov or Perron exponents are achieved on one of the sequences in this class.

The paper is organized in the following way: in the next paragraph, we establish certain properties of partial limits of real sequences, in particular their Perron and Lyapunov exponents. In the third section, containing the main results of the work, the theorems from the second section are applied to obtain properties of the Perron and Lyapunov exponents of the solutions of system (1). The work ends with a paragraph containing conclusions and suggestions for further research.

2. Preliminaries

Denote by $\mathcal{S}$ the set of all sequences of positive real numbers $a = (a_n)_{n\in\mathbb{N}}$ such that there exist constants c_1, c_2 (in general depending on the sequence a) such that

$$c_1 \le \frac{a(n+1)}{a(n)} \le c_2, \quad n \in \mathbb{N}. \tag{6}$$

By $\mathcal{C}$ we denote the set of all increasing sequences of natural numbers. If $b = (b_n)_{n\in\mathbb{N}}$ is any sequence of real numbers and $m = (m_l)_{l\in\mathbb{N}} \in \mathcal{C}$ are such that there exists a finite limit

$$\beta = \lim_{l\to\infty} b(m_l), \tag{7}$$

then the number β we will be called a partial limit or limit point of sequence b and we will say that it is achieved on the sequence m.

The next theorem shows that each number between the Perron and Lyapunov exponents of the sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathcal{S}$ is a partial limit of the sequence $((1/n)\ln a(n))_{n\in\mathbb{N}}$.

Theorem 1. *For each sequence $a \in \mathcal{S}$ and each number $\alpha \in [\pi(a), \lambda(a)]$ there exists sequence $(m_l)_{l\in\mathbb{N}} \in \mathcal{C}$ such that*

$$\alpha = \lim_{l\to\infty} \frac{1}{m_l} \ln a(m_l). \tag{8}$$

Proof. If $\alpha = \pi(a)$ or $\alpha = \lambda(a)$ then the conclusion follows from the properties of the upper and lower limits. Suppose that $\alpha \in (\pi(a), \lambda(a))$. Let us define sequence $(n_l)_{l\in\mathbb{N}}$ in the following way:

$$\begin{aligned} n_0 &= 0, \\ n_1 &= \min\left\{n \in \mathbb{N} : n > 0,\ \frac{1}{n}\ln a(n) > \alpha\right\}, \\ n_2 &= \min\left\{n \in \mathbb{N} : n > n_1,\ \frac{1}{n}\ln a(n) < \alpha\right\}, \\ n_{2l+1} &= \min\left\{n \in \mathbb{N} : n > n_{2l},\ \frac{1}{n}\ln a(n) > \alpha\right\}, \\ &\qquad l = 1, 2, \ldots, \\ n_{2l+2} &= \min\left\{n \in \mathbb{N} : n > n_{2l+1},\ \frac{1}{n}\ln a(n) < \alpha\right\}, \\ &\qquad l = 1, 2, \ldots. \end{aligned} \tag{9}$$

By the inequality $\pi(a) < \alpha < \lambda(a)$ and by the definition of the upper and lower limits, it follows that the definition of $(n_l)_{l\in\mathbb{N}}$ is correct, the sequence $(n_l)_{l\in\mathbb{N}}$ is increasing, and

$$\begin{aligned} \frac{1}{n_{2l+1}} \ln a(n_{2l+1}) &> \alpha, \quad l = 1, 2, \ldots, \\ \frac{1}{n_{2l+1}-1} \ln a(n_{2l+1}-1) &\le \alpha, \quad l = 1, 2, \ldots. \end{aligned} \tag{10}$$

From the above two inequalities we get

$$\liminf_{l\to\infty} \frac{1}{n_{2l+1}} \ln a(n_{2l+1}) \ge \alpha, \tag{11}$$

$$a(n_{2l+1}-1) \le \exp(\alpha(n_{2l+1}-1)), \quad l = 1, 2, \ldots. \tag{12}$$

Since $a \in \mathcal{S}$, therefore there exists a constant $c_2 \in \mathbb{R}$, $c_2 > 0$ such that

$$a(n_{2l+1}) < c_2 a(n_{2l+1}-1), \quad l = 1, 2, \ldots. \tag{13}$$

By the last two inequalities, we obtain

$$a(n_{2l+1}) < c_2 \exp(\alpha(n_{2l+1}-1)), \quad l = 1, 2, \ldots, \tag{14}$$

$$\frac{1}{n_{2l+1}} \ln a(n_{2l+1}) < \frac{\ln c_2}{n_{2l+1}} + \frac{\alpha(n_{2l+1}-1)}{n_{2l+1}}, \quad l = 1, 2, \ldots. \tag{15}$$

Passing to the upper limit and taking into account that $\lim_{l\to\infty} n_{2l+1} = \infty$, we have

$$\limsup_{l\to\infty} \frac{1}{n_{2l+1}} \ln a(n_{2l+1}) \le \alpha. \tag{16}$$

Inequalities (11) and (16) imply that

$$\lim_{l\to\infty} \frac{1}{n_{2l+1}} \ln a(n_{2l+1}) = \alpha. \tag{17}$$

It means that the sequence $m_l = n_{2l+1}$, $l = 1, 2, \ldots$ is that one from the theorem's thesis. The proof is completed. □

It is easy to construct an example showing that the theorem is no longer true without the assumption that $a \in \mathcal{S}$.

Example 2. Let us define sequence $a = (a(n))_{n\in\mathbb{N}}$ in the following way:

$$a(n) = \begin{cases} e^{n} & \text{for even } n \\ e^{-n} & \text{for odd } n. \end{cases} \tag{18}$$

It is clear that $a \notin \mathcal{S}$. Moreover,

$$\frac{1}{n}\ln a(n) = \begin{cases} 1 & \text{for even } n \\ -1 & \text{for odd } n. \end{cases} \tag{19}$$

Therefore, each convergent subsequence of the sequence $((1/n)\ln a(n))_{n\in\mathbb{N}}$ may have as a limit only 1 or -1.

Theorem 1 may be generalized as follows.

Theorem 3. *If $(T_n)_{n\in\mathbb{N}} \in \mathcal{C}$ is such that*

$$\lim_{n\to\infty}\frac{T_{n+1}-T_n}{T_n} = 0, \tag{20}$$

then for each sequence $a \in \mathcal{S}$ and each number $\alpha \in [\pi(a), \lambda(a)]$ there exists sequence $(n_l)_{l\in\mathbb{N}} \in \mathcal{C}$ such that

$$\alpha = \lim_{l\to\infty}\frac{1}{T_{n_l}}\ln a\left(T_{n_l}\right). \tag{21}$$

Proof. This theorem may be obtained from the general fact from the functional analysis as it was shown in [32, Lemma 7.5]. Repeating the construction from the proof of Theorem 1, we obtain instead of inequality (15) the following one:

$$\begin{aligned}\frac{1}{n_{2l+1}}\ln a\left(n_{2l+1}\right) &< \frac{\left(T_{n_{2l+1}} - T_{n_{2l+1}-1}\right)\ln c_2}{T_{n_{2l+1}}} \\ &\quad + \frac{\alpha T_{n_{2l+1}-1}}{T_{n_{2l+1}}}, \quad l = 1, 2, \ldots.\end{aligned} \tag{22}$$

From this inequality and by assumption (20) the thesis follows. The proof is completed. □

Let us now introduce certain relation in the set $\mathcal{C}$ (Definition 4). It will appear to be an equivalence relation (Theorem 5) and if two sequences belong to the same equivalence class, then corresponding to them subsequences of $(a(n))_{n\in\mathbb{N}}$ have the same exponents (Theorem 6).

Definition 4. We say that the sequence $m = (m_l)_{l\in\mathbb{N}} \in \mathcal{C}$ is close to the sequence $n = (n_l)_{l\in\mathbb{N}} \in \mathcal{C}$ if

$$\lim_{k\to\infty}\frac{\min\left\{\left|m_k - n_l\right| : l \in \mathbb{N}\right\}}{m_k} = 0. \tag{23}$$

This fact will be denoted in the following way $m \sim n$.

Theorem 5. *The relation $\sim$ is an equivalence relation in the set $\mathcal{C}$.*

Proof. Reflexivity of the relation $\sim$ is obvious. Suppose that $m \sim n$. For a $k \in \mathbb{N}$, denote by $l(k)$ any natural number satisfying the condition

$$\left|m_k - n_{l(k)}\right| = \min\left\{\left|m_k - n_l\right| : l \in \mathbb{N}\right\}. \tag{24}$$

We will show that the set $\{l(k) : k \in \mathbb{N}\}$ is infinite. On the contrary, suppose that it is finite and denote its elements by $l_1, \ldots, l_p$. Then, there exists infinite set $A \subset \mathbb{N}$ such that

$$l(k) = l_i \tag{25}$$

for all $k \in A$ and certain $i = 1, \ldots, p$. For $k \in A$, denote

$$a = \left|m_k - n_{l(k)}\right| = \left|m_k - n_{l_i}\right|. \tag{26}$$

Then,

$$\begin{aligned} m_k &= n_{l_i} + a \\ \text{or } m_k &= n_{l_i} - a. \end{aligned} \tag{27}$$

The last two equalities are in contradiction with the facts that A is infinite and m tends to infinity. Now we show symmetry of the relation $\sim$. Suppose that $m \sim n$ but the sequence n is not close to the sequence m. Denote

$$\begin{aligned}\alpha' &= \limsup_{k\to\infty}\frac{\min\left\{\left|n_k - m_l\right| : l \in \mathbb{N}\right\}}{n_k} \\ &= \limsup_{k\to\infty}\min\left\{\left|1 - \frac{m_l}{n_k}\right| : l \in \mathbb{N}\right\}.\end{aligned} \tag{28}$$

The fact that the sequence n is not close to the sequence m implies that $\alpha' > 0$. Let us fix $\alpha \in (0, \alpha')$, $\alpha < 1$. By the definition of upper limit we know that there exists sequence $(p(k))_{k\in\mathbb{N}} \in \mathcal{C}$ such that

$$\min\left\{\left|1 - \frac{m_l}{n_{p(k)}}\right| : l \in \mathbb{N}\right\} > \alpha. \tag{29}$$

It means that

$$\left|1 - \frac{m_l}{n_{p(k)}}\right| > \alpha \tag{30}$$

or equivalently that

$$\begin{aligned} 1 - \frac{m_l}{n_{p(k)}} &> \alpha \\ \text{or } 1 - \frac{m_l}{n_{p(k)}} &< -\alpha \end{aligned} \tag{31}$$

for all $l, k \in \mathbb{N}$. For the fixed $l \in \mathbb{N}$, the second inequality may hold only for finite many $k \in \mathbb{N}$ (in the opposite case, after passing to the limit with $k \to \infty$ we obtain $-1 > \alpha$). Moreover, if for certain $l, k \in \mathbb{N}$ the first inequality holds, then

$$1 - \frac{m_l}{n_q} > \alpha \tag{32}$$

for all $q \in \mathbb{N}$, $q \geq p(k)$. Therefore, for all $l \in \mathbb{N}$ there exists $q(l) \in \mathbb{N}$ such that

$$1 - \frac{m_l}{n_k} > \alpha \tag{33}$$

for all $k \in \mathbb{N}$, $k \geq q(l)$. By the definition of the relation $\sim$, the fact that $m \sim n$ and the definition of the limit it follows that there exists $k_1 \in \mathbb{N}$ such that

$$\min\left\{\left|1 - \frac{n_l}{m_k}\right| : l \in \mathbb{N}\right\} < \alpha \tag{34}$$

for all $k \in \mathbb{N}$, $k > k_1$. The last inequality implies that

$$\frac{\alpha}{1+\alpha} > 1 - \frac{m_k}{n_{l(k)}} \tag{35}$$

for all $k > k_1$. Finally, notice that $\alpha > \alpha/(1+\alpha)$. It means that the inequalities (33) and (35) are in contradiction. Therefore, the relation $\sim$ is in fact symmetric.

Now we show the transitivity of the relation $\sim$. Suppose that we have three sequences $m, n, p \in \mathscr{C}$ such that $n \sim m$ and $m \sim p$. From the fact $n \sim m$, we conclude that

$$\lim_{k\to\infty}\left|1 - \frac{m_{l_1(k)}}{n_k}\right| = 0, \tag{36}$$

where $l_1(k)$ is any natural number satisfying the condition

$$\left|n_k - m_{l_1(k)}\right| = \min\left\{\left|n_k - m_l\right| : l \in \mathbb{N}\right\}. \tag{37}$$

Let us fix an arbitrary $\varepsilon \in (0,1)$. By the definition of the limit and the equality (36), it follows that there exists $k_1 \in \mathbb{N}$ such that

$$\left|1 - \frac{m_{l_1(k)}}{n_k}\right| < \varepsilon \tag{38}$$

for all $k \in \mathbb{N}$, $k \geq k_1$. The last inequality implies that

$$1 + \varepsilon > \frac{m_{l_1(k)}}{n_k} > 1 - \varepsilon. \tag{39}$$

From the fact that $m \sim p$, it follows that

$$\begin{aligned} 0 &= \lim_{k\to\infty}\min\left\{\left|1 - \frac{p_l}{m_k}\right| : l \in \mathbb{N}\right\} \\ &= \lim_{k\to\infty}\min\left\{\left|1 - \frac{p_l}{m_{l_1(k)}}\right| : l \in \mathbb{N}\right\}. \end{aligned} \tag{40}$$

To obtain the last equality nondecreaseness of the sequence $(l_1(k))_{k\in\mathbb{N}}$ is necessary. If this is not the case, then we can choose the nondecreasing subsequence from $(l_1(k))_{k\in\mathbb{N}}$ and the further reasoning lead for it. Denote by $l_2(k)$ any natural number such that

$$\left|p_{l_2(k)} - m_{l_1(k)}\right| = \min\left\{\left|p_l - m_{l_1(k)}\right| : l \in \mathbb{N}\right\}. \tag{41}$$

Applying the introduced notation and the definition of the limit, we conclude that there exists $k_2 \in \mathbb{N}$ such that

$$\left|1 - \frac{p_{l_2(k)}}{m_{l_1(k)}}\right| < \varepsilon \tag{42}$$

for all $k \in \mathbb{N}$, $k \geq k_2$. The last inequality implies that

$$1 + \varepsilon > \frac{p_{l_2(k)}}{m_{l_1(k)}} > 1 - \varepsilon. \tag{43}$$

From inequalities (39) and (43), we get

$$(1+\varepsilon)^2 > \frac{p_{l_2(k)}}{n_k} > (1-\varepsilon)^2 \tag{44}$$

for $k \in \mathbb{N}$, $k \geq \max\{k_1, k_2\}$; that is,

$$2\varepsilon + \varepsilon^2 > 1 - \frac{p_{l_2(k)}}{n_k} > 2\varepsilon - \varepsilon^2. \tag{45}$$

Due to arbitrariness of selection of $\varepsilon \in (0,1)$, the last inequality means that

$$\lim_{k\to\infty}\left|1 - \frac{p_{l_2(k)}}{n_k}\right| = 0. \tag{46}$$

However, since

$$\left|1 - \frac{p_{l_2(k)}}{n_k}\right| \geq \min\left\{\left|1 - \frac{p_l}{n_k}\right| : l \in \mathbb{N}\right\}, \tag{47}$$

then

$$\lim_{k\to\infty}\min\left\{\left|1 - \frac{p_l}{n_k}\right| : l \in \mathbb{N}\right\} = 0, \tag{48}$$

that is, $n \sim p$. The proof of transitivity of the relation $\sim$ is finished. □

Theorem 6. *If $(a(n))_{n\in\mathbb{N}} \in \mathscr{S}$, $(m_l)_{l\in\mathbb{N}}$, $(n_l)_{l\in\mathbb{N}} \in \mathscr{C}$, and $(m_l)_{l\in\mathbb{N}} \sim (n_l)_{l\in\mathbb{N}}$ and there exists the limit*

$$\lim_{l\to\infty}\frac{1}{m_l}\ln a(m_l), \tag{49}$$

then there exists the limit

$$\lim_{l\to\infty}\frac{1}{n_l}\ln a(n_l) \tag{50}$$

and the limits are equal.

Proof. Since $(a(n))_{n\in\mathbb{N}} \in \mathscr{S}$, then there exists a constant c such that

$$\frac{a(n)}{a(m)} < c^{|n-m|}. \tag{51}$$

For $k \in \mathbb{N}$, denote by $l(k)$ any natural number satisfying the condition

$$\left|m_k - n_{l(k)}\right| = \min\left\{\left|m_k - n_l\right| : l \in \mathbb{N}\right\}. \tag{52}$$

We have

$$\begin{aligned}\left|\frac{1}{n_{l(k)}}\ln a\left(n_{l(k)}\right)-\frac{1}{m_k}\ln a\left(m_k\right)\right| &= \left|\frac{1}{n_{l(k)}}\ln a\left(n_{l(k)}\right)\right.\\ &\left.-\frac{1}{m_l}\ln a\left(n_{l(k)}\right)+\frac{1}{m_l}\ln a\left(n_{l(k)}\right)-\frac{1}{m_l}\ln a\left(m_l\right)\right|\\ &\le\left|\frac{1}{n_{l(k)}}\ln a\left(n_{l(k)}\right)-\frac{1}{m_l}\ln a\left(n_{l(k)}\right)\right|\\ &+\left|\frac{1}{m_l}\ln a\left(n_{l(k)}\right)-\frac{1}{m_l}\ln a\left(m_l\right)\right|=\left|\frac{m_l-n_{l(k)}}{m_l}\right|\\ &\cdot\frac{1}{n_{l(k)}}\ln a\left(n_{l(k)}\right)+\frac{1}{m_l}\left|\ln\frac{a\left(n_{l(k)}\right)}{a\left(m_l\right)}\right|\le\left|\frac{m_l-n_{l(k)}}{m_l}\right|\\ &\cdot\frac{1}{n_{l(k)}}\ln a\left(n_{l(k)}\right)+\frac{\left|m_l-n_{l(k)}\right|}{m_l}\left|\ln c\right|.\end{aligned}\tag{53}$$

To obtain the last inequality, we used inequality (51). Since

$$\lim_{l\to\infty}\frac{m_l-n_{l(k)}}{m_l}=0\tag{54}$$

and there exists the limit

$$\lim_{k\to\infty}\frac{1}{n_{l(k)}}\ln a\left(n_{l(k)}\right),\tag{55}$$

then inequality (53) implies the thesis of the theorem. The proof is completed. □

Denote by $[x]$ the greatest integer no greater than x. The next theorem describes $\lambda(a)$ and $\pi(a)$ by the partial limits of a which correspond to time subsequences of the form $([\theta^n])_{n\in\mathbb{N}}$, where $\theta>1$, $\theta\in\mathbb{R}$.

Theorem 7. *For any sequence $(a(n))_{n\in\mathbb{N}}\in\mathcal{S}$, the following equalities hold:*

$$\lambda(a)=\lim_{\theta\to1^+}\limsup_{n\to\infty}\frac{1}{[\theta^n]}\ln a\left([\theta^n]\right),\tag{56}$$

$$\pi(a)=\lim_{\theta\to1^+}\liminf_{n\to\infty}\frac{1}{[\theta^n]}\ln a\left([\theta^n]\right).\tag{57}$$

Proof. Let $(n_l)_{l\in\mathbb{N}}\in\mathcal{C}$ be such that

$$\lambda(a)=\lim_{l\to\infty}\frac{1}{n_l}\ln a\left(n_l\right).\tag{58}$$

Without loss of generality, for further consideration, we may assume that, for fixed $\theta>1$ in each interval $[[\theta^n],[\theta^{n+1}])$, $n\in\mathbb{N}$, there are no more than one element of the sequence $(n_l)_{l\in\mathbb{N}}$. For $l\in\mathbb{N}$, denote by $m(l)\in\mathbb{N}$ such a number that

$$n_l\in\left[\left[\theta^{m(l)}\right],\left[\theta^{m(l)+1}\right]\right).\tag{59}$$

Additionally, denote by $f:(1,\infty)\to\mathbb{R}$ a function given by

$$f(\theta)=\limsup_{n\to\infty}\frac{1}{[\theta^n]}\ln a\left([\theta^n]\right).\tag{60}$$

Since $(a(n))_{n\in\mathbb{N}}\in\mathcal{S}$, then there exists a constant c, such that

$$\frac{a(n)}{a(m)}<c^{(n-m)}\quad\text{for } n,m\in\mathbb{N},\ n\ge m.\tag{61}$$

In particular, taking $n=n_l$ and $m=[\theta^{n_1(l)}]$, we get

$$a\left(n_l\right)\le a\left(\left[\theta^{m(l)}\right]\right)c^{n_l-[\theta^{m(l)}]}.\tag{62}$$

Using this inequality, the introduced notation, and the definition of upper limit, we have

$$\begin{aligned}f(\theta)\le\lambda(a)&=\lim_{l\to\infty}\frac{1}{n_l}\ln a\left(n_l\right)\\ &\le\lim_{l\to\infty}\left(\frac{1}{n_l}\ln a\left(\left[\theta^{m(l)}\right]\right)+\frac{n_l-\left[\theta^{m(l)}\right]}{n_l}\ln c\right)\\ &\le(1-\theta)\ln c+\limsup_{l\to\infty}\frac{1}{n_l}\ln a\left(\left[\theta^{m(l)}\right]\right).\end{aligned}\tag{63}$$

Denoting

$$r_\theta(l)=\frac{\left[\theta^{m(l)}\right]}{n_l}\tag{64}$$

we get

$$\begin{aligned}f(\theta)&\le(1-\theta)\ln c\\ &+\left(\limsup_{l\to\infty}r_\theta(l)\frac{1}{\left[\theta^{m(l)}\right]}\ln a\left(\left[\theta^{m(l)}\right]\right)\right)\\ &\le(1-\theta)\ln c\\ &+\left(\limsup_{l\to\infty}r_\theta(l)\right)\left(\limsup_{l\to\infty}\frac{1}{\left[\theta^{m(l)}\right]}\ln a\left(\left[\theta^{m(l)}\right]\right)\right)\\ &\le(1-\theta)\ln c+r_\theta f(\theta),\end{aligned}\tag{65}$$

where

$$r_\theta=\limsup_{l\to\infty}r_\theta(l).\tag{66}$$

From inequalities (63) and (65), we have

$$f(\theta)\le\lambda(a)\le(1-\theta)\ln c+r_\theta f(\theta).\tag{67}$$

Passing in the last inequality to upper limit with $\theta\to1^+$ and taking into account that

$$\lim_{\theta\to1^+}r_\theta=1\tag{68}$$

we get

$$\lambda(a)=\limsup_{\theta\to1^+}f(\theta).\tag{69a}$$

Analogically, passing to the lower limit with $\theta\to1^+$, we obtain

$$\lambda(a)=\liminf_{\theta\to1^+}f(\theta).\tag{69b}$$

Equality (56) follows from the equalities (69a) and (69b). In the same way, one can prove (57). The proof is completed. □

3. Main Results

Consider a solution $(x(n,x_0))_{n\in\mathbb{N}}$ of system (1) and denote by c a common bound for the sequences $(\|A^{-1}(n)\|)_{n\in\mathbb{N}}$ and $(\|A(n)\|)_{n\in\mathbb{N}}$. We have

$$\begin{aligned}
\frac{\|x(n+1,x_0)\|}{\|x(n,x_0)\|} &= \frac{\|A(n)\,x(n,x_0)\|}{\|x(n,x_0)\|} \le \|A(n)\| \le c,\\
\frac{\|x(n+1,x_0)\|}{\|x(n,x_0)\|} &= \frac{\|x(n+1,x_0)\|}{\|A^{-1}(n)\,A(n)\,x(n,x_0)\|}\\
&= \frac{\|x(n+1,x_0)\|}{\|A^{-1}(n)\,x(n+1,x_0)\|}\\
&\ge \frac{\|x(n+1,x_0)\|}{\|A^{-1}(n)\|\,\|x(n+1,x_0)\|} \ge \frac{1}{c}.
\end{aligned} \tag{70}$$

The two above inequalities show that $(x(n,x_0))_{n\in\mathbb{N}} \in \mathcal{S}$.

Applying Theorems 1, 3, and 6 to the sequence $(\|x(n,x_0)\|)_{n\in\mathbb{N}}$, we get the following result.

Theorem 8. *The set of limit points of the sequence* $((1/n)\ln\|x(n,x_0)\|)_{n\in\mathbb{N}}$ *is the interval* $[\pi(x_0),\lambda(x_0)]$. *If the sequence* $(T_n)_{n\in\mathbb{N}} \in \mathcal{C}$ *satisfies assumption (20), then for any number* $\alpha \in$ *there exists a sequence* $(n_l)_{l\in\mathbb{N}} \in \mathcal{C}$ *such that*

$$\alpha = \lim_{l\to\infty}\frac{1}{T_{n_l}}\ln\left\|x\left(T_{n_l},x_0\right)\right\|. \tag{71}$$

Moreover, if $m = (m_l)_{l\in\mathbb{N}} \in \mathcal{C}$ *and* $n \sim m$, *then*

$$\alpha = \lim_{l\to\infty}\frac{1}{T_{m_l}}\ln\left\|x\left(T_{m_l},x_0\right)\right\|. \tag{72}$$

Notice that each arithmetic sequence satisfies condition (20). Then, taking in the above theorem $\alpha = \pi(x_0)$ or $\alpha = \lambda(x_0)$ we conclude that the Lyapunov and Perron exponents are achieved at a certain subsequence of any arithmetic sequence. We do not know whether the analogous statement is true for geometric sequences. But, applying Theorem 7 to the sequence $(\|x(n,x_0)\|)_{n\in\mathbb{N}}$, we may formulate the following result.

Theorem 9. *For any solution* $(x(n,x_0))_{n\in\mathbb{N}}$ *of system (1), we have*

$$\begin{aligned}
\lambda(x_0) &= \lim_{\theta\to1^+}\limsup_{n\to\infty}\frac{1}{[\theta^n]}\ln\left\|x\left([\theta^n],x_0\right)\right\|,\\
\pi(x_0) &= \lim_{\theta\to1^+}\liminf_{n\to\infty}\frac{1}{[\theta^n]}\ln\left\|x\left([\theta^n],x_0\right)\right\|.
\end{aligned} \tag{73}$$

4. Conclusions

In the paper for the discrete time-varying linear system, we described the limits points of the sequence $((1/n)\ln\|x(n,x_0)\|)_{n\in\mathbb{N}}$. This set is equal to the interval $[\pi(x_0),\lambda(x_0)]$. Moreover, we proved that each partial limit of this sequence is achievable on a certain subsequence of any sequence satisfying condition (20), in particular on certain subsequence of any arithmetic sequence (Theorem 8). Finally, we showed that the Perron and Lyapunov exponents may be approximated by subsequences in certain sense similar to geometric sequences (Theorem 9). The objective of future works will be the investigation of the possibility of omitting limits with $\theta \to 1^+$ in equalities (73).

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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