Research Article

Sharp One-Parameter Mean Bounds for Yang Mean

Wei-Mao Qian,¹ Yu-Ming Chu,² and Xiao-Hui Zhang²

¹School of Distance Education, Huzhou Broadcast and TV University, Huzhou 313000, China

²School of Mathematics and Computation Sciences, Hunan City University, Yiyang 413000, China

Correspondence should be addressed to Yu-Ming Chu; chuyuming2005@126.com

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We prove that the double inequality $J_\alpha(a, b) < U(a, b) < J_\beta(a, b)$ holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \sqrt{2}/(\pi - \sqrt{2}) = 0.8187 \dots$ and $\beta \geq 3/2$, where $U(a, b) = (a-b)/[\sqrt{2} \arctan((a-b)/\sqrt{2ab})]$, and $J_p(a, b) = p(a^{p+1} - b^{p+1})/[(p+1)(a^p - b^p)]$ ($p \neq 0, -1$), $J_0(a, b) = (a-b)/(\log a - \log b)$, and $J_{-1}(a, b) = ab(\log a - \log b)/(a-b)$ are the Yang and p th one-parameter means of a and b , respectively.

1. Introduction

Let $p \in \mathbb{R}$ and $a, b > 0$ with $a \neq b$. Then the p th one-parameter mean $J_p(a, b)$, p th power mean $M_p(a, b)$, harmonic mean $H(a, b)$, geometric mean $G(a, b)$, logarithmic mean $L(a, b)$, first Seiffert mean $P(a, b)$, identric mean $I(a, b)$, arithmetic mean $A(a, b)$, Yang mean $U(a, b)$, second Seiffert mean $T(a, b)$, and quadratic mean $Q(a, b)$ are, respectively, defined by

$$J_p(a, b) = \begin{cases} \frac{p(a^{p+1} - b^{p+1})}{(p+1)(a^p - b^p)}, & p \neq 0, -1, \\ \frac{a-b}{\log a - \log b}, & p = 0, \\ \frac{ab(\log a - \log b)}{a-b}, & p = -1, \end{cases}$$

$$M_p(a, b) = \left[\frac{a^p + b^p}{2} \right]^{1/p} \quad (p \neq 0),$$

$$M_0(a, b) = \sqrt{ab},$$

$$H(a, b) = \frac{2ab}{a+b},$$

$$G(a, b) = \sqrt{ab},$$

$$L(a, b) = \frac{b-a}{\log b - \log a},$$

$$P(a, b) = \frac{a-b}{2 \arcsin((a-b)/(a+b))},$$

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)},$$

$$A(a, b) = \frac{a+b}{2},$$

$$U(a, b) = \frac{a-b}{\sqrt{2} \arctan((a-b)/\sqrt{2ab})},$$

$$T(a, b) = \frac{a-b}{2 \arctan((a-b)/(a+b))},$$

$$Q(a, b) = \sqrt{\frac{a^2 + b^2}{2}}.$$

(1)

It is well known that both the means $J_p(a, b)$ and $M_p(a, b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Recently, the one-parameter mean $J_p(a, b)$ and Yang mean $U(a, b)$ have attracted the attention of many researchers.

Alzer [1] proved that the inequalities

$$\begin{aligned} G(a, b) &< \sqrt{J_p(a, b) J_{-p}(a, b)} < L(a, b) \\ &< \frac{J_p(a, b) + J_{-p}(a, b)}{2} < A(a, b) \end{aligned} \quad (2)$$

hold for all $a, b > 0$ with $a \neq b$ and $p \neq 0$.

In [2, 3], the authors discussed the monotonicity and logarithmic convexity properties of the one-parameter mean $J_p(a, b)$.

In [4, 5], the authors proved that the double inequalities

$$\begin{aligned} J_{p_1}(a, b) &< \alpha A(a, b) + (1 - \alpha) L(a, b) < J_{q_1}(a, b) \\ J_{p_2}(a, b) &< \alpha A(a, b) + (1 - \alpha) H(a, b) < J_{q_2}(a, b), \end{aligned} \quad (3)$$

hold for all $a, b > 0$ with $a \neq b$ and $\alpha \in (0, 1)$ if and only if $p_1 \leq \alpha/(2 - \alpha)$, $q_1 \geq \alpha$, $p_2 \leq 3\alpha - 2$, and $q_2 \geq \alpha/(2 - \alpha)$.

Xia et al. [6] proved that the double inequality

$$\begin{aligned} J_{(3\alpha-1)/2}(a, b) &< \alpha A(a, b) + (1 - \alpha) G(a, b) \\ &< J_{\alpha/(2-\alpha)}(a, b) \end{aligned} \quad (4)$$

holds for all $a, b > 0$ with $a \neq b$ if $\alpha \in (0, 2/3)$, and inequality (4) is reversed if $\alpha \in (2/3, 1)$.

Gao and Niu [7] presented the best possible parameters p and q such that the double inequality $J_p(a, b) < A^\alpha(a, b) G^\beta(a, b) H^{1-\alpha-\beta}(a, b) < J_q(a, b)$ holds for all $a, b > 0$ with $a \neq b$ and $\alpha + \beta \in (0, 1)$.

In [8, 9], the authors proved that the double inequalities

$$\begin{aligned} J_{\lambda_1}(a, b) &< T(a, b) < J_{\mu_1}(a, b), \\ J_{\lambda_2}(a, b) &< I(a, b) < J_{\mu_2}(a, b) \end{aligned} \quad (5)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 2/(2 - \pi)$, $\mu_1 \geq 2$, $\lambda_2 \leq 1/2$, and $\mu_2 \geq 1/(e - 1)$.

Xia et al. [10] found that $M_{(1+2p)/3}(a, b)$ is the best possible lower power mean bound for the one-parameter mean $J_p(a, b)$ if $p \in (-2, -1/2) \cup (1, \infty)$ and $M_{(1+2p)/3}(a, b)$ is the best possible upper power mean bound for the one-parameter mean $J_p(a, b)$ if $p \in (-\infty, -2) \cup (-1/2, 1)$.

For all $a, b > 0$ with $a \neq b$, Yang [11] provided the bounds for the Yang mean $U(a, b)$ in terms of other bivariate means as follows:

$$\begin{aligned} P(a, b) &< U(a, b) < T(a, b), \\ \frac{G(a, b) T(a, b)}{A(a, b)} &< U(a, b) < \frac{P(a, b) Q(a, b)}{A(a, b)}, \\ Q^{1/2}(a, b) \left[\frac{2G(a, b) + Q(a, b)}{3} \right]^{1/2} &< U(a, b) \\ &< Q^{2/3}(a, b) \left[\frac{G(a, b) + Q(a, b)}{2} \right]^{1/3}, \\ \frac{G(a, b) + Q(a, b)}{2} &< U(a, b) \\ &< \left[\frac{2}{3} \left(\frac{G(a, b) + Q(a, b)}{2} \right)^{1/2} + \frac{1}{3} Q^{1/2}(a, b) \right]^2. \end{aligned} \quad (6)$$

In [12, 13], the authors proved that the double inequalities

$$\begin{aligned} \left[\frac{2}{3} \left(\frac{G(a, b) + Q(a, b)}{2} \right)^p + \frac{1}{3} Q^p(a, b) \right]^{1/p} &< U(a, b) \\ &< \left[\frac{2}{3} \left(\frac{G(a, b) + Q(a, b)}{2} \right)^q + \frac{1}{3} Q^q(a, b) \right]^{1/q} \\ \frac{2^{1-\lambda} (G(a, b) + Q(a, b))^\lambda Q(a, b) + G(a, b) Q^\lambda(a, b)}{2^{1-\lambda} (G(a, b) + Q(a, b))^\lambda + Q^\lambda(a, b)} &< U(a, b) \\ &< \frac{2^{1-\mu} (G(a, b) + Q(a, b))^\mu Q(a, b) + G(a, b) Q^\mu(a, b)}{2^{1-\mu} (G(a, b) + Q(a, b))^\mu + Q^\mu(a, b)}, \\ M_\alpha(a, b) &< U(a, b) < M_\beta(a, b), \end{aligned} \quad (7)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $p \leq p_0$, $q \geq 1/5$, $\lambda \geq 1/5$, $\mu \leq p_1$, $\alpha \leq 2 \log 2 / (2 \log \pi - \log 2)$, and $\beta \geq 4/3$, where $p_0 = 0.1941 \dots$ is the unique solution of the equation $p \log(2/\pi) - \log(1 + 2^{1-p}) + \log 3 = 0$ on the interval $(1/10, \infty)$, and $p_1 = \log(\pi - 2) / \log 2 = 0.1910 \dots$.

Very recently, Zhou et al. [14] proved that $\alpha = 1/2$ and $\beta = \log 3 / (1 + \log 2) = 0.6488 \dots$ are the best possible parameters such that the double inequality

$$\begin{aligned} \left[\frac{a^\alpha + (ab)^{\alpha/2} + b^\alpha}{3} \right]^{1/\alpha} &< U(a, b) \\ &< \left[\frac{a^\beta + (ab)^{\beta/2} + b^\beta}{3} \right]^{1/\beta} \end{aligned} \quad (8)$$

holds for all $a, b > 0$ with $a \neq b$.

The aim of this paper is to present the best possible parameters α and β such that the double inequality $J_\alpha(a, b) < U(a, b) < J_\beta(a, b)$ holds for all $a, b > 0$ with $a \neq b$.

2. Main Result

In order to prove our main result we need a lemma, which we present in this section.

Lemma 1. Let $p \in \mathbb{R}$, and

$$\begin{aligned} f(x, p) &= px^{4p+6} - (p+1)x^{4p+5} + px^{4p+4} \\ &\quad - (p+1)x^{4p+1} - p(p+1)x^{2p+7} \\ &\quad + 2(p+1)^2 x^{2p+5} - 2px^{2p+4} \\ &\quad - 2p(p+1)x^{2p+3} - 2px^{2p+2} \\ &\quad + 2(p+1)^2 x^{2p+1} - p(p+1)x^{2p-1} \\ &\quad - (p+1)x^5 + px^2 - (p+1)x + p. \end{aligned} \quad (9)$$

Then the following statements are true:

- (1) if $p = 3/2$, then $f(x, p) > 0$ for all $x \in (1, \infty)$;

(2) if $p = \sqrt{2}/(\pi - \sqrt{2}) = 0.8187 \dots$, then there exists $\lambda \in (1, \infty)$ such that $f(x, p) < 0$ for $x \in (1, \lambda)$ and $f(x, p) > 0$ for $x \in (\lambda, \infty)$.

Proof. For part (1), if $p = 3/2$, then (9) becomes

$$f(x, p) = \frac{1}{4}(x-1)^6(x^2+2x+2)(2x^2+2x+1) \cdot (3x^2+4x+3). \quad (10)$$

Therefore, part (1) follows from (10).

For part (2), let $p = \sqrt{2}/(\pi - \sqrt{2})$, $f_1(x, p) = \partial f(x, p)/\partial x$, $f_2(x, p) = (1/2)(\partial f_1(x, p)/\partial x)$, $f_3(x, p) = (1/(p+1)x^2)(\partial f_2(x, p)/\partial x)$, $f_4(x, p) = (x^{7-2p}/2p)(\partial f_3(x, p)/\partial x)$, $f_5(x, p) = (1/2x)(\partial f_4(x, p)/\partial x)$, $f_6(x, p) = \partial f_5(x, p)/\partial x$, $f_7(x, p) = (1/2)(\partial f_6(x, p)/\partial x)$, $f_8(x, p) = \partial f_7(x, p)/\partial x$, $f_9(x, p) = (1/2(p+1))(\partial f_8(x, p)/\partial x)$, and $f_{10}(x, p) = \partial f_9(x, p)/\partial x$. Then elaborated computations lead to

$$\lim_{x \rightarrow 1} f(x, p) = 0, \quad (11)$$

$$\lim_{x \rightarrow +\infty} f(x, p) = +\infty,$$

$$\lim_{x \rightarrow 1} f_1(x, p) = 0, \quad (12)$$

$$\lim_{x \rightarrow +\infty} f_1(x, p) = +\infty,$$

$$\lim_{x \rightarrow 1} f_2(x, p) = 0, \quad (13)$$

$$\lim_{x \rightarrow +\infty} f_2(x, p) = +\infty,$$

$$\lim_{x \rightarrow 1} f_3(x, p) = 0, \quad (14)$$

$$\lim_{x \rightarrow +\infty} f_3(x, p) = +\infty,$$

$$\lim_{x \rightarrow 1} f_4(x, p) = -48(p+1)\left(\frac{3}{2} - p\right) < 0, \quad (15)$$

$$\lim_{x \rightarrow +\infty} f_4(x, p) = +\infty,$$

$$\lim_{x \rightarrow 1} f_5(x, p) = -192(p+1)^2\left(\frac{3}{2} - p\right) < 0, \quad (16)$$

$$\lim_{x \rightarrow +\infty} f_5(x, p) = +\infty,$$

$$\lim_{x \rightarrow 1} f_6(x, p) = 2(p+1)(368p^3 + 332p^2 - 484p - 963) < 0, \quad (17)$$

$$\lim_{x \rightarrow +\infty} f_6(x, p) = +\infty, \quad (18)$$

$$\lim_{x \rightarrow 1} f_7(x, p) = (p+1)(1024p^4 + 2096p^3 + 1844p^2 - 2876p - 5193) < 0, \quad (19)$$

$$\lim_{x \rightarrow +\infty} f_7(x, p) = +\infty, \quad (20)$$

$$\lim_{x \rightarrow 1} f_8(x, p) = (p+1)(2560p^5 + 6336p^4 + 14176p^3 + 11028p^2 - 12680p - 22005) < 0, \quad (21)$$

$$\lim_{x \rightarrow +\infty} f_8(x, p) = +\infty, \quad (22)$$

$$\lim_{x \rightarrow 1} f_9(x, p) = 6(512p^6 + 1184p^5 + 4064p^4 + 6372p^3 + 4068p^2 - 3495p - 5775) = 15.2085 \dots > 0, \quad (23)$$

$$\begin{aligned} f_{10}(x, p) = & (p+2)(2p+1)(2p+3)^2(2p+5)(2p \\ & + 7)(4p+1)(4p+5)x^{2p} - 8p(p+1)(p+2)(p \\ & + 3)(2p+1)(2p+3)(4p+3)(4p+5)x^{2p-1} \\ & + p(2p-1)(2p+1)^2(2p+3)(2p+5)(4p-1) \\ & \cdot (4p+3)x^{2p-2} - 8p(p-1)^2(p-2)(2p-1)(2p \\ & - 3)(16p^2-1)x^{2p-5} - 720(p+3)(2p+5)(2p \\ & + 7)x. \end{aligned} \quad (24)$$

Note that

$$\begin{aligned} 2p > 1 > 2p-1 > 0 > 2p-2 > 2p-5, \\ 1536p^7 + 15040p^6 + 59440p^5 + 122280p^4 \\ + 137144p^3 + 61850p^2 - 49845p - 72450 \\ = 85165.4405 \dots > 0. \end{aligned} \quad (25)$$

It follows from (24) and (25) that

$$\begin{aligned} f_{10}(x, p) > & [(p+2)(2p+1)(2p+3)^2(2p+5) \\ & \cdot (2p+7)(4p+1)(4p+5) - 8p(p+1)(p+2) \\ & \cdot (p+3)(2p+1)(2p+3)(4p+3)(4p+5) \\ & - 720(p+3)(2p+5)(2p+7)]x + [p(2p-1) \\ & \cdot (2p+1)^2(2p+3)(2p+5)(4p-1)(4p+3) \\ & - 8p(p-1)^2(p-2)(2p-1)(2p-3)(16p^2-1)] \\ & \cdot x^{2p-2} = (1536p^7 + 15040p^6 + 59440p^5 \\ & + 122280p^4 + 137144p^3 + 61850p^2 - 49845p \\ & - 72450)x + p(2p-1)(4p-1)(704p^4 + 136p^3 \\ & + 1120p^2 + 248p - 3)x^{2p-2} > 0, \end{aligned} \quad (26)$$

for $x \in (1, \infty)$.

From (23) and (26) we clearly see that $f_8(x, p)$ is strictly increasing with respect to x on the interval $(1, \infty)$. Then (21) and (22) lead to the conclusion that there exists $\lambda_1 > 1$ such

that the function $x \rightarrow f_7(x, p)$ is strictly decreasing on $(1, \lambda_1]$ and strictly increasing on $[\lambda_1, \infty)$.

It follows from (19) and (20) together with the piecewise monotonicity of the function $x \rightarrow f_7(x, p)$ that there exists $\lambda_2 > 1$ such that the function $x \rightarrow f_6(x, p)$ is strictly decreasing on $(1, \lambda_2]$ and strictly increasing on $[\lambda_2, \infty)$.

Making use of (13)–(18) and the same method as the above we know that there exists $\lambda_i > 1$ ($i = 3, 4, 5, 6, 7$) such that the function $x \rightarrow f_{8-i}(x, p)$ is strictly decreasing on $(1, \lambda_i]$ and strictly increasing on $[\lambda_i, \infty)$.

It follows from (12) and the piecewise monotonicity of the function $x \rightarrow f_1(x, p)$ that there exists $\lambda^* > 1$ such that the function $x \rightarrow f(x, p)$ is strictly decreasing on $(1, \lambda^*]$ and strictly increasing on $[\lambda^*, \infty)$.

Therefore, part (2) follows easily from (11) and the piecewise monotonicity of the function $x \rightarrow f(x, p)$. $\square$

Theorem 2. *The double inequality*

$$J_\alpha(a, b) < U(a, b) < J_\beta(a, b) \quad (27)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \sqrt{2}/(\pi - \sqrt{2}) = 0.8187 \dots$ and $\beta \geq 3/2$.

Proof. Since $U(a, b)$ and $J_p(a, b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a = x^2 > 1$ and $b = 1$. Let $p \in \mathbb{R}$ and $p \neq 0, -1$. Then (1) lead to

$$\begin{aligned} J_p(a, b) - U(a, b) &= J_p(x^2, 1) - U(x^2, 1) \\ &= \frac{p(x^{2p+2} - 1)}{(p+1)(x^{2p} - 1) \arctan((x^2 - 1)/\sqrt{2}x)} F(x, p), \end{aligned} \quad (28)$$

where

$$\begin{aligned} F(x, p) &= \arctan\left(\frac{x^2 - 1}{\sqrt{2}x}\right) \\ &\quad - \frac{(p+1)(x^2 - 1)(x^{2p} - 1)}{\sqrt{2}p(x^{2p+2} - 1)}, \end{aligned} \quad (29)$$

$$\lim_{x \rightarrow 1} F(x, p) = 0, \quad (30)$$

$$\frac{\partial F(x, p)}{\partial x} = \frac{\sqrt{2}}{p(x^4 + 1)(x^{2p+2} - 1)^2} f(x, p), \quad (31)$$

where $f(x, p)$ is defined by (9).

We divide the proof into four cases.

Case 1 ($p = \sqrt{2}/(\pi - \sqrt{2})$). Then it follows from Lemma 1(2), (29), and (31) that there exists $\lambda > 1$ such that the function $x \rightarrow F(x, p)$ is strictly decreasing on $(1, \lambda]$ and strictly increasing on $[\lambda, \infty)$, and

$$\lim_{x \rightarrow \infty} F(x, p) = 0. \quad (32)$$

Therefore,

$$J_{\sqrt{2}/(\pi - \sqrt{2})}(a, b) < U(a, b) \quad (33)$$

follows easily from (28), (30), and (32) together with the piecewise monotonicity of the function $x \rightarrow F(x, p)$.

Case 2 ($p > \sqrt{2}/(\pi - \sqrt{2})$). Then (1) leads to

$$\lim_{x \rightarrow \infty} \frac{J_p(x, 1)}{U(x, 1)} = \frac{\sqrt{2}p}{2(p+1)}\pi > 1. \quad (34)$$

Inequality (34) implies that there exists large enough $X = X(p) > 1$ such that $U(a, b) < J_p(a, b)$ for all $a, b > 0$ with $a/b \in (0, 1/X) \cup (X, \infty)$.

Case 3 ($p = 3/2$). Then from Lemma 1(1) and (31) we know that the function $x \rightarrow F(x, p)$ is strictly increasing on the interval $(1, \infty)$. Therefore,

$$U(a, b) < J_{3/2}(a, b) \quad (35)$$

follows from (28) and (30) together with the monotonicity of the function $x \rightarrow F(x, p)$.

Case 4 ($0 < p < 3/2$). Let $x > 0$ and $x \rightarrow 0$; then making use of Taylor expansion we get

$$\begin{aligned} U(1, 1+x) - J_p(1, 1+x) &= \frac{x}{\sqrt{2} \arctan(x/\sqrt{2}(1+x))} \\ &\quad - \frac{p[1 - (1+x)^{p+1}]}{(p+1)[1 - (1+x)^p]} = \frac{3-2p}{24}x^2 + o(x^2). \end{aligned} \quad (36)$$

Equation (36) implies that there exists small enough $\delta > 0$ such that $U(1, 1+x) > J_p(1, 1+x)$ for all $x \in (0, \delta)$. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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