Research Article

An MDADT-Based Approach for $L_2$-Gain Analysis of Discrete-Time Switched Delay Systems

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Received 8 October 2015; Accepted 14 February 2016

We study the $L_2$-gain analysis problem for a class of discrete-time switched systems with time-varying delays. A mode-dependent average dwell time (MDADT) approach is applied to analyze the $L_2$-gain performance for these discrete-time switched delay systems. Combining a multiple Lyapunov functional method with the MDADT approach, sufficient conditions expressed in terms of a set of feasible linear matrix inequalities (LMIs) are established to guarantee the $L_2$-gain performance. Finally, a numerical example will be provided to demonstrate the validity and usefulness of the obtained results.

1. Introduction

Switched systems consist of a finite number of subsystems and a logical law which orchestrates the switching behaviors between these subsystems. These dynamical systems can mathematically model many practical engineering applications with switching characteristics in a variety of disciplines; see, for example, [1–7].

A constrained switching signal can be regarded as a powerful tool to stabilize and control these switched systems [8–10]. Among them, the average dwell time (ADT) switching is the most common and typical one. It guarantees that the number of types of switching in a finite interval be bounded and the average time between any two types of consecutive switching be not less than a positive constant [11, 12]. In recent years, it has been recognised that ADT is flexible and efficient for dynamics analysis of many switched systems [8, 13–16]. However, the ADT switching's property that the average time interval between any two types of consecutive switching should be greater than a positive number $\tau_0$ makes the dwell time independent of the system modes. Hence whether the dwelling at some classes of subsystems will deteriorate the disturbance attenuation cannot be predicted.

As shown in [17], the minimum of admissible ADT is computed by two mode-independent parameters: the increase coefficient of the Lyapunov-like function and the decay rate of the Lyapunov function, which will cause certain conservativeness. To solve the problem, more recently, a new mode-dependent ADT concept has been introduced in [18]. Two mode-independent parameters can be set in a mode-dependent manner, which will reduce the conservativeness. Even though stability analysis for the switched systems with MDADT has been investigated extensively (see, e.g., [17, 18]), how to solve the $L_2$-gain problem of the switched systems with MDADT is interesting and worthwhile to study. This has motivated our study in this paper.

The rest of the paper is as follows. In Section 2, we introduce the class of discrete-time switched system, some necessary definitions, and lemmas. In Section 3, sufficient conditions for ensuring $L_2$-gain for the discrete-time switched delay system are constructed. In Section 4, a numerical example is presented to illustrate the obtained results. Conclusion remarks are given in Section 5.
2. Preliminaries and Problem Statement

Consider a discrete-time switched system with a time-varying delay:

\[
L_i: \begin{cases}
    x(t+1) = A_i x(t) + B_i x(t-d(t)) + C_i w(t), \\
    x_{i_0}(t) = x(t_0 + l) = \phi(l), \\
    z(t) = D_i x(t) + E_i w(t),
\end{cases}
\]

where \(x(t) \in \mathbb{R}^n\) is the system state, \(z(t) \in \mathbb{R}^m\) is the controlled output, \(\phi(l)\) is the disturbance input which belongs to \(L_2[0, \infty)\). \(d(t)\) is the time-varying delay and satisfies \(0 < d_m < d(t) \leq d_M\), where \(d_m\) and \(d_M\) denote the upper and lower bounds of the delays. \(i\) is the switching signal, which takes its values in the finite set \(S = \{1, \ldots, M\}\), where \(M\) is the number of subsystems. When \(t \in [t_i, t_{i+1}), i \in \mathbb{N}\), we call the \(i\)th subsystem active. \(A_p, B_p, C_p, D_p, E_p\) are constant matrices with appropriate dimension. When \(i = p = 1, \ldots, m\), it represents the \(p\)th subsystem or \(p\)th mode of (1).

To proceed, we will need the following definitions and lemmas.

Definition 1 (see [11]). For any \(T_0 > T_1 \geq 0\) and any switching signal \(i\), \(T_1 \leq t < T_2\), let \(N_I(T_1, T_2)\) denote the number of switches of type \(i\) over \((T_1, T_2)\). If \(N_I(T_1, T_2) \leq N_0 + T_2 - T_1/T_a\) holds for \(N_0 \geq 0\) and \(T_a > 0\), then \(T_a\) is the average dwell time and \(N_0\) is the chatter bound. Without loss of generality, we choose \(N_0 = 0\).

Definition 2 (see [18]). For a switching signal \(i\) and any \(T \geq t \geq 0\), let \(N_{I_p}(T, t)\) be the switching numbers in which the \(p\)th subsystem is activated over the interval \([t, T]\) and let \(T_p(T, t)\) denote the total running time of the \(p\)th subsystem over the interval \([t, T]\), \(p \in S\). We say that \(i\) has a mode-dependent average dwell time (MDADT) \(\tau_{ap}\) if there exist positive numbers \(N_{op}\) and \(\tau_{ap}\) such that

\[
N_{I_p}(T, t) \leq N_{op} + \frac{T_p(T, t)}{\tau_{ap}}, \quad \forall T \geq t \geq 0
\]

and we call \(N_{op}\) the mode-dependent chatter bounds. Here, we choose \(N_{op} = 0\) as well.

Definition 3. For \(y > 0\), the switched delay system (1) is said to have \(L_2\)-gain property, if, under zero initial condition \(\phi(l) = 0, l \in [t_0 - d_M, t_0]\), it holds that

\[
\int_0^\infty z^T(s) z(s) ds \leq y^2 \int_0^\infty w^T(s) w(s) ds.
\]

Lemma 4. For any given matrices \(X, Y \in \mathbb{R}^{n \times n}\), it holds that

\[
X^T Y + Y^T X \leq \delta X^T X + \delta^{-1} Y^T Y,
\]

where \(\delta\) is any given positive constant.

Lemma 5 (see [6]). Let \(A, D, E, F,\) and \(P\) be real matrices of appropriate dimensions with \(P > 0 \) and \( F \) satisfying \(F^T F \leq I\). Then for any scalar \(\varepsilon > 0\) satisfying \(P^{-1} - \varepsilon^{-1} DD^T > 0\), one has

\[
(A + DFE)^T P (A + DFE) \leq A^T (P^{-1} - \varepsilon^{-1} DD^T)^{-1} A + \varepsilon E^T E.
\]

Lemma 6 (Schur complement). Let \(M, P,\) and \(Q\) be given matrices such that \(Q > 0\). Then

\[
\begin{bmatrix}
    P & M \\
    * & -Q
\end{bmatrix} \leq 0 \iff P + M Q^{-1} M^T < 0.
\]

Lemma 7 (see [13]). Let \(\phi(k) \in \mathbb{R}^r\) be a vector-valued function. If there exist any matrices \(R > 0, G_1, G_2,\) and a scalar \(\delta \geq 0\), then the following inequality

\[
- \sum_{s=k-d}^{k-1} N^T(s) R N(s) \leq \eta^T(k) \begin{bmatrix} G_1 + G_1^T - G_1^T G_2 + G_2 & \* \\ \* & -G_2 - G_2^T \end{bmatrix} \eta(k) + \eta^T(k) \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix} dR^{-1} [G_1, G_2] \eta(k)
\]

holds, where \(N(s) = \phi(s+1) - \phi(s)\) and \(\eta(t) = \begin{bmatrix} \phi(t) \\ \phi(t-d) \end{bmatrix}\).

3. \(L_2\)-Gain Analysis

Firstly, we will introduce two important lemmas for the \(L_2\)-gain analysis of the switched delay system (1). The first lemma will provide the decay estimation of the Lyapunov functional \(V_i(t)\) along the trajectory of the switched delay system without disturbances.

Lemma 8. Consider the switched delay system (1) with \(w(t) = 0\). For given positive integers \(d_M, d_m,\) and \(\lambda_i\), suppose that there exist matrices \(G_1, G_2, \Omega_1, \Omega_2,\) and \(\Omega_3\) such that

\[(i) \quad \Omega_3 \leq 0.\]
\( \Omega_1 - \Omega_2 \Omega_3^{-1} \Omega_2^T \leq 0, \) \hspace{1cm} (9)

where

\[
\Omega_1 = A_i^T P_i A_i - P_i + (d_M - d_m) \lambda_i^{-2} Q_i + \lambda_i^{-2} d_M \left[ \lambda_i^2 A_i^T R_i A_i - \lambda_i R_i A_i - \lambda_i A_i^T R_i + R_i \right] \\
+ \lambda_i^{-2} \left( G_i + G_i^T + d_M G_i^T R_i^{-1} G_i \right),
\]

\[
\Omega_2 = B_i^T P_i B_i + d_M \left( A_i^T R_i B_i - \lambda_i^{-1} R_i B_i \right)
\]

\[
\Omega_3 = B_i^T P_i B_i - \lambda_i^{-2 (1 + d_M)} Q_i + d_M B_i^T R_i B_i \\
+ \lambda_i^{-2} \left( -G_i + G_i^T + d_M G_i^T R_i^{-1} G_i \right),
\]

with \( P_i, Q_i, \) and \( R_i \) being symmetric positive definite matrices; then the Lyapunov functional \( V_i(t) \) along the trajectory of the switched delay system (1) will satisfy

\[
V_i(t) \leq \lambda_i^{-2(t-t_0)} V_i(t_0). \hspace{1cm} (11)
\]

**Proof.** Choose the following Lyapunov functional candidate:

\[
V_i(t) = V_{i_1}(t) + V_{i_2}(t) + V_{i_3}(t) + V_{i_4}(t). \hspace{1cm} (12)
\]

Here,

\[
V_{i_1}(t) = x^T(t) P_i x(t),
\]

\[
V_{i_2}(t) = \sum_{s=-d(t)}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s),
\]

\[
V_{i_3}(t) = \sum_{s=-d(t)+2}^{t-1} \sum_{s=-t+1+\theta}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s),
\]

\[
V_{i_4}(t) = \sum_{s=-d(t)+1}^{t-1} \sum_{s=-t+1+\theta}^{t-1} \lambda_i^{2(s-t)} y^T(s) R_i y(s),
\]

where \( P_i, Q_i, \) and \( R_i \) are symmetric positive definite matrices, \( \lambda_i > 1 \) is a given constant, and \( y(s) = \lambda_i x(s+1) - x(s) \). Next, we will estimate the difference of \( V_i(t) \) along the trajectory of the switched delay system (1):

\[
\Delta V_{i_1}(t) = V_{i_1}(t+1) - V_{i_1}(t) \\
= x^T(t+1) P_i x(t+1) - x^T(t) P_i x(t)
\]

Then, we have

\[
\Delta V_i(t) = \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}^T \begin{bmatrix} A_i^T P_i A_i - P_i & A_i^T P_i B_i \\ B_i^T P_i A_i & B_i^T P_i B_i \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \end{bmatrix}.
\]

\[
\Delta V_{i_2}(t) = V_{i_2}(t+1) - V_{i_2}(t) \leq V_{i_2}(t+1) - V_{i_2}(t) = \lambda_i^{-2} V_{i_2}(t)
\]

Since the delay \( d(t) \) satisfies \( 0 < d_m < d(t) \leq d_M \), we can consider the following two cases.

When \( d_m > 1 \), it holds that

\[
\sum_{s=t+1-d(t+1)}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s)
\]

\[
\leq \sum_{s=t+1-d_M}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s)
\]

\[
+ \sum_{s=t+1-d(t+1)}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s)
\]

\[
\leq \sum_{s=t+1-d(t+1)}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s)
\]

\[
+ \sum_{s=t+1-d_M}^{t-1} \lambda_i^{2(s-t)} x^T(s) Q_i x(s)
\]
When \( d_m = 1 \),
\[
\sum_{s=t+1-d(t)}^{t-1} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
\leq \sum_{s=t+1-d(t)}^{t-1} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
+ \sum_{s=t+1-d_M}^{t-d_m} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
\]
is satisfied as well.

So from (16) and (17) we can obtain
\[
\Delta V_i(t) \leq \lambda_i^{-2} x^T(t) Q_i x(t)
+ \sum_{s=t+1-d(t)}^{t-1} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
+ \sum_{s=t+1-d_M}^{t-d_m} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
\]
\[
\leq \lambda_i^{-2} x^T(t) Q_i x(t)
- \lambda_i^{2(1-d(t))} x^T(t-d(t)) Q_i x(t-d(t))
+ \sum_{s=t+1-d_M}^{t-d_m} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
\]
\[
(18)
\]

Since \(-\lambda_i^{2(1-d(t))} \leq -\lambda_i^{2(1-d_M)}\), we get
\[
\Delta V_i(t) \leq \left[ x(t) \right]^T \left[ \begin{array}{cc}
\lambda_i^{-2} Q_i & 0 \\
0 & -\lambda_i^{2(1-d_M)} Q_i
\end{array} \right] \left[ x(t) \right]
+ \sum_{s=t+1-d_M}^{t-d_m} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
\]
\[
(19)
\]

The derivation process of \( \Delta V_i(t) \) is similar to \( \Delta V'_i(t) \), and then we have
\[
\Delta V_i(t) = V_i(t + 1) - V_i(t) \leq V_i(t + 1) - \lambda_i^{-2} V_i(t)
\]
\[
= - \sum_{\theta=-d_M+2}^{-d_M+1} \sum_{s=t+1+\theta}^{t-1} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
+ \sum_{\theta=-d_M+2}^{-d_M+1} \sum_{s=t+1+\theta}^{t-1} \lambda_i^{2(s-t-1)} x^T(s) Q_i x(s)
\]
\[
(21)
\]

which is equal to
\[
\Delta V_i \leq \left[ \begin{array}{c}
x(t) \\
x(t-d(t))
\end{array} \right]^T \left[ \begin{array}{cc}
(d_M-d_m) \lambda_i^{-2} Q_i & 0 \\
0 & 0
\end{array} \right] \left[ \begin{array}{c}
x(t) \\
x(t-d(t))
\end{array} \right]
\]
\[
(22)
\]

\[
\Delta V_i(t) = V_i(t + 1) - V_i(t) \leq V_i(t + 1) - \lambda_i^{-2} V_i(t)
\]
\[
\leq \sum_{\theta=-d_M+1}^{0} \sum_{s=t+1+\theta}^{t-1} \lambda_i^{2(s-t-1)} y^T(s) R_i y(s) + d_M
\]
\[
\cdot \lambda_i^{-2} y^T(t) R_i y(t)
\]
\[
(23)
\]

\[
\Delta V_i(t) = V_i(t + 1) - V_i(t) \leq V_i(t + 1) - \lambda_i^{-2} V_i(t)
\]
\[
\leq \sum_{\theta=-d_M+1}^{0} \sum_{s=t+1+\theta}^{t-1} \lambda_i^{2(s-t-1)} y^T(s) R_i y(s) \leq d_M
\]
\[
\cdot \lambda_i^{-2} y^T(t) R_i y(t) - \sum_{s=t+1-d_M}^{t-d_m} \lambda_i^{2(s-t-1)} y^T(s) R_i y(s)
\]
Since \( y(s) = \lambda_i x(s+1) - x(s) \), we substitute it into (23) and obtain
\[
\Delta V_i(t) \leq \lambda_i^{-2} d_M \left[ \begin{array}{c}
x(t) \\
x(t-d(t))
\end{array} \right]^T \left[ \begin{array}{cc}
(\lambda_i A_i^T - I) R_i (\lambda_i A_i - I) & \lambda_i (\lambda_i A_i^T - I) R_i B_i \\
B_i^T R_i (\lambda_i A_i - I) & \lambda_i^2 B_i^T R_i B_i
\end{array} \right] \left[ \begin{array}{c}
x(t) \\
x(t-d(t))
\end{array} \right]
\]
\[
(24)
\]
where we apply the transformation $\phi(s) = \lambda_{i}^{t} x(s)$. Then we have $\lambda_{i}^{t-1} y(s) = \phi(s+1) - \phi(s)$; by Lemma 7 we continue to have

$$
- \sum_{s=t-d}^{t-1} \lambda_{i}^{s+1} y(s) T R_i y(s) \leq \left[ \phi(t) \left[ \phi(t-d(t)) \right]^T \right] T + \left[ \phi(t) \left[ \phi(t-d(t)) \right]^T \right] \cdot \left[ \begin{array}{c}
G_1 \\
G_2 \end{array} \right] d_M R_i^{-1} [G_1 \ G_2] \left[ \begin{array}{c}
\phi(t) \\
\phi(t-d(t)) \end{array} \right].
$$

Due to the fact that $\phi(t) = \lambda_{i}^{-1} x(t), \phi(t-d(t)) = \lambda_{i}^{-1} x(t-d(t))$, it holds that

$$
\Delta V_i(t) \leq \lambda_{i}^{-2} d_M \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right]^T \left[ \begin{array}{c}
\lambda_i A_i^T I R_i (\lambda_i A_i - I) \lambda_i (\lambda_i A_i^T I R_i) \lambda_i^2 B_i^T R_i B_i \\
B_i^T R_i (\lambda_i A_i - I) \lambda_i (\lambda_i A_i^T I R_i) \lambda_i^2 B_i^T R_i B_i \\
\end{array} \right] \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right] + \lambda_{i}^{-2} \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right]^T \left[ \begin{array}{c}
G_1 + G_1^T & -G_1^T + G_2 \\
* & -G_2 - G_2^T \\
\end{array} \right] \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right] + \lambda_{i}^{-2} \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right]^T \left[ \begin{array}{c}
G_2^T \\
G_2^T \end{array} \right] d_M R_i^{-1} [G_1 \ G_2] \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right].
$$

Let $\xi(t) = \left[ \begin{array}{c}
x(t) \\
x(t-d(t)) \end{array} \right]$; then we add (15), (20), (22), and (26) together to yield

$$
\Delta V_i \leq \xi^T(t) \Xi_i \xi(t),
$$

where $\Xi_i = \left[ \begin{array}{cc}
\Omega_1 & \Omega_2 \\
\Omega_1 & \Omega_3 \end{array} \right].$

$$
\Omega_1 = A_i^T P_i A_i - P_i + \lambda_{i}^{-2} Q_i + (d_M - d_m) \lambda_{i}^{-2} Q_i \\
+ \lambda_{i}^{-2} d_M \left[ \begin{array}{c}
\lambda_i A_i^T I R_i (\lambda_i A_i - I) \lambda_i (\lambda_i A_i^T I R_i) R_i \\
\end{array} \right] + \lambda_{i}^{-2} \left( G_1 + G_1^T + d_M G_1^T R_i^{-1} G_1 \right),
$$

$$
\Omega_2 = A_i^T P_i B_i + d_M \left( A_i^T R_i B_i - \lambda_{i}^{-1} R_i B_i \right) + \lambda_{i}^{-2} \left( -G_1^T + G_2 + d_M G_1^T R_i^{-1} G_2 \right),
$$

$$
\Omega_3 = B_i^T P_i B_i - \lambda_{i}^{-1} (d_M + d_M) Q_i + d_M B_i^T R_i B_i + \lambda_{i}^{-2} \left( -G_2 - G_2^T + d_M G_2^T R_i^{-1} G_2 \right).\tag{28}
$$

By (8) and (9) and Lemma 6, we can obtain

$$
\Omega = \left[ \begin{array}{cc}
\Omega_1 & \Omega_2 \\
\Omega_1 & \Omega_3 \end{array} \right] \leq 0.
$$

It follows from (27) and (29) that

$$
V_i(t+1) \leq \lambda_{i}^{-2} V_i(t).
$$

Therefore,

$$
V_i(t) \leq \lambda_{i}^{-2} V_i(t-1) \leq \cdots \leq \lambda_{i}^{-2(t-1)} V_i(t_0).
$$

This completes the proof. \qed

**Lemma 9.** For given constants $\lambda_i$ and $\gamma_0$, suppose that there exist matrices $\Xi_1, \Xi_2,$ and $\Xi_3$ such that

(i) $\Xi_3 \leq 0$

(ii) $\Xi_1 - \Xi_2 \Xi_3^{-1} \Xi_1^T \leq 0$

and $\gamma_0 > 0, \epsilon_1 > 0$, and $\epsilon_2 > 0$ satisfying

$$
\gamma_0^2 I \geq \epsilon_1^{-1} I + \epsilon_2^{-1} I + C_i^T P_i C_i + d_M C_i^T R_i C_i + E_i^T E_i,
$$

then along the trajectory of system (1), one has

$$
V_i(t+1) \leq \lambda_{i}^{-2} V_i(t) + \gamma_0^2 \omega^T(t) \omega(t) - Z^T(t) Z(t),
$$

where

$$
\Xi_1 = \Xi_1 + \epsilon_1 \psi_1 \psi_1^T D_i, \\
\Xi_2 = \Xi_2, \\
\Xi_3 = \Xi_3 + \epsilon_2 \psi_2 \psi_2^T, \\
\phi_1 = C_i^T P_i A_i + d_M (C_i^T R_i A_i - \lambda_{i}^{-1} C_i^T R_i) + E_i^T D_i \\
\phi_2 = C_i^T P_i B_i + d_M C_i^T R_i B_i.$$


Proof. Using Lemma 8 and (1), we have

\[
V_i(t+1) - \lambda_i^{-2}V_i(t) + Z^T(t)Z(t) - \gamma_0^2 w^T(t)w(t) \\
\leq \xi^T(t)\Omega_k(t) + x^T(t) \\
\cdot \left[ A_i^T P_i C_i + d_M \left( A_i^T R_i C_i - \lambda_i^{-1} R_i C_i \right) + D_i^T E_i \right] \\
\cdot w(t) + w^T(t) \\
\cdot \left[ C_i^T P_i A_i + d_M \left( C_i^T R_i A_i - \lambda_i^{-1} C_i^T R_i \right) + E_i^T D_i \right] \\
\cdot x(t) + x^T(t) (t-d(t)) \left[ B_i^T P_i C_i + d_M B_i^T R_i C_i \right] w(t) \\
+ w^T(t) \left[ C_i^T P_i B_i + d_M C_i^T R_i B_i \right] x(t-d(t)) \\
+ x^T(t) D_i^T D_i x(t) + w^T(t) \\
\cdot \left( C_i^T P_i C_i + d_M C_i^T R_i C_i + E_i^T E_i - \gamma_0^2 I \right) w(t).
\]

(37)

Based on Lemmas 4 and 5, it holds that

\[
x^T(t) \left[ A_i^T P_i C_i + d_M \left( A_i^T R_i C_i - \lambda_i^{-1} R_i C_i \right) + D_i^T E_i \right] \\
\cdot w(t) + w^T(t) \\
\cdot \left[ C_i^T P_i A_i + d_M \left( C_i^T R_i A_i - \lambda_i^{-1} C_i^T R_i \right) + E_i^T D_i \right] \\
\cdot x(t) \leq \epsilon_1 x^T(t) \left( \varphi(t) \right) \varphi(t) + \epsilon_1^2 w^T(t) w(t). \\
x^T(t-d(t)) \left[ B_i^T P_i C_i + d_M B_i^T R_i C_i \right] w(t) + w^T(t) \\
\cdot \left[ C_i^T P_i B_i + d_M C_i^T R_i B_i \right] x(t-d(t)) \\
\leq \epsilon_2 x^T(t-d(t)) \left( \varphi(t) \varphi(t) \right) x(t-d(t)) + \epsilon_2^2 w^T(t) w(t). \\
\cdot w(t).
\]

(38)

Then, it follows from (35) and (38) that

\[
V_i(t+1) - \lambda_i^{-2}V_i(t) + Z^T(t)Z(t) - \gamma_0^2 w^T(t)w(t) \\
\leq \xi^T(t) \\
\cdot \left[ \Omega_1 + \epsilon_1 \varphi(t) \varphi(t) + D_i^T D_i \Omega_2 \right] \xi(t) \\
+ w^T(t) \left[ \epsilon_1^2 I + \epsilon_2^2 I + C_i^T P_i C_i + d_M C_i^T R_i C_i \right] \\
+ E_i^T E_i - \gamma_0^2 I \right] w(t).
\]

(39)

Combining (32), (33) with (34) will lead to

\[
V_i(t+1) \leq \lambda_i^{-2}V_i(t) + \gamma_0^2 w^T(t)w(t) - Z^T(t)Z(t).
\]

(40)

This completes the proof. □

Now, our \(L_2\)-gain analysis results can be presented as follows.

**Theorem 10.** For given constants \(\lambda_i\) and \(\gamma_0\), suppose that there exist matrices \(\Xi_1, \Xi_2, \Xi_3\) such that

(i) \[\Xi_3 \leq 0\] (41)

(ii) \[\Xi_1 - \Xi_2 \Xi_3^2 \Xi_1^T \leq 0\] (42)

and \(\gamma_0 > 0, \epsilon_1 > 0, \) and \(\epsilon_2 > 0\) satisfying

\[
\gamma_0^2 I \geq \epsilon_1^2 I + \epsilon_2^2 I + C_i^T P_i C_i + d_M C_i^T R_i C_i + E_i^T E_i.
\]

(43)

Then the switched delay system (1) has a \(L_2\)-gain with MDADT \(\tau_{ap} > \tau_{ap}'\) if \(\varphi(t), \Xi_1, \Xi_2, \Xi_3\) are defined in Lemma 9.

Proof. Choose the Lyapunov functional candidate (12). From (41) and (42) and Lemma 9, we have

\[
V_i(t+1) \leq \lambda_i^{-2}V_i(t) + \gamma_0^2 w^T(t)w(t) - Z^T(t)Z(t).
\]

(44)

Let \(\Gamma(t) = \gamma_0^2 w(t)w(t) - Z^T(t)Z(t)\). From (35), since \(t_{j-1} = t_j - 1\), we have

\[
\Gamma_{\sigma(t)}(t) \leq \lambda_{\sigma(t)}^{-2(t-1)}V_{\sigma(t)}(t_j) + \sum_{j=1}^{t_j-1} \lambda_{\sigma(t)}^{-2(t-j-1)} \Gamma(j) \\
\leq \mu_{\sigma(t)} \lambda_{\sigma(t)}^{-2(t-j)}V_{\sigma(t)}(t_j) + \sum_{j=1}^{t_j-1} \lambda_{\sigma(t)}^{-2(t-j-1)} \Gamma(j) \\
\leq \mu_{\sigma(t)} \lambda_{\sigma(t)}^{-2(t-j)} \left( \lambda_{\sigma(t)}^{-2(t-j-1)}V_{\sigma(t)}(t_j) \right) + \sum_{j=1}^{t_j-1} \lambda_{\sigma(t)}^{-2(t-j-1)} \Gamma(j) \leq \cdots \\
\leq \left( \sum_{j=1}^{t_j} \mu_{\sigma(t)} \right) \cdot \left( 2 \sum_{j=1}^{t_j} \ln \lambda_{\sigma(t)}(t_j - t_{j-1}) \right).
\]

(45)
which combined with Definition 2 and the MDADT scheme \( N_{\sigma}(T, t) \leq T_p(T, t)/\tau_{ap} \) yields

\[
V_{\sigma(t)}(t) \leq \exp \left[ \sum_{p=1}^{m} \left( \ln \mu_p^{N_{\sigma}(t, t_0)} - 2 \ln \lambda_p T_p(t, t_0) \right) \right] \\
\cdot V_{\sigma(t_0)}(t_0) \\
+ \sum_{k=1}^{i} \left\{ \exp \left[ \sum_{p=1}^{m} \left( \ln \mu_p^{N_{\sigma}(t, t_{k-1})} - 2 \ln \lambda_p T_p(t, t_{k-1}) \right) \right] \right\} \\
\cdot \Gamma(t_{k-1}) \leq e^{-\beta T_{r}(t,t_{k})} V_{\sigma(t_0)}(t_0) + \sum_{k=1}^{i} e^{-\beta T_{r}(t,t_{k})} \Gamma(t_{k-1}),
\]

where \( \beta = \sum_{p=1}^{m} \left( 2 \ln \lambda_p - \ln \mu_p / \tau_{ap} \right) > 0. \)

Under zero initial condition, from (46), one obtains

\[
0 \leq V_{\sigma(t)}(t) \leq \sum_{k=1}^{i} e^{-\beta T_{r}(t,t_{k-1})} \Gamma(t_{k-1}) \tag{47}
\]

which implies that

\[
\sum_{k=1}^{i} e^{-\beta T_{r}(t,t_{k-1})} Z^T(t_{k-1}) Z(t_{k-1}) \\
\leq \sum_{k=1}^{i} e^{-\beta T_{r}(t,t_{k-1})} \gamma_0^2 w^T(t_{k-1}) w(t_{k-1}).
\]

Then, we multiply both sides by \( e^{-\beta T_{r}(t,t_{k-1})} \) to get

\[
\sum_{k=1}^{i} e^{-\beta T_{r}(t,t_{k-1})} Z^T(t_{k-1}) Z(t_{k-1}) \\
\leq \sum_{k=1}^{i} e^{-\beta T_{r}(t,t_{k-1})} \gamma_0^2 w^T(t_{k-1}) w(t_{k-1}).
\]

Thus,

\[
\sum_{k=0}^{i} Z^T(t_k) Z(t_k) \leq \sum_{k=0}^{i} \gamma_0^2 w^T(t_k) w(t_k). \tag{50}
\]

This completes the proof.

### 4. A Numerical Example

Consider the switched delay system (1) with the following specifications:

\[
\begin{align*}
A_1 &= \begin{bmatrix} -0.2 & 0.3 \\ 0.1 & -0.5 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0.4 & 0 \\ 0.1 & -0.5 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} -0.1 & 1 \\ 0 & -0.6 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} -0.7 & 0.1 \\ 1 & 0.2 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
D_1 &= [1, 1], \\
D_2 &= [0, 1], \\
E_1 &= E_2 = [0.2, 0.8], \\
d(t) &= \sin(t \pi/2) + 1, \text{ so that } d_M = 2, d_m = 0.
\end{align*}
\]

The disturbance input is defined as

\[
w(t) = \begin{cases} 
1, & 0 < t \leq 20, \\
0, & t > 20.
\end{cases}
\]

Let \( \mu_1 = \mu_2 = 12; \) by the LMI Control Toolbox and Theorem 10, we obtain

\[
\begin{align*}
P_1 &= \begin{bmatrix} 12.5739 & 5.0613 \\ 5.0613 & 4.8703 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} 2.6676 & 1.2445 \\ 1.2445 & 0.9452 \end{bmatrix}, \\
R_1 &= \begin{bmatrix} 4.4703 & -0.6996 \\ -0.6996 & 9.3862 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 12.3445 & 18.3565 \\ 18.3565 & 30.2222 \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} 2.3941 & 3.7907 \\ 3.7907 & 6.1097 \end{bmatrix}, \\
R_2 &= \begin{bmatrix} 16.1813 & 18.8134 \\ 18.8134 & 22.4268 \end{bmatrix},
\end{align*}
\]

and \( \lambda_1 = 27.2485, \lambda_2 = 38.7807, \) where \( r_{a1}^* = \ln \mu_1 / 2 \ln \lambda_1 = 0.3759 \) and \( r_{a2}^* = \ln \mu_2 / 2 \ln \lambda_2 = 0.3397. \) Now, we choose
the switching periods $\tau_{s1} = 2$, $\tau_{s2} = 1$ and take the initial state condition $\mathbf{y}(l) = [1;2]$ for all $l = -2,-1,0$. Then the numerical simulations can be shown in Figure 1.

It can be seen from Figure 1 that under the designed MDADT switching signals the switched delay system can achieve better dynamics performance and disturbance tolerance capability, which shows the potentiality of our results in practice.

5. Conclusions

In this paper, the problem of $L_2$-gain analysis for discrete-time switched systems with MDADT switching has been investigated. By combining with the multiple Lyapunov function method, sufficient conditions are established to ensure $L_2$-gain performance for discrete-time switched delay system, and the admissible MDADT switching signals are also designed accordingly. Finally, a numerical example is given to demonstrate the usefulness of the obtained results.

Competing Interests

The authors declare that they have no competing interests.

References
