Research Article

A Two-Level Additive Schwarz Preconditioning Algorithm for the Weak Galerkin Method for the Second-Order Elliptic Equation

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This paper proposes a two-level additive Schwarz preconditioning algorithm for the weak Galerkin approximation of the second-order elliptic equation. In the algorithm, a P_1 conforming finite element space is defined on the coarse mesh, and a stable intergrid transfer operator is proposed to exchange the information between the spaces on the coarse mesh and the fine mesh. With the framework of the Schwarz method, it is proved that the condition number of the preconditioned system only depends on the rate of the coarse mesh size and the overlapping size. Some numerical experiments are carried out to verify the theoretical results.

1. Introduction

The weak Galerkin (WG) finite element method (FEM) is a new efficient numerical method for solving partial differential equations. It was first introduced for the second-order elliptic problem in [1]. The central idea of the WG method is using weak derivatives in place of strong derivatives that define the weak formulation for the underlying partial differential equations. Due to the weak derivative, the degrees of freedom can be discontinuous from the element interior to the element boundary, which makes the weak Galerkin method have more choices of the finite elements than the standard FEMs. The WG FEM also has some other properties, such as high-order accuracy and multiphysics capability. The idea of the weak derivative has been generalized to some other problems, such as the biharmonic equation [2] and the two-phase subsurface flow problems [3]. Chen et al. introduced a posteriori error estimates for the WG method for the second-order elliptic problems in [4].

Although many works focus on developing WG methods for different problems, there are few works on efficiently solving the discretized systems, of which the condition number is $O(h^{-2})$, where h denotes the mesh size. Recently,

Li and Xie [5] proposed a multigrid algorithm for the WG method using P_1 conforming element on the coarser meshes. In [6], Chen et al. also constructed a fast auxiliary space multigrid preconditioner for the WG method, as well as a corresponding reduced system involving only the degrees of freedom on edges/faces. It is proved that the condition number of the preconditioned system is independent of the mesh size.

Due to its high parallelizability and scalability, the domain decomposition method is one of the most efficient algorithms for the numerical solutions of partial differential equations (see, e.g., [7]). In this paper, we study an overlapping domain decomposition algorithm and propose a two-level additive Schwarz preconditioner for the WG method for the second-order elliptic equation. The preconditioner is consisted of overlapping local subproblems on a fine mesh and a global subproblem on a coarse mesh. The coarse subproblem is defined by the P_1 conforming element method, and an intergrid transfer operator, which has stability and approximability, is introduced to transmit information between the coarse mesh and the fine mesh. Under the Schwarz framework, we prove that the condition number of the preconditioned system is independent of the fine mesh size h and is of $O((1 +$

$H/\delta)^2$), where H denotes the coarse mesh size and δ stands for the overlapping size.

The rest of this paper is organized as follows. In Section 2, we recall the WG method and give some notations. In Section 3, we propose an intergrid transfer operator and a two-level additive Schwarz preconditioner. Moreover, we prove the stability and approximation of the intergrid transfer operator. Then we estimate the upper bound of the maximum eigenvalue and the lower bound of minimum eigenvalue, respectively. Section 4 is devoted to numerical experiments, which are carried out to confirm our theoretical results.

2. The Weak Galerkin Method for the Second-Order Elliptic Equation

For the sake of simplicity, we consider the model problem as follows:

$$\begin{aligned} -\nabla \cdot (\mathbb{A} \nabla u) &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where $\Omega \subset \mathbb{R}^2$ is bounded polygonal domain and $f \in L^2(\Omega)$. Assume that the matrix $\mathbb{A}$ is symmetric positive definite (SPD); namely, there exist positive constants α, β such that

$$\alpha \xi^T \xi \leq \xi^T \mathbb{A}(\xi) \leq \beta \xi^T \xi, \quad \forall \xi \in \mathbb{R}^2, \quad x \in \Omega. \quad (2)$$

The variational formulations of (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (3)$$

where $a(u, v) = (\mathbb{A} \nabla u, \nabla v)$. It follows from Lax-Milgram's theorem that problem (3) has a unique solution.

Let $\mathcal{T}_h$ be a quasiuniform triangulation of the domain Ω with the mesh size h . On each $T \in \mathcal{T}_h$, the space $W(T)$ denotes the collections of weak functions, each of which is consisted of an interior part and a boundary part; that is,

$$W(T) := \{v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in H^{1/2}(\partial T)\}. \quad (4)$$

For any weak function $v \in W(T)$, the weak gradient of v , denoted by $\nabla_w v \in H(\text{div}, T)$, satisfies

$$\begin{aligned} \int_T \nabla_w v \cdot \mathbf{q} \, dx &= - \int_T v_0 \nabla \cdot \mathbf{q} \, dx + \int_{\partial T} v_b \mathbf{q} \cdot \mathbf{n} \, ds \\ \forall \mathbf{q} &\in H(\text{div}, T), \end{aligned} \quad (5)$$

where $\mathbf{n}$ is the unit outer normal of ∂T . It is easy to see that ∇_w is the classical gradient if it actions on a function $u \in H^1(T)$.

For each element $T \in \mathcal{T}_h$, we denote by T_0 and ∂T the interior and the boundary of T . The notation $\mathcal{F}_T$ stands for the set of edges of T . We also use $\mathcal{F}_h$ and $\mathcal{F}_h^\partial$ to denote the edges of $\mathcal{T}_h$ in Ω and on $\partial\Omega$. Then the discrete WG space is defined as

$$\begin{aligned} W_h &:= \{v : v|_{T_0} \in P_j(T_0) \quad \forall T \in \mathcal{T}_h, v|_F \in P_l(F) \quad \forall F \\ &\in \mathcal{F}_h, v|_F = 0 \quad \forall F \in \mathcal{F}_h^\partial\}, \end{aligned} \quad (6)$$

where $l = j$ or $l = j + 1$ and $P_j(T_0)$ denotes the set of polynomials on T_0 with degree no more than $j \geq 0$. To define a discrete weak gradient, we use $G_j(T)$ to denote the RT element space $[P_j(T)]^2 + \widehat{P}_j(T)x$ if $l = j$ and the BDM element space $[P_{j+1}(T)]^2$ if $l = j + 1$, where $\widehat{P}_j(T)$ is the homogeneous polynomial with the degree j .

For each $v \in W_h$, the discrete weak gradient $\nabla_w v \in G_j(T)$ is defined as

$$\begin{aligned} \int_T \nabla_w v \cdot \mathbf{q} \, dx &= - \int_T v_0 (\nabla \cdot \mathbf{q}) \, dx + \int_{\partial T} v_b (\mathbf{q} \cdot \mathbf{n}) \, ds \\ \forall \mathbf{q} &\in G_j(T), \end{aligned} \quad (7)$$

where v_0 and v_b denote the values of v in the interior and on the boundary of T . Then the discrete problem for (3) is to seek $u_h = \{u_0, u_b\} \in W_h$ such that

$$a_h(u_h, v_h) = (f, v_0), \quad \forall v_h = \{v_0, v_b\} \in W_h, \quad (8)$$

where $a_h(u_h, v_h) := (\mathbb{A} \nabla_w u_h, \nabla_w v_h)$.

For any $v_h = \{v_0, v_b\}$ and $u_h = \{u_0, u_b\}$ in W_h , we use $((u_h, v_h))$ to indicate a special inner product defined as

$$((v_h, u_h)) := \sum_{T \in \mathcal{T}_h} [(v_0, u_0)_T + h(v_0 - v_b, u_0 - u_b)_{\partial T}], \quad (9)$$

where $(\cdot, \cdot)_T$ and $(\cdot, \cdot)_{\partial T}$ denote the L^2 inner product on T and ∂T . Accordingly, a norm and seminorm are introduced for any $v_h = \{v_0, v_b\} \in W_h$ by

$$\begin{aligned} \|v_h\|_{0,h} &:= \left(\sum_{T \in \mathcal{T}_h} \|v_h\|_{0,h,T}^2 \right)^{1/2}, \\ |v_h|_{1,h} &:= \left(\sum_{T \in \mathcal{T}_h} |v_h|_{1,h,T}^2 \right)^{1/2}, \end{aligned} \quad (10)$$

where $\|v_h\|_{0,h,T} := (\|v_0\|_{0,T}^2 + h\|v_0 - v_b\|_{\partial T}^2)^{1/2}$ and $|v_h|_{1,h,T} := (|v_0|_{1,T}^2 + h^{-1}\|v_0 - v_b\|_{\partial T}^2)^{1/2}$.

Let $A_h : W_h \rightarrow W_h$ be an operator

$$((A_h u_h, v_h)) = (\mathbb{A} \nabla_w u_h, \nabla_w v_h), \quad \forall u_h, v_h \in W_h. \quad (11)$$

It can be proved that A_h is symmetric and positive definite and the condition number is of $O(h^{-2})$ (see, e.g., [8]), which brings difficulty to solve the discrete problem when the mesh size is small. In the next section, we will present a preconditioner to overcome this difficulty.

3. The Overlapping Domain Decomposition Method

3.1. A Two-Level Additive Schwarz Preconditioner. To introduce our preconditioner, we first divide the domain Ω by J overlapping subdomains $\Omega_1, \Omega_2, \dots, \Omega_J$ such that each point in Ω belongs to no more than N_C subdomains. We assume that the boundary of each subdomain does not cut through any elements in the triangulation $\mathcal{T}_h$, and there

exist nonnegative C^∞ functions $\theta_1, \theta_2, \dots, \theta_J$ satisfying the following properties: (1) $\theta_j = 0$ in $\Omega \setminus \Omega_j$; (2) $\sum_{j=1}^J \theta_j = 1$; (3) there exists a positive constant δ , such that $|\nabla \theta_j| \leq C/\delta$, where C is a constant independent of δ, h , and J .

On each subdomain Ω_j , the notation $\mathcal{T}_{h,j}$ stands for the triangulation inherited from $\mathcal{T}_h$. The corresponding weak Galerkin subspace on $\mathcal{T}_{h,j}$ is defined as

$$V_j = \{v \in W_h : v|_{\Omega \setminus \Omega_j} = 0\}. \quad (12)$$

We introduce an operator $A_j : V_j \rightarrow V_j$ by

$$((A_j u_j, v_j)) = (\mathbb{A} \nabla_w u_j, \nabla_w v_j), \quad \forall u_j, v_j \in V_j. \quad (13)$$

It is easy to see that A_j is symmetric and positive definite. Since V_j is a subset of W_h , we use I_j to denote a natural injection from V_j to W_h .

To define a coarse subproblem, we define a coarse triangulation $\mathcal{T}_H$ with mesh size H such that each element in $\mathcal{T}_H$ is a subdivision of the one in $\mathcal{T}_h$. The notation $V_H \subset H_0^1(\Omega)$ stands for P_1 conforming finite element space associated with $\mathcal{T}_H$. We also introduce an operator $A_H : V_H \rightarrow V_H$ satisfying

$$(A_H u_H, v_H) = (\mathbb{A} \nabla u_H, \nabla v_H), \quad \forall u_H, v_H \in V_H. \quad (14)$$

Note that the coarse space V_H is a subspace of W_h if the degree of the piecewise polynomials in W_h is greater than 0, and we can choose a natural injection as the intergrid transfer operator I_H from V_H to W_h . For the piecewise constant space case, the intergrid transfer operator $I_H : V_H \rightarrow W_h$ is defined as

$$I_H v_H = \{I_H^0 v_H, I_H^b v_H\}, \quad \forall v_H \in V_H, \quad (15)$$

where I_H^0 and I_H^b satisfy

$$\begin{aligned} (I_H^0 v_H)|_T &= \frac{1}{|T|} \int_T v_H dx, \\ (I_H^b v_H)|_F &= \frac{1}{|F|} \int_F v_H ds, \end{aligned} \quad (16)$$

for any $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$. We also need the transpose of the intergrid transfer operators I_j^t from W_h to the subspace V_j for $j = H, 1, 2, \dots, J$, defined as

$$((I_j^t u_h, v_j)) = ((u_h, I_j v_j)), \quad \forall u_h \in W_h, v_j \in V_j. \quad (17)$$

Then our two-level additive Schwarz preconditioner B_h is stated as

$$B_h = I_H A_H^{-1} I_H^t + \sum_{j=1}^J I_j A_j^{-1} I_j^t. \quad (18)$$

3.2. Analysis. Our analysis is based on the standard Schwarz framework (see, e.g., [7, 9]).

Lemma 1 ([9, Theorem 7.1.20]). *The eigenvalues of $B_h A_h$ are positive, and one has the following characterizations of the maximum and minimum eigenvalues:*

$$\begin{aligned} \lambda_{\max}(B_h A_h) &= \max_{v \in W_h, v \neq 0} \frac{((A_h v, v))}{\min_{v = \sum_{j \in V_j} I_j v_j} \sum_j ((A_j v_j, v_j))}, \\ \lambda_{\min}(B_h A_h) &= \min_{v \in W_h, v \neq 0} \frac{((A_h v, v))}{\min_{v = \sum_{j \in V_j} I_j v_j} \sum_j ((A_j v_j, v_j))}, \end{aligned} \quad (19)$$

where the sum is taken over $j = H, 1, \dots, J$.

Lemma 2. *For the intergrid transfer operator I_H , it holds for any $v_H \in V_H$ that*

$$\|v_H - I_H v_H\|_{0,h}^2 + h^2 |I_H v_H|_{1,h}^2 \lesssim h^2 |v_H|_1^2. \quad (20)$$

Proof. We only need consider the case that the functions in W_h are the constants in the interior and on the boundary of each element.

It follows from the triangle inequality, the trace inequality, and the Poincaré-Friedrichs inequality that

$$\begin{aligned} |I_H v_H|_{1,h}^2 &= \sum_{T \in \mathcal{T}_h} h^{-1} \|I_H^0 v_H - I_H^b v_H\|_{0,\partial T}^2 \\ &\leq \sum_{T \in \mathcal{T}_h} h^{-1} (\|I_H^0 v_H - v_H\|_{0,\partial T}^2 + \|I_H^b v_H - v_H\|_{0,\partial T}^2) \\ &= |v_H|_1^2. \end{aligned} \quad (21)$$

Similarly, for the lower-order term, we have

$$\begin{aligned} \|v_H - I_H v_H\|_{0,h}^2 &= \sum_{T \in \mathcal{T}_h} (\|v_H - I_H^0 v_H\|_{0,T}^2 + h \|I_H^0 v_H - I_H^b v_H\|_{0,\partial T}^2) \\ &\leq h^2 |v_H|_1^2, \end{aligned} \quad (22)$$

which completes the proof. $\square$

Lemma 3. *There exists an operator Q^H from W_h to V_H such that for any v_h in W_h it holds that*

$$\|v_h - Q^H v_h\|_{0,h}^2 + H^2 |Q^H v_h|_1^2 \lesssim H^2 |v_h|_{1,h}^2. \quad (23)$$

Proof. Let V_h be the P_1 conforming finite element space defined on $\mathcal{T}_h$. Then, for any v_h in W_h , we can construct a $\tilde{v}_h \in V_h$ satisfying ([6, Lemma 3.5])

$$\|v_h - \tilde{v}_h\|_{0,h} + h |\tilde{v}_h|_{1,h} \lesssim h |v_h|_{1,h}. \quad (24)$$

Define $Q^H v_h = Q_H \tilde{v}_h$, where Q_H is the L^2 projection operator from V_h to V_H . Using the standard properties of Q_H (see, e.g., [7]) and the inequality (24), we have

$$\begin{aligned} |Q^H v_h|_1^2 &= |Q_H \tilde{v}_h|_1^2 \lesssim |\tilde{v}_h|_1^2 \lesssim |v_h|_{1,h}^2, \\ \|v_h - Q^H v_h\|_{0,h}^2 &= \|v_h - Q_H \tilde{v}_h\|_{0,h}^2 \\ &\lesssim \|v_h - \tilde{v}_h\|_{0,h}^2 + \|Q_H \tilde{v}_h - \tilde{v}_h\|_{0,h}^2 \\ &\lesssim h^2 |v_h|_{1,h}^2 + H^2 |\tilde{v}_h|_1^2 \lesssim H^2 |v_h|_{1,h}^2. \end{aligned} \quad (25)$$

The proof is completed. $\square$

Lemma 4. Given any $v_h \in W_h$, there exists a decomposition

$$v_h = I_H v_H + \sum_{j=1}^J I_j v_j, \quad (26)$$

where $v_H \in V_H$ and $v_j \in V_j$, such that

$$\begin{aligned} (A_H v_H, v_H) + \sum_{j=1}^J ((A_j v_j, v_j)) \\ \lesssim \left(1 + \frac{H}{\delta}\right)^2 ((A_h v_h, v_h)). \end{aligned} \quad (27)$$

Proof. Let $v_H = Q^H v$, $w = v_h - I_H v_H$, and $v_j = Q_h(\theta_j w) = \{Q_h^0(\theta_j w_0), Q_h^b(\theta_j w_b)\}$, where Q_h^0 and Q_h^b are the piecewise L^2 projection to the interior polynomial space $P_j(T_0)$ and the edge polynomial space $P_l(F)$, respectively. It is easy to check that

$$\begin{aligned} I_H v_H + \sum_{j=1}^J v_j &= I_H v_H + \sum_{j=1}^J Q_h(\theta_j w) \\ &= I_H v_H + Q_h \left(\sum_{j=1}^J (\theta_j w) \right) \\ &= I_H v_H + Q_h w = v_h. \end{aligned} \quad (28)$$

It follows from Lemma 3 that

$$(A_H v_H, v_H) \lesssim |v_H|_1^2 \lesssim |v_h|_{1,h}^2 \lesssim ((A_h v_h, v_h)). \quad (29)$$

On the other hand, we have for v_j that

$$\begin{aligned} ((A_j v_j, v_j)) &\lesssim \sum_{T \in \mathcal{T}_{h,j}} |v_j|_{1,h,T}^2 = \sum_{T \in \mathcal{T}_{h,j}} (|Q_h^0(\theta_j w_0)|_{1,T}^2 \\ &\quad + h^{-1} \|Q_h^0(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2). \end{aligned} \quad (30)$$

By the inverse inequality, the stability of L^2 projection, the Poincaré inequality, and the scaling argument, we deduce that

$$\begin{aligned} |Q_h^0(\theta_j w_0)|_{1,T}^2 &= \inf_{c \in \mathbb{R}} |Q_h^0(\theta_j w_0 - c)|_{1,T}^2 \\ &\lesssim \inf_{c \in \mathbb{R}} h^{-1} \|Q_h^0(\theta_j w_0 - c)\|_{0,T}^2 \\ &\lesssim \inf_{c \in \mathbb{R}} h^{-1} \|\theta_j w_0 - c\|_{0,T}^2 \lesssim |\theta_j w_0|_{1,T}^2, \end{aligned} \quad (31)$$

which, together with the triangle inequality and the assumption of θ_j , yields

$$\begin{aligned} ((A_j v_j, v_j)) &\lesssim \sum_{T \in \mathcal{T}_{h,j}} (|\theta_j w_0|_{1,T}^2 \\ &\quad + h^{-1} \|Q_h^0(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2) \\ &\lesssim \sum_{T \in \mathcal{T}_{h,j}} \left(\int_T |(\nabla \theta_j \cdot w_0 + \nabla w_0 \cdot \theta_j)|^2 dx \right. \\ &\quad \left. + h^{-1} \|Q_h^0(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2 \right) \\ &\lesssim \sum_{T \in \mathcal{T}_{h,j}} \left(2 \|\nabla \theta_j \cdot w_0\|_{0,T}^2 + 2 \|\nabla w_0 \cdot \theta_j\|_{0,T}^2 \right. \\ &\quad \left. + h^{-1} \|Q_h^0(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2 \right) \\ &\lesssim \sum_{T \in \mathcal{T}_{h,j}} \left(\frac{1}{\delta^2} \|w_0\|_{0,T}^2 + |w_0|_{1,T}^2 \right. \\ &\quad \left. + h^{-1} \|Q_h^0(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2 \right). \end{aligned} \quad (32)$$

For the last term, the triangle inequality gives

$$\begin{aligned} \|Q_h^0(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2 \\ \leq \|Q_h^0(\theta_j w_0) - \theta_j w_0\|_{0,\partial T}^2 \\ + \|\theta_j w_0 - Q_h^b(\theta_j w_0)\|_{0,\partial T}^2 \\ + \|Q_h^b(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2 = \text{I} + \text{II} + \text{III}. \end{aligned} \quad (33)$$

We estimate I, II, and III, respectively, as follows.

From the trace theorem, the scaling argument, and the stability and approximation of Q_h^0 , we have

$$\begin{aligned} \text{I} &= \|Q_h^0(\theta_j w_0) - \theta_j w_0\|_{0,\partial T}^2 \\ &\lesssim h^{-1} \|Q_h^0(\theta_j w_0) - \theta_j w_0\|_{0,T}^2 \\ &\quad + h |Q_h^0(\theta_j w_0) - \theta_j w_0|_{1,T}^2 \lesssim h |\theta_j w_0|_{1,T}^2 \\ &\lesssim \frac{h}{\delta^2} \|w_0\|_{0,T}^2 + h |w_0|_{1,T}^2. \end{aligned} \quad (34)$$

Denote $\overline{\theta_j w_0} = (1/|T|) \int_T \theta_j w_0 dx$. The trace theorem, the L^2 stability of Q_h^b , and the Poincaré inequality imply that

$$\begin{aligned} \text{II} &= \|\theta_j w_0 - Q_h^b(\theta_j w_0)\|_{0,\partial T}^2 \\ &\leq \|\theta_j w_0 - \overline{\theta_j w_0}\|_{0,\partial T}^2 + \|Q_h^b(\theta_j w_0 - \overline{\theta_j w_0})\|_{0,\partial T}^2 \\ &\leq \|\theta_j w_0 - \overline{\theta_j w_0}\|_{0,\partial T}^2 \leq h |\theta_j w_0|_{1,T}^2 \\ &\leq \frac{h}{\delta^2} \|w_0\|_{0,T}^2 + h |w_0|_{1,T}^2. \end{aligned} \quad (35)$$

Simiarly, we obtain

$$\begin{aligned} \text{III} &= \|Q_h^b(\theta_j w_0) - Q_h^b(\theta_j w_b)\|_{0,\partial T}^2 \\ &\leq \|\theta_j w_0 - \theta_j w_b\|_{0,\partial T}^2 \leq \|w_0 - w_b\|_{0,\partial T}^2. \end{aligned} \quad (36)$$

Combining the above five inequalities, we find

$$\begin{aligned} &((A_j v_j, v_j)) \\ &\leq \sum_{T \in \mathcal{T}_{h,j}} \left[\frac{1}{\delta^2} \|w_0\|_{0,T}^2 + |w_0|_{1,T}^2 + h^{-1} \|w_0 - w_b\|_{0,\partial T}^2 \right], \end{aligned} \quad (37)$$

which leads to

$$\begin{aligned} \sum_{j=1}^J ((A_j v_j, v_j)) &\leq \sum_{j=1}^J \sum_{T \in \mathcal{T}_{h,j}} \left[\frac{1}{\delta^2} \|w_0\|_{0,T}^2 + |w_0|_{1,T}^2 \right. \\ &\quad \left. + h^{-1} \|w_0 - w_b\|_{0,\partial T}^2 \right] \leq \left(\frac{1}{\delta^2} \|v_h - I_H v_H\|_{0,h}^2 + |v_h - I_H v_H|_{1,h}^2 \right) \\ &\leq \left(\frac{1}{\delta^2} \|v_h - v_H\|_{0,h}^2 + \frac{1}{\delta^2} \|v_H\|_{0,h}^2 \right) \\ &\leq \left(1 + \frac{H^2}{\delta^2} \right) |v_h|_{1,h}^2 \\ &\leq \left(1 + \frac{H}{\delta} \right)^2 ((A_h v_h, v_h)). \end{aligned} \quad (38)$$

Using (27) and (38), we achieve

$$\begin{aligned} &(A_H v_H, v_H) + \sum_{j=1}^J ((A_j v_j, v_j)) \\ &\leq \left(1 + \frac{H}{\delta} \right)^2 ((A_h v_h, v_h)). \end{aligned} \quad (39)$$

This ends the proof. $\square$

Lemma 5. Let $v_H \in V_H$ and $v_j \in V_j$, $1 \leq j \leq J$. For any $v_h = I_H v_H + \sum_{j=1}^J v_j$, it is true that

$$((A_h v_h, v_h)) \leq (A_H v_H, v_H) + \sum_{j=1}^J ((A_j v_j, v_j)). \quad (40)$$

TABLE 1: The iteration numbers of the PCG and CG.

h	1/8	1/16	1/32	1/64	1/128
PCG ($H/\delta = 1$)	10	11	11	11	12
PCG ($H/\delta = 2$)	14	15	15	16	16
CG	60	119	231	436	864

Proof. The inequality can be obtained directly by using the triangle inequality and the assumption on the finite cover of subdomains. $\square$

As an immediate consequence, we have the following theorem.

Theorem 6. There exists a positive constant C , independent of H , h , δ , and J , such that

$$\kappa(B_h A_h) = \frac{\lambda_{\max}(B_h A_h)}{\lambda_{\min}(B_h A_h)} \leq C \left(1 + \frac{H}{\delta} \right)^2. \quad (41)$$

Remark 7. According the theorem, the two-level additive preconditioner is optimal if H/δ is bounded above by a constant. In particular, when $\delta = O(H)$, we have

$$\kappa(B_h A_h) \leq C. \quad (42)$$

4. Numerical Experiments

In this section, we give some numerical results to demonstrate the efficiency of our preconditioner. For convenience, we consider a simple two-dimensional Poisson equation with homogeneous boundary as follows:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (43)$$

We choose f to satisfy the exact solution $u(x_1, x_2) = x_1(1-x_1)x_2(1-x_2)$ on $\Omega = [0, 1] \times [0, 1]$. Let $\mathcal{T}_H$ be a uniform triangulation with the mesh size $H = 1/4$, and let $\mathcal{T}_h$ be the refinement of $\mathcal{T}_H$. All the tests are stopped when the relative error is less than 10^{-6} .

In Table 1, we list the iteration numbers of the preconditioning conjugate gradient (PCG) method and the conjugate gradient (CG) method with different meshes. From the second and third rows of the table, we see that the iteration numbers of the PCG method are almost the same if the overlapping factors H/δ are fixed. If the overlap becomes small, that is, the rate H/δ is increasing, one needs a little more steps to achieve the tolerance. This indicates that the condition number of the preconditioned system is almost a constant which is independent of the mesh size h and only depends on the rate of the coarse mesh size and the overlap.

Finally, we will do some experiments to show the efficiency of our algorithm for problems with checkboard distributed coefficient, which equals 1 and 10^i ($i = 1, 3, 6$) on adjacent subdomains. We assume that the coefficient is constant on each coarser element, and the coarser mesh size and the overlap size are fixed ($H = 1/4$, $\delta = 1/8$). The

TABLE 2: The iteration numbers with different mesh sizes h and coefficient jumps (10^4).

i	h				
	1/8	1/16	1/32	1/64	1/128
1	14	15	15	16	16
3	15	15	15	15	16
6	15	15	15	15	15

iteration numbers, with different mesh sizes h and jumps in the coefficient, are reported in Table 2, from which, we conclude that our preconditioner also works well for problems with discontinuous coefficients at least in two dimensions.

Competing Interests

The authors declare that they have no competing interests.

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