Research Article

On the General Solution of Impulsive Systems with Hadamard Fractional Derivatives

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This paper is concerned with the solution for impulsive differential equations with Hadamard fractional derivatives. The general solution of this impulsive fractional system is found by considering the limit case in which impulses approach zero. Next, an example is provided to expound the theoretical result.

1. Introduction

The theory of fractional calculus has been applied in widespread fields of science and engineering [1–3], and some properties of the solution were researched for fractional differential equations in [4–11]. Moreover, the Hadamard-type fractional calculus which is a branch of fractional calculus was developed in [12–17]. Next, the Caputo-Hadamard fractional derivative which is a Caputo-type modification of the Hadamard fractional derivative is suggested in [18], some fundamental theorems of this fractional derivative were proved in [19], and Cauchy problems of a differential equation with a left Caputo-Hadamard fractional derivative were studied in spaces of continuously differentiable functions in [20].

On the other hand, impulsive differential equations are often used for description some processes or system with impulsive effects, impulsive (partial) differential equations with Caputo fractional derivative were widely studied in [21–32], and the existence of solutions was considered for impulsive differential equations with Hadamard fractional derivative in [33].

Furthermore, the general solution for some impulsive fractional differential equations was found in [34–39]. Therefore, we will seek the general solution for the impulsive systems with Hadamard fractional derivatives in present paper:

$${}_H D_{a^+}^q z(t) = g(t, z(t)), \quad (1a)$$

$$t \in [a, T], \quad t \neq t_i \quad (i = 1, 2, \dots, m),$$

$$\Delta \left({}_H \mathcal{J}_{a^+}^{1-q} z \right) \Big|_{t=t_i} = {}_H \mathcal{J}_{a^+}^{1-q} z(t_i^+) - {}_H \mathcal{J}_{a^+}^{1-q} z(t_i^-) \quad (1b)$$

$$= J_i(z(t_i^-)) \in \mathbb{C}, \quad i = 1, 2, \dots, m,$$

$${}_H \mathcal{J}_{a^+}^{1-q} z(a^+) = z_a, \quad z_a \in \mathbb{C}, \quad (1c)$$

where $q \in \mathbb{C}$ and $\Re(q) \in (0, 1)$, ${}_H D_{a^+}^q$ denotes left-sided Hadamard fractional derivative of order q and $a > 0$, and ${}_H \mathcal{J}_{a^+}^{1-q}$ denotes left-sided Hadamard fractional integral of order $1 - q$. $g : [a, T] \times \mathbb{C} \rightarrow \mathbb{C}$ and $J_i : \mathbb{C} \rightarrow \mathbb{C}$ ($i = 1, \dots, m$) are some appropriate functions and impulsive points $a = t_0 < t_1 < \dots < t_m < t_{m+1} = T$. ${}_H \mathcal{J}_{a^+}^{1-q} z(t_i^+) = \lim_{\varepsilon \rightarrow 0^+} {}_H \mathcal{J}_{a^+}^{1-q} z(t_i + \varepsilon)$ and

${}_H\mathcal{J}_{a^+}^{1-q} z(t_i^-) = \lim_{\varepsilon \rightarrow 0^+} {}_H\mathcal{J}_{a^+}^{1-q} z(t_i + \varepsilon)$ represent the right and left limits of ${}_H\mathcal{J}_{a^+}^{1-q} z(t)$ at $t = t_i$, respectively.

Considering limited case in system (1a)–(1c), we have

$$\lim_{\substack{J_1(z(t_1^-)) \rightarrow 0, \\ \vdots \\ J_m(z(t_m^-)) \rightarrow 0}} \begin{cases} {}_H D_{a^+}^q z(t) = g(t, z(t)), & t \in [a, T], \quad t \neq t_i \quad (i = 1, 2, \dots, m), \\ \Delta({}_H\mathcal{J}_{a^+}^{1-q} z)|_{t=t_i} = {}_H\mathcal{J}_{a^+}^{1-q} z(t_i^+) - {}_H\mathcal{J}_{a^+}^{1-q} z(t_i^-) = J_i(z(t_i^-)) \in \mathbb{C}, & i = 1, 2, \dots, m, \\ {}_H\mathcal{J}_{a^+}^{1-q} z(a^+) = z_a, & z_a \in \mathbb{C} \end{cases} \quad (2)$$

$$= \begin{cases} {}_H D_{a^+}^q z(t) = f(t, z(t)), & t \in (a, T], \\ {}_H\mathcal{J}_{a^+}^{1-q} z(a) = z_a, & z_a \in \mathbb{C}. \end{cases}$$

Therefore,

$$\lim_{\substack{J_1(z(t_1^-)) \rightarrow 0, \\ \vdots \\ J_m(z(t_m^-)) \rightarrow 0}} \{ \text{the solution of impulsive system (1a)–(1c)} \} \quad (3)$$

$$= \{ \text{the solution of system (2)} \}.$$

Next, using the definition of Hadamard fractional derivative, system (1a)–(1c) is transformed into

$$\{ \text{system (1a)–(1c)} \} \iff \begin{cases} {}_H D_{a^+}^q z(t) = \left(t \frac{d}{dt} \right) \frac{1}{\Gamma(1-q)} \int_a^t \left(\ln \frac{t}{s} \right)^{-q} z(s) \frac{ds}{s} = g(t, z(t)), & t \in [a, t_1], \\ \Delta({}_H\mathcal{J}_{a^+}^{1-q} z)|_{t=t_1} = {}_H\mathcal{J}_{a^+}^{1-q} z(t_1^+) - {}_H\mathcal{J}_{a^+}^{1-q} z(t_1^-) = J_1(z(t_1^-)) \in \mathbb{C}, \\ {}_H D_{a^+}^q z(t) = \left(t \frac{d}{dt} \right) \frac{1}{\Gamma(1-q)} \int_a^t \left(\ln \frac{t}{s} \right)^{-q} z(s) \frac{ds}{s} = g(t, z(t)), & t \in (t_1, t_2], \\ \vdots \\ \Delta({}_H\mathcal{J}_{a^+}^{1-q} z)|_{t=t_k} = {}_H\mathcal{J}_{a^+}^{1-q} z(t_k^+) - {}_H\mathcal{J}_{a^+}^{1-q} z(t_k^-) = J_k(z(t_k^-)) \in \mathbb{C}, \\ {}_H D_{a^+}^q z(t) = \left(t \frac{d}{dt} \right) \frac{1}{\Gamma(1-q)} \int_a^t \left(\ln \frac{t}{s} \right)^{-q} z(s) \frac{ds}{s} = g(t, z(t)), & t \in (t_k, t_{k+1}], \\ \vdots \\ \Delta({}_H\mathcal{J}_{a^+}^{1-q} z)|_{t=t_m} = {}_H\mathcal{J}_{a^+}^{1-q} z(t_m^+) - {}_H\mathcal{J}_{a^+}^{1-q} z(t_m^-) = J_m(z(t_m^-)) \in \mathbb{C}, \\ {}_H D_{a^+}^q z(t) = \left(t \frac{d}{dt} \right) \frac{1}{\Gamma(1-q)} \int_a^t \left(\ln \frac{t}{s} \right)^{-q} z(s) \frac{ds}{s} = g(t, z(t)), & t \in (t_m, T], \\ {}_H\mathcal{J}_{a^+}^{1-q} z(a^+) = z_a, & z_a \in \mathbb{C}. \end{cases} \quad (4)$$

According to each interval $(t_k, t_{k+1}]$ (here $k = 0, 1, 2, \dots, m$) in (4), define a function

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s},$$

$$\text{for } t \in (t_k, t_{k+1}], \quad (5)$$

$$\tilde{z}(t) = \frac{1}{\Gamma(q)} \left({}_H\mathcal{J}_{a^+}^{1-q} z(t_k^+) \right) \left(\ln \frac{t}{t_k} \right)^{q-1}$$

$$\text{with } {}_H\mathcal{J}_{a^+}^{1-q} z(t_k^+) = {}_H\mathcal{J}_{a^+}^{1-q} z(t_k^-) + J_k(z(t_k^-)).$$

By Definition 2, we have

$$\begin{aligned} \left[{}_H D_{a^+}^q \tilde{z}(t) \right]_{t \in (t_k, t_{k+1}]} &= {}_H D_{a^+}^q \left\{ \frac{1}{\Gamma(q)} \left({}_H \mathcal{J}_{a^+}^{1-q} z(t_k^+) \right) \right. \\ &\quad \cdot \left(\ln \frac{t}{t_k} \right)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_k}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \Big\} \\ &= \frac{1}{\Gamma(q) \Gamma(1-q)} \left(t \frac{d}{dt} \right) \int_{t_k}^t \left(\ln \frac{t}{\eta} \right)^{1-q-1} \\ &\quad \cdot \left[\int_{t_k}^{\eta} \left(\ln \frac{\eta}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right] \frac{d\eta}{\eta} \\ &= g(t, z(t)) \Big|_{t \in (t_k, t_{k+1}]} . \end{aligned} \quad (6)$$

Therefore, $\tilde{z}(t)$ satisfies condition (1a), and $\tilde{z}(t)$ also satisfies conditions (1b) and (1c). But $\tilde{z}(t)$ does not satisfy condition (3). Thus, $\tilde{z}(t)$ is not a solution of system (1a)–(1c) and will be considered as an approximate solution to seek the exact solution of system (1a)–(1c).

Next, we will give some definitions and conclusions in Section 2 and prove the formula of general solution for impulsive differential equations with Hadamard fractional derivative in Section 3. Finally, we will provide an example to expound the main result in Section 4.

2. Preliminaries

Definition 1 (see [2]). Let $0 \leq a \leq T \leq \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{C}$ ($\Re(\alpha) > 0$) of function $\varphi(x)$ is defined by

$$\left({}_H \mathcal{J}_{a^+}^{\alpha} \varphi \right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{s} \right)^{\alpha-1} \varphi(s) \frac{ds}{s}, \quad (a < x < T), \quad (7)$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (see [2]). The left-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ on (a, T) is defined by

$$\begin{aligned} \left({}_H D_{a^+}^{\alpha} \varphi \right)(x) &= \delta^n \left({}_H \mathcal{J}_{a^+}^{n-\alpha} \varphi \right)(x) \\ &= \left(x \frac{d}{dx} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln \frac{x}{s} \right)^{n-\alpha-1} \varphi(s) \frac{ds}{s}, \end{aligned} \quad (8)$$

$(a < x < T),$

where $n = [\Re(\alpha)] + 1$ and differential operator $\delta = x(d/dx)$ with $\delta^0 y(x) = y(x)$.

Lemma 3 (see [2]). Let $\Re(\alpha) \geq 0$ and $n = [\Re(\alpha)] + 1$, $0 < a < T < \infty$. The fractional Cauchy problem

$$\begin{aligned} {}_H D_{a^+}^{\alpha} y(x) &= h(x, y(x)), \quad x \in (a, T], \\ {}_H D_{a^+}^{\alpha-i} y(a^+) &= b_i \in \mathbb{C}, \quad i = 1, 2, \dots, n \end{aligned} \quad (9)$$

is equivalent to the following nonlinear integral equation:

$$\begin{aligned} y(x) &= \sum_{i=1}^n \frac{b_i}{\Gamma(\alpha-i+1)} \left(\ln \frac{x}{a} \right)^{\alpha-i} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{s} \right)^{\alpha-1} h(s, y(s)) \frac{ds}{s}, \end{aligned} \quad (10)$$

$(0 < a < x).$

3. Main Result

Theorem 4. Let $q \in \mathbb{C}$, $\Re(q) \in (0, 1)$, and $\sum_{i=1}^0 x_i = 0$, and let ξ be an arbitrary constant. System (1a)–(1c) is equivalent to the integral equation:

$$\begin{aligned} z(t) &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} + \sum_{i=1}^k \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} - \frac{\xi}{\Gamma(q)} \sum_{i=1}^k J_i(z(t_i^-)) \\ &\quad \cdot \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} - \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_i} \right)^{q-1} - \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \end{aligned} \quad (11)$$

for $t \in (t_k, t_{k+1}]$, $k \in \{0, 1, 2, \dots, m\}$

provided that the integral in (11) exists.

Proof.

Necessity. Letting $J_i(z(t_i^-)) = 0$ ($i = 1, 2, \dots, k$) in (11), we have

$$\lim_{J_1(z(t_1^-)) \rightarrow 0, \dots, J_m(z(t_m^-)) \rightarrow 0} z(t)$$

$$\begin{aligned} &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} \\ &\quad + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &\quad \text{for } t \in (t_k, t_{k+1}], \quad k \in \{0, 1, 2, \dots, m\}. \end{aligned} \quad (12)$$

According to Lemma 3, (12) is equivalent to (2). Therefore, (11) satisfies condition (3).

Next, using Definition 2 for (11), we have

$$\begin{aligned}
 {}_H D_{a^+}^q z(t) \Big|_{t \in (t_k, t_{k+1}]} &= {}_H D_{a^+}^q \left\{ \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &+ \sum_{i=1}^k \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} - \frac{\xi}{\Gamma(q)} \sum_{i=1}^k J_i(z(t_i^-)) \\
 &\cdot \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right. \\
 &- \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_i} \right)^{q-1} \\
 &\left. \left. - \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \right\} = g(t, \\
 z(t)) \Big|_{t \geq a} \Big|_{t \in (t_k, t_{k+1}]} &- \left\{ \sum_{i=1}^k \xi J_i(z(t_i^-)) [g(t, z(t))]_{t \geq a} \right. \\
 &\left. - g(t, z(t)) \Big|_{t \geq t_i} \right\} \Big|_{t \in (t_k, t_{k+1}]} = g(t, z(t)) \Big|_{t \in (t_k, t_{k+1}]} .
 \end{aligned} \tag{13}$$

So (11) satisfies (1a).

By (11), we get

$$\begin{aligned}
 {}_H \mathcal{J}_{a^+}^{1-q} z(t_k^+) - {}_H \mathcal{J}_{a^+}^{1-q} z(t_k^-) &= \left\{ \frac{1}{\Gamma(1-q)} \right. \\
 &\cdot \int_a^t \left(\ln \frac{t}{\eta} \right)^{1-q-1} z(\eta) \frac{d\eta}{\eta} \Big|_{t \rightarrow t_k^+} - \left\{ \frac{1}{\Gamma(1-q)} \right. \\
 &\cdot \int_a^t \left(\ln \frac{t}{\eta} \right)^{1-q-1} z(\eta) \frac{d\eta}{\eta} \Big|_{t \rightarrow t_k^-} \Big\} = J_k(z(t_k^-)) \\
 &- \xi J_k(z(t_k^-)) \left[z_a + \int_a^t g(s, z(s)) \frac{ds}{s} \right. \\
 &- \left(z_a + \int_a^{t_k} g(s, z(s)) \frac{ds}{s} \right) \\
 &\left. \left. - \int_{t_k}^t g(s, z(s)) \frac{ds}{s} \right] \Big|_{t \rightarrow t_k^+} = J_k(z(t_k^-)) .
 \end{aligned} \tag{14}$$

Therefore, (11) satisfies (1b). Thus, (11) satisfies all conditions of system (1a)–(1c).

Sufficiency (by Mathematical Induction). By Lemma 3, the solution of system (1a)–(1c) satisfies

$$\begin{aligned}
 z(t) &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} \\
 &+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\text{for } t \in (a, t_1] .
 \end{aligned} \tag{15}$$

Using (15), we obtain

$$\begin{aligned}
 {}_H \mathcal{J}_{a^+}^{1-q} z(t_1^+) &= {}_H \mathcal{J}_{a^+}^{1-q} z(t_1^-) + J_1(z(t_1^-)) \\
 &= z_a + J_1(z(t_1^-)) + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} .
 \end{aligned} \tag{16}$$

Therefore, the approximate solution $\tilde{z}(t)$ is provided by

$$\begin{aligned}
 \tilde{z}(t) &= \frac{1}{\Gamma(q)} \left({}_H \mathcal{J}_{a^+}^{1-q} z(t_1^+) \right) \left(\ln \frac{t}{t_1} \right)^{q-1} + \frac{1}{\Gamma(q)} \\
 &\cdot \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &= \frac{1}{\Gamma(q)} \left[z_a + J_1(z(t_1^-)) + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \\
 &\cdot \left(\ln \frac{t}{t_1} \right)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\text{for } t \in (t_1, t_2] .
 \end{aligned} \tag{17}$$

Let $e_1(t) = z(t) - \tilde{z}(t)$ for $t \in (t_1, t_2]$. Moreover, by (15), the exact solution $z(t)$ of system (1a)–(1c) satisfies

$$\begin{aligned}
 \lim_{J_1(z(t_1^-)) \rightarrow 0} z(t) &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} \\
 &+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s}, \\
 &\text{for } t \in (t_1, t_2] .
 \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned}
 \lim_{J_1(z(t_1^-)) \rightarrow 0} e_1(t) &= \lim_{J_1(z(t_1^-)) \rightarrow 0} \{z(t) - \tilde{z}(t)\} \\
 &= \frac{1}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &- \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_1} \right)^{q-1} \\
 &\left. - \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} .
 \end{aligned} \tag{19}$$

By (19), let

$$\begin{aligned}
 e_1(t) &= \sigma[J_1(z(t_1^-))] \lim_{J_1(z(t_1^-)) \rightarrow 0} e_1(t) \\
 &= \frac{\sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &\quad + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\quad - \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_1} \right)^{q-1} \\
 &\quad \left. - \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\},
 \end{aligned} \quad (20)$$

where $\sigma(\cdot)$ is an undetermined function with $\sigma(0) = 1$. Thus,

$$\begin{aligned}
 z(t) &= \tilde{z}(t) + e_1(t) = \frac{\sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &\quad + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\quad + \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} \\
 &\quad + \frac{1 - \sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left\{ \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \right. \\
 &\quad \cdot \left(\ln \frac{t}{t_1} \right)^{q-1} + \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \left. \right\} \\
 &\quad \left. \text{for } t \in (t_1, t_2] \right].
 \end{aligned} \quad (21)$$

Using (21), we get

$$\begin{aligned}
 {}_H\mathcal{J}_{a^+}^{1-q} z(t_2^+) &= {}_H\mathcal{J}_{a^+}^{1-q} z(t_2^-) + J_2(z(t_2^-)) \\
 &= z_a + J_1(z(t_1^-)) + J_2(z(t_2^-)) \\
 &\quad + \int_a^{t_2} g(s, z(s)) \frac{ds}{s}.
 \end{aligned} \quad (22)$$

Then the approximate solution $\tilde{z}(t)$ is given by

$$\begin{aligned}
 \tilde{z}(t) &= \frac{1}{\Gamma(q)} ({}_H\mathcal{J}_{a^+}^{1-q} z(t_2^+)) \left(\ln \frac{t}{t_2} \right)^{q-1} + \frac{1}{\Gamma(q)} \\
 &\quad \cdot \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} = \frac{1}{\Gamma(q)} \left[z_a \right. \\
 &\quad + J_1(z(t_1^-)) + J_2(z(t_2^-)) + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \\
 &\quad \cdot \left(\ln \frac{t}{t_2} \right)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\quad \left. \text{for } t \in (t_2, t_3] \right].
 \end{aligned} \quad (23)$$

Let $e_2(t) = z(t) - \tilde{z}(t)$ for $t \in (t_2, t_3]$. By (21), the exact solution $z(t)$ of (1a)–(1c) satisfies

$$\begin{aligned}
 \lim_{J_1(z(t_1^-)) \rightarrow 0} z(t) &= \frac{\sigma[J_2(z(t_2^-))]}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &\quad + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\quad + \frac{J_2(z(t_2^-))}{\Gamma(q)} \left(\ln \frac{t}{t_2} \right)^{q-1} \\
 &\quad + \frac{1 - \sigma[J_2(z(t_2^-))]}{\Gamma(q)} \left\{ \left[z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right] \right. \\
 &\quad \cdot \left(\ln \frac{t}{t_2} \right)^{q-1} + \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \left. \right\} \\
 &\quad \left. \text{for } t \in (t_2, t_3] \right],
 \end{aligned}$$

$$\begin{aligned}
 \lim_{J_2(z(t_2^-)) \rightarrow 0} z(t) &= \frac{\sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &\quad + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
 &\quad + \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} \\
 &\quad + \frac{1 - \sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left\{ \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \right. \\
 &\quad \cdot \left(\ln \frac{t}{t_1} \right)^{q-1} + \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \left. \right\} \\
 &\quad \left. \text{for } t \in (t_2, t_3] \right],
 \end{aligned} \quad (24)$$

$$\lim_{\substack{J_1(z(t_1^-)) \rightarrow 0, \\ J_2(z(t_1^-)) \rightarrow 0}} z(t) = \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)}$$

$$\cdot \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \quad \text{for } t \in (t_2, t_3].$$

Therefore,

$$\begin{aligned}
 \lim_{J_1(z(t_1^-)) \rightarrow 0} e_2(t) &= \lim_{J_1(z(t_1^-)) \rightarrow 0} \{z(t) - \tilde{z}(t)\} \\
 &= \frac{\sigma[J_2(z(t_2^-))]}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
 &\quad + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s}
 \end{aligned}$$

$$\begin{aligned}
& - \left(z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right) \left(\ln \frac{t}{t_2} \right)^{q-1} \\
& - \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s}, \\
\lim_{J_2(z(t_2^-)) \rightarrow 0} e_2(t) &= \lim_{J_2(z(t_2^-)) \rightarrow 0} \{z(t) - \tilde{z}(t)\} \\
&= \frac{\sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&+ \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} - \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_2} \right)^{q-1} \\
&+ \frac{1 - \sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left\{ \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \right. \\
&\cdot \left. \left(\ln \frac{t}{t_1} \right)^{q-1} + \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \\
&- \frac{1}{\Gamma(q)} \left[z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_2} \right)^{q-1} \\
&- \frac{1}{\Gamma(q)} \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s}, \\
\lim_{\substack{J_1(z(t_1^-)) \rightarrow 0, \\ J_2(z(t_2^-)) \rightarrow 0}} e_2(t) &= \lim_{\substack{J_1(z(t_1^-)) \rightarrow 0, \\ J_2(z(t_2^-)) \rightarrow 0}} \{z(t) - \tilde{z}(t)\} \\
&= \frac{1}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&- \left(z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right) \left(\ln \frac{t}{t_2} \right)^{q-1} \\
&- \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \Big]. \tag{25}
\end{aligned}$$

Then, by (25), we obtain

$$\begin{aligned}
e_2(t) &= \frac{\sigma[J_1(z(t_1^-))] + \sigma[J_2(z(t_2^-))] - 1}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&\quad \left. + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right. \\
&\quad \left. - \left(z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right) \left(\ln \frac{t}{t_2} \right)^{q-1} \right. \\
&\quad \left. - \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right]
\end{aligned}$$

$$\begin{aligned}
& + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \Big] \\
& + \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} - \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_2} \right)^{q-1} \\
& + \frac{1 - \sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left[\left(z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right) \right. \\
&\cdot \left. \left(\ln \frac{t}{t_1} \right)^{q-1} + \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right] \\
& - \frac{\sigma[J_2(z(t_2^-))]}{\Gamma(q)} \left[\left(z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right) \right. \\
&\cdot \left. \left(\ln \frac{t}{t_2} \right)^{q-1} + \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right]. \tag{26}
\end{aligned}$$

Thus,

$$\begin{aligned}
z(t) &= \tilde{z}(t) + e_2(t) \\
&= \frac{\sigma[J_1(z(t_1^-))] + \sigma[J_2(z(t_2^-))] - 1}{\Gamma(q)} \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&+ \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} + \frac{J_2(z(t_2^-))}{\Gamma(q)} \left(\ln \frac{t}{t_2} \right)^{q-1} \\
&+ \frac{1 - \sigma[J_1(z(t_1^-))]}{\Gamma(q)} \left[\left(z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right) \right. \\
&\cdot \left. \left(\ln \frac{t}{t_1} \right)^{q-1} + \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right] \\
&+ \frac{1 - \sigma[J_2(z(t_2^-))]}{\Gamma(q)} \left[\left(z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right) \right. \\
&\cdot \left. \left(\ln \frac{t}{t_2} \right)^{q-1} + \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right] \tag{27}
\end{aligned}$$

for $t \in (t_2, t_3]$.

In addition,

$$\lim_{t_2 \rightarrow t_1} \begin{cases} {}_H D_{a^+}^q z(t) = g(t, z(t)), & t \in (a, t_3], \quad t \neq t_1, \quad t \neq t_2, \\ \Delta({}_H \mathcal{J}_{a^+}^{1-q} z)|_{t=t_i} = {}_H \mathcal{J}_{a^+}^{1-q} z(t_i^+) - {}_H \mathcal{J}_{a^+}^{1-q} z(t_i^-) = J_i(z(t_i^-)) \in \mathbb{C}, & i = 1, 2, \\ {}_H \mathcal{J}_{a^+}^{1-q} z(a^+) = z_a, & z_a \in \mathbb{C} \end{cases} \tag{28}$$

$$= \begin{cases} {}_H D_{a^+}^q z(t) = g(t, z(t)), & t \in (a, t_3], \quad t \neq t_1, \\ \Delta \left({}_H \mathcal{J}_{a^+}^{1-q} z \right) \Big|_{t=t_1} = J_1(z(t_1^-)) + J_2(z(t_2^-)), \\ {}_H \mathcal{J}_{a^+}^{1-q} z(a^+) = z_a, & z_a \in \mathbb{C}. \end{cases} \quad (29)$$

By using (21) and (27) for systems (29) and (28), we obtain

$$\begin{aligned} 1 - \sigma [J_1(z(t_1^-)) + J_2(z(t_2^-))] \\ = 1 - \sigma [J_1(z(t_1^-))] + 1 - \sigma [J_2(z(t_2^-))]. \end{aligned} \quad (30)$$

So $1 - \sigma(x) = \xi x$ for $\forall x \in \mathbb{C}$, where ξ is an arbitrary constant. Thus, (21) and (27) are rewritten into

$$\begin{aligned} z(t) &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)} \\ &\cdot \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &+ \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} \\ &- \frac{\xi J_1(z(t_1^-))}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\ &+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &- \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_1} \right)^{q-1} \\ &\left. - \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \quad \text{for } t \in (t_1, t_2], \\ z(t) &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)} \\ &\cdot \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &+ \frac{J_1(z(t_1^-))}{\Gamma(q)} \left(\ln \frac{t}{t_1} \right)^{q-1} + \frac{J_2(z(t_2^-))}{\Gamma(q)} \left(\ln \frac{t}{t_2} \right)^{q-1} \\ &- \frac{\xi J_1(z(t_1^-))}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\ &+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &- \left[z_a + \int_a^{t_1} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_1} \right)^{q-1} \\ &\left. - \int_{t_1}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \end{aligned}$$

$$\begin{aligned} &- \frac{\xi J_2(z(t_2^-))}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\ &+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &- \left[z_a + \int_a^{t_2} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_2} \right)^{q-1} \\ &\left. - \int_{t_2}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \quad \text{for } t \in (t_2, t_3]. \end{aligned} \quad (31)$$

For $t \in (t_n, t_{n+1}]$, let

$$\begin{aligned} z(t) &= \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} \\ &\cdot g(s, z(s)) \frac{ds}{s} + \sum_{i=1}^n \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} \\ &- \frac{\xi}{\Gamma(q)} \sum_{i=1}^n J_i(z(t_i^-)) \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\ &+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\ &- \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_i} \right)^{q-1} \\ &\left. - \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \quad \text{for } t \in (t_n, t_{n+1}]. \end{aligned} \quad (32)$$

Using (32), we have

$$\begin{aligned} {}_H \mathcal{J}_{a^+}^{1-q} z(t_{n+1}^+) &= {}_H \mathcal{J}_{a^+}^{1-q} z(t_{n+1}^-) + J_{n+1}(z(t_{n+1}^-)) \\ &= z_a + \sum_{i=1}^{n+1} J_i(z(t_i^-)) \\ &\quad + \int_a^{t_{n+1}} g(s, z(s)) \frac{ds}{s}. \end{aligned} \quad (33)$$

Therefore, the approximate solution $\tilde{z}(t)$ is provided as

$$\begin{aligned} \tilde{z}(t) &= \frac{1}{\Gamma(q)} \left({}_H \mathcal{J}_{a^+}^{1-q} z(t_{n+1}^+) \right) \left(\ln \frac{t}{t_{n+1}} \right)^{q-1} + \frac{1}{\Gamma(q)} \\ &\cdot \int_{t_{n+1}}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\Gamma(q)} \left[z_a + \sum_{i=1}^{n+1} J_i(z(t_i^-)) + \int_a^{t_{n+1}} g(s, z(s)) \frac{ds}{s} \right] \\
&\cdot \left(\ln \frac{t}{t_{n+1}} \right)^{q-1} + \frac{1}{\Gamma(q)} \\
&\cdot \int_{t_{n+1}}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&\quad \text{for } t \in (t_{n+1}, t_{n+2}].
\end{aligned} \tag{34}$$

Let $e_{n+1}(t) = z(t) - \tilde{z}(t)$ for $t \in (t_{n+1}, t_{n+2}]$. On the other hand, by (32), the exact solution $z(t)$ of (1a)–(1c) satisfies

$$\begin{aligned}
&\lim_{J_1(z(t_1^-)) \rightarrow 0, \dots, J_{n+1}(z(t_{n+1}^-)) \rightarrow 0} z(t) = \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} \\
&+ \frac{1}{\Gamma(q)} \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&\quad \text{for } t \in (t_{n+1}, t_{n+2}], \\
&\lim_{J_j(z(t_j^-)) \rightarrow 0, \dots, J_{n+1}(z(t_{n+1}^-)) \rightarrow 0} z(t) = \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)} \\
&\cdot \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&+ \sum_{\substack{1 \leq i \leq n+1, \\ i \neq j}} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} \\
&- \xi \sum_{\substack{1 \leq i \leq n+1, \\ i \neq j}} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&- \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_i} \right)^{q-1} \\
&\left. - \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right\} \\
&\quad \text{for } t \in (t_{n+1}, t_{n+2}].
\end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned}
&\lim_{J_1(z(t_1^-)) \rightarrow 0, \dots, J_{n+1}(z(t_{n+1}^-)) \rightarrow 0} e_{n+1}(t) = \lim_{J_1(z(t_1^-)) \rightarrow 0, \dots, J_{n+1}(z(t_{n+1}^-)) \rightarrow 0} \{z(t) - \tilde{z}(t)\} \\
&= \frac{1}{\Gamma(q)} \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} \right. \\
&\cdot g(s, z(s)) \frac{ds}{s} - \left[z_a + \int_a^{t_{n+1}} g(s, z(s)) \frac{ds}{s} \right]
\end{aligned}$$

$$\begin{aligned}
&\cdot \left(\ln \frac{t}{t_{n+1}} \right)^{q-1} - \int_{t_{n+1}}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \Big\}, \\
&\lim_{\substack{J_j(z(t_j^-)) \rightarrow 0, \\ j \in \{1, 2, \dots, n+1\}}} e_{n+1}(t) = \lim_{\substack{J_j(z(t_j^-)) \rightarrow 0, \\ j \in \{1, 2, \dots, n+1\}}} \{z(t) - \tilde{z}(t)\} \\
&= \frac{1}{\Gamma(q)} \left(1 - \xi \sum_{\substack{1 \leq i \leq n+1, \\ i \neq j}} J_i(z(t_i^-)) \right) \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&+ \sum_{\substack{1 \leq i \leq n+1, \\ i \neq j}} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} \\
&+ \xi \sum_{\substack{1 \leq i \leq n+1, \\ i \neq j}} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left\{ \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \right. \\
&\cdot \left(\ln \frac{t}{t_i} \right)^{q-1} + \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \Big\} \\
&- \frac{1}{\Gamma(q)} \left(z_a + \sum_{\substack{1 \leq i \leq n+1, \\ i \neq j}} J_i(z(t_i^-)) \right. \\
&+ \left. \int_a^{t_{n+1}} g(s, z(s)) \frac{ds}{s} \right) \left(\ln \frac{t}{t_{n+1}} \right)^{q-1} - \frac{1}{\Gamma(q)} \\
&\cdot \int_{t_{n+1}}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s}.
\end{aligned} \tag{36}$$

By (36), we obtain

$$\begin{aligned}
e_{n+1}(t) &= \frac{1}{\Gamma(q)} \left[1 - \xi \sum_{i=1}^{n+1} J_i(z(t_i^-)) \right] \left[z_a \left(\ln \frac{t}{a} \right)^{q-1} \right. \\
&+ \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
&+ \sum_{i=1}^{n+1} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} \\
&- \sum_{i=1}^{n+1} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_{n+1}} \right)^{q-1} \\
&\left. + \xi \sum_{i=1}^{n+1} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left\{ \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(\ln \frac{t}{t_i} \right)^{q-1} + \int_{t_i}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \Bigg\} \\
& - \frac{1}{\Gamma(q)} \left[z_a + \int_a^{t_{n+1}} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_{n+1}} \right)^{q-1} \\
& - \int_{t_{n+1}}^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
& \text{for } t \in (t_{n+1}, t_{n+2}].
\end{aligned} \tag{37}$$

Then

$$\begin{aligned}
z(t) &= \tilde{z}(t) + e_{n+1}(t) = \frac{z_a}{\Gamma(q)} \left(\ln \frac{t}{a} \right)^{q-1} + \frac{1}{\Gamma(q)} \\
& \cdot \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \\
& + \sum_{i=1}^{n+1} \frac{J_i(z(t_i^-))}{\Gamma(q)} \left(\ln \frac{t}{t_i} \right)^{q-1} - \frac{\xi}{\Gamma(q)} \sum_{i=1}^{n+1} J_i(z(t_i^-)) \\
& \cdot \left\{ z_a \left(\ln \frac{t}{a} \right)^{q-1} + \int_a^t \left(\ln \frac{t}{s} \right)^{q-1} g(s, z(s)) \frac{ds}{s} \right. \\
& \left. - \left[z_a + \int_a^{t_i} g(s, z(s)) \frac{ds}{s} \right] \left(\ln \frac{t}{t_i} \right)^{q-1} \right\}
\end{aligned}$$

Thus, the solution of system (1a)–(1c) satisfies (11).

So system (1a)–(1c) is equivalent to (11). The proof is now completed. $\square$

4. An Example

In this section, an example will be provided to illustrate the usefulness of Theorem 4, which will show that there is general solution for impulsive systems with Hadamard fractional derivatives.

Example. The analytical solution of (1a)–(1c) is difficultly obtained when f is a nonlinear function. Therefore, let us consider an impulsive fractional linear system:

$$\begin{aligned}
{}_H D_{1^+}^{1/2} z(t) &= \ln t, \quad t \in (1, 3], \quad t \neq 2, \\
\Delta({}_H \mathcal{J}_{1^+}^{1-1/2} z)|_{t=2} &= {}_H \mathcal{J}_{1^+}^{1-1/2} z(2^+) \\
&\quad - {}_H \mathcal{J}_{1^+}^{1-1/2} z(2^-) = l \in \mathbb{C}, \\
{}_H \mathcal{J}_{1^+}^{1-1/2} z(1^+) &= z_1 \in \mathbb{C}.
\end{aligned} \tag{39}$$

Using Theorem 4, we give general solution of system (39) by

$z(t)$

$$\begin{aligned}
& \left[\frac{z_1}{\Gamma(1/2)} (\ln t)^{-1/2} + \frac{4}{3} \frac{1}{\Gamma(1/2)} (\ln t)^{3/2} \right] \quad \text{for } t \in (1, 2], \\
& = \begin{cases} \frac{1}{\Gamma(1/2)} \left[z_1 (\ln t)^{-1/2} \Big|_{t \geq 1} + \frac{4}{3} (\ln t)^{3/2} \Big|_{t \geq 1} + l \left(\ln \frac{t}{2} \right)^{-1/2} \Big|_{t \geq 2} \right] \\ \quad - \frac{\xi l}{\Gamma(1/2)} \left[z_1 (\ln t)^{-1/2} \Big|_{t \geq 1} + \frac{4}{3} (\ln t)^{3/2} \Big|_{t \geq 1} - \left(z_1 + \frac{1}{2} \ln^2 2 \right) \left(\ln \frac{t}{2} \right)^{-1/2} \Big|_{t \geq 2} - \left(\frac{4}{3} \left(\ln \frac{t}{2} \right)^{3/2} + 2 \left(\ln \frac{t}{2} \right)^{1/2} \ln 2 \right) \Big|_{t \geq 2} \right] \end{cases} \quad \text{for } t \in (2, 3],
\end{aligned} \tag{40}$$

where ξ is a constant. Next, it will be verified that (40) satisfies all conditions of system (39).

(i) For $t \in (1, 2]$ in (40), we have

$${}_H D_{1^+}^{1/2} z(t) \Big|_{t \in (1, 2]} = \left\{ \frac{1}{\Gamma(1-1/2)} \left(t \frac{d}{dt} \right) \right.$$

$$\begin{aligned}
& \cdot \int_1^t \left(\ln \frac{t}{s} \right)^{1-1/2-1} \left[\frac{z_1}{\Gamma(1/2)} (\ln t)^{-1/2} \right. \\
& \left. + \frac{4}{3} \frac{1}{\Gamma(1/2)} (\ln t)^{3/2} \right] \frac{ds}{s} \Bigg\}_{t \in (1, 2]} = \ln t \Big|_{t \in (1, 2]}.
\end{aligned} \tag{41}$$

(ii) For $t \in (2, 3]$ in (40), we get

$$\begin{aligned}
{}_H D_{1^+}^{1/2} z(t) \Big|_{t \in (2,3]} &= \frac{1}{\Gamma(1-1/2)} \left(t \frac{d}{dt} \right) \int_1^t \left(\ln \frac{t}{s} \right)^{1-1/2-1} \left\{ \frac{1}{\Gamma(1/2)} \left[z_1 (\ln s)^{-1/2} \Big|_{t \geq 1} + \frac{4}{3} (\ln s)^{3/2} \Big|_{t \geq 1} + l \left(\ln \frac{s}{2} \right)^{-1/2} \Big|_{t \geq 2} \right] \right. \\
&\quad \left. - \frac{\xi l}{\Gamma(1/2)} \left[z_1 (\ln s)^{-1/2} \Big|_{t \geq 1} + \frac{4}{3} (\ln s)^{3/2} \Big|_{t \geq 1} - \left(z_1 + \frac{1}{2} \ln^2 2 \right) \left(\ln \frac{s}{2} \right)^{-1/2} \Big|_{t \geq 2} - \left(\frac{4}{3} \left(\ln \frac{s}{2} \right)^{3/2} + 2 \left(\ln \frac{s}{2} \right)^{1/2} \ln 2 \right) \Big|_{t \geq 2} \right] \right\} \frac{ds}{s} \Big|_{t \in (2,3]} \\
&= \frac{1}{\Gamma(1-1/2) \Gamma(1/2)} \left(t \frac{d}{dt} \right) \int_1^t \left(\ln \frac{t}{s} \right)^{1-1/2-1} \left[z_1 (\ln s)^{-1/2} + \frac{4}{3} (\ln s)^{3/2} - \xi l z_1 (\ln s)^{-1/2} - \xi l \frac{4}{3} (\ln s)^{3/2} \right] \frac{ds}{s} \Big|_{t \in (2,3]} \\
&\quad + \frac{l}{\Gamma(1-1/2) \Gamma(1/2)} \left(t \frac{d}{dt} \right) \int_2^t \left(\ln \frac{t}{\eta} \right)^{1-1/2-1} \left[\left(\ln \frac{\eta}{2} \right)^{-1/2} + \xi \left(z_1 + \frac{1}{2} \ln^2 2 \right) \left(\ln \frac{\eta}{2} \right)^{-1/2} \right. \\
&\quad \left. + \xi \left(\frac{4}{3} \left(\ln \frac{\eta}{2} \right)^{3/2} + 2 \left(\ln \frac{\eta}{2} \right)^{1/2} \ln 2 \right) \right] \frac{d\eta}{\eta} \Big|_{t \in (2,3]} = (\ln t - \xi l \ln t) \Big|_{t \geq 1} \Big|_{t \in (2,3]} + (\xi l \ln t) \Big|_{t \geq 2} \Big|_{t \in (2,3]} = \ln t \Big|_{t \in (2,3]}.
\end{aligned} \tag{42}$$

So (40) satisfies Hadamard fractional derivative of system (39).

By the definition of Hadamard fractional integral, we have

$$\begin{aligned}
{}_H \mathcal{J}_{1^+}^{1-1/2} \left\{ \frac{\xi l}{\Gamma(1/2)} \left[z_1 (\ln t)^{-1/2} \Big|_{t \geq 1} + \frac{4}{3} (\ln t)^{3/2} \Big|_{t \geq 1} \right. \right. \\
&\quad \left. \left. - \left(z_1 + \frac{1}{2} \ln^2 2 \right) \left(\ln \frac{t}{2} \right)^{-1/2} \Big|_{t \geq 2} - \left(\frac{4}{3} \left(\ln \frac{t}{2} \right)^{3/2} + 2 \left(\ln \frac{t}{2} \right)^{1/2} \ln 2 \right) \Big|_{t \geq 2} \right] \right\} \Big|_{t \rightarrow 2^+} \\
&= \frac{\xi l}{\Gamma(1/2) \Gamma(1/2)} \left\{ \int_1^t \left(\ln \frac{t}{\eta} \right)^{-1/2} \left[z_1 (\ln \eta)^{-1/2} \right. \right. \\
&\quad \left. \left. + \frac{4}{3} (\ln \eta)^{3/2} \right] \frac{d\eta}{\eta} - \int_2^t \left(\ln \frac{t}{\eta} \right)^{-1/2} \right. \\
&\quad \left. \cdot \left[\left(z_1 + \frac{1}{2} \ln^2 2 \right) \left(\ln \frac{\eta}{2} \right)^{-1/2} \right. \right. \\
&\quad \left. \left. + \left(\frac{4}{3} \left(\ln \frac{\eta}{2} \right)^{3/2} + 2 \left(\ln \frac{\eta}{2} \right)^{1/2} \ln 2 \right) \right] \frac{d\eta}{\eta} \right\} \Big|_{t \rightarrow 2^+} \\
&= \xi l z_1 + \left\{ \frac{4 \xi l (\ln t - \ln 1)^2}{3 \Gamma(1/2) \Gamma(1/2)} \int_1^t \left(1 \right. \right. \\
&\quad \left. \left. - \frac{\ln \eta - \ln 1}{\ln t - \ln 1} \right)^{-1/2} \right. \\
&\quad \left. \cdot \left(\frac{\ln \eta - \ln 1}{\ln t - \ln 1} \right)^{3/2} d \frac{\ln \eta - \ln 1}{\ln t - \ln 1} \right\} \Big|_{t \rightarrow 2^+}
\end{aligned}$$

$$\begin{aligned}
&- \xi l \left\{ z_1 + \frac{1}{2} \ln^2 2 + \frac{4 (\ln t - \ln 2)^2}{3 \Gamma(1/2) \Gamma(1/2)} \int_2^t \left(1 \right. \right. \\
&\quad \left. \left. - \frac{\ln \eta - \ln 2}{\ln t - \ln 2} \right)^{-1/2} \right. \\
&\quad \cdot \left(\frac{\ln \eta - \ln 2}{\ln t - \ln 2} \right)^{3/2} d \frac{\ln \eta - \ln 2}{\ln t - \ln 2} \\
&\quad \left. + \frac{2 (\ln t - \ln 2)}{\Gamma(1/2) \Gamma(1/2)} \int_2^t \left(1 - \frac{\ln \eta - \ln 2}{\ln t - \ln 2} \right)^{-1/2} \right. \\
&\quad \left. \cdot \left(\frac{\ln \eta - \ln 2}{\ln t - \ln 2} \right)^{1/2} d \frac{\ln \eta - \ln 2}{\ln t - \ln 2} \right\} \Big|_{t \rightarrow 2^+} \\
&= \xi l z_1 + \frac{4 \xi l (\ln 2 - \ln 1)^2}{3 \Gamma(1/2) \Gamma(1/2)} \frac{\Gamma(1/2) \Gamma(5/2)}{\Gamma(3)} - \xi l \left[z_1 \right. \\
&\quad \left. + \frac{1}{2} \ln^2 2 \right] = 0.
\end{aligned} \tag{43}$$

Therefore,

$$\begin{aligned}
{}_H \mathcal{J}_{1^+}^{1-1/2} z(2^+) - {}_H \mathcal{J}_{1^+}^{1-1/2} z(2^-) \\
&= \left\{ \frac{1}{\Gamma(1-1/2)} \int_1^t \left(\ln \frac{t}{\eta} \right)^{1-1/2-1} z(\eta) \frac{d\eta}{\eta} \right\} \Big|_{t \rightarrow 2^+} \\
&\quad - \left\{ \frac{1}{\Gamma(1-1/2)} \int_1^t \left(\ln \frac{t}{\eta} \right)^{1-1/2-1} z(\eta) \frac{d\eta}{\eta} \right\} \Big|_{t \rightarrow 2^-} \\
&= l.
\end{aligned} \tag{44}$$

That is, (40) satisfies impulsive condition in system (39).

Finally, for system (39), there exists an implicit condition

$$\lim_{l \rightarrow 0} \begin{cases} {}_H D_{1^+}^{1/2} z(t) = \ln t, & t \in (1, 3], \quad t \neq 2, \\ \Delta \left({}_H \mathcal{J}_{1^+}^{1-1/2} z \right) \Big|_{t=2} = {}_H \mathcal{J}_{1^+}^{1-1/2} z(2^+) - {}_H \mathcal{J}_{1^+}^{1-1/2} z(2^-) = l \in \mathbb{C}, \\ {}_H \mathcal{J}_{1^+}^{1-1/2} z(1^+) = z_1 \in \mathbb{C}, \end{cases} \quad (45)$$

$$\rightarrow \begin{cases} {}_H D_{1^+}^{1/2} z(t) = \ln t, & t \in (1, 3], \\ {}_H \mathcal{J}_{1^+}^{1-1/2} z(1^+) = z_1 \in \mathbb{C}. \end{cases}$$

By (40), we have

$$\lim_{l \rightarrow 0} z(t) = \begin{cases} \frac{z_1}{\Gamma(1/2)} \left(\int_1^t \frac{ds}{s} \right)^{1/2-1} + \frac{1}{\Gamma(1/2)} \int_1^t \left(\ln \frac{t}{s} \right)^{1/2-1} \ln s \frac{ds}{s} & \text{for } t \in (1, 2], \\ \frac{z_1}{\Gamma(1/2)} \left(\int_1^t \frac{ds}{s} \right)^{1/2-1} + \frac{1}{\Gamma(1/2)} \int_1^t \left(\ln \frac{t}{s} \right)^{1/2-1} \ln s \frac{ds}{s} & \text{for } t \in (2, 3] \end{cases} \quad (46)$$

which shows that (40) satisfies condition (3). So (40) is the general solution of system (39).

Competing Interests

The authors declare that they have no competing interests.

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