Research Article

On a Stochastic Lotka-Volterra Competitive System with Distributed Delay and General Lévy Jumps

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This paper considers a stochastic competitive system with distributed delay and general Lévy jumps. Almost sufficient and necessary conditions for stability in time average and extinction of each population are established under some assumptions. And two facts are revealed: both stability in time average and extinction have closer relationships with the general Lévy jumps, firstly; and secondly, the distributed delay has no effect on the stability in time average and extinction of the stochastic system. Some simulation figures, which are obtained by the split-step \( \theta \)-method to discretize the stochastic model, are introduced to support the analytical findings.

1. Introduction

In recent years, delay differential equations has been used in the study of population dynamics. A famous competitive system with distributed delay can be expressed by

\[
\begin{align*}
\frac{d y_1(t)}{dt} &= y_1(t) \left[ b_1 - a_{11} y_1(t) - a_{12} \int_{-\tau_1}^{0} y_2(t + \theta) \, d\mu_1(\theta) \right], \\
\frac{d y_2(t)}{dt} &= y_2(t) \left[ b_2 - a_{21} \int_{-\tau_2}^{0} y_1(t + \theta) \, d\mu_2(\theta) - a_{22} y_2(t) \right],
\end{align*}
\]

(1)

where \( y_i(t) \) denotes the size of the \( i \)th population, \( b_i, a_{ij} \), and \( \tau_2 \) are all positive constants, and \( \mu_i \) is a probability measure on \([ -\tau_i, 0] \). There is an extensive literature concerned with the dynamics of (1) and we here only mention Kuang and Smith [1], Faria [2], Freedman and Wù [3], Bereketoglu and Györi [4], and Gopalsamy [5] among many others. In particular, Kuang (see [6, p. 231]) claimed that if \( \Psi_1 > 0 \) and \( \Psi_2 > 0 \), then model (1) has a positive equilibrium \( x^* = (x_1^*, x_2^*) = (\Psi_1/\Psi, \Psi_2/\Psi) \) which is globally asymptotically stable, where

\[
\Psi = a_{11}a_{22} - a_{12}a_{21}, \quad \Psi_1 = b_1a_{22} - b_2a_{12}, \quad \text{and} \quad \Psi_2 = b_2a_{11} - b_1a_{21}.
\]

It is important to point out that if \( \Psi_1 > 0 \) and \( \Psi_2 > 0 \), then \( \Psi > 0 \).

In the real world, the intrinsic growth rates of many species are always disturbed by environmental noises (see, e.g., [7–10]), which was recognized by many scholars in recent years (see, e.g., [11–14]). In particular, May [7] has pointed out that, due to environmental noises, the birth rates, carrying capacity, and other parameters involved in the system should be stochastic. In this paper, we assume that the parameters \( b_1 \) and \( b_2 \) are stochastic; then by the central limit theorem, we can replace \( b_1 \) and \( b_2 \) by

\[
\begin{align*}
b_1 &\rightarrow b_1 + \sigma_1 B_1(t), \\
b_2 &\rightarrow b_2 + \sigma_2 B_2(t),
\end{align*}
\]

(2)

where, for \( i = 1, 2 \), \( B_i(t) \) represents a standard Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathcal{P})\) and \( \sigma_i^2 \) is the intensity of the noise.

On the other hand, the population systems may suffer sudden environmental perturbations, that is, some jump type stochastic perturbations, for example, earthquakes, hurricanes, and epidemics. Some scholars have concentrated on the population systems with compensator jumps, and some significant and interesting results have been obtained (see,
e.g., [15–19]). Bao et al. [15, 16] did pioneering work in this field. In addition, Zou et al. [20–22] introduce a general Lévy jumps, which is more reasonable and complicated than the compensator jumps from the viewpoint of biomathematics (see [20]), into population models for the first time. However, there are no articles introducing the general Lévy jumps into population models with distributed delay, to the best of our knowledge. Motivated by these, we consider the famous stochastic competitive system with distributed delay and general Lévy jumps:

\[
\begin{align*}
\dot{y}_1(t) &= y_1(t^+) \\
&\quad \cdot \left[ b_1 - a_{11}y_1(t^+) - a_{12} \int_{-\tau_1}^{0} y_2(s^+) d\mu_1(s^+) \right] dt \\
&\quad + \sigma_1 y_1(t^+) \int_{\mathbb{Y}} Y_1(u) N(dt, du) \\
&\quad + \sigma_2 y_2(t^+) \int_{\mathbb{Y}} Y_2(u) N(dt, du), \\
\dot{y}_2(t) &= y_2(t^+) \\
&\quad \cdot \left[ b_2 - a_{21} \int_{-\tau_1}^{0} y_1(s^+) d\mu_2(s^+) - a_{22} y_2(t^+) \right] dt \\
&\quad + \sigma_2 y_1(t^+) \int_{\mathbb{Y}} Y_1(u) N(dt, du),
\end{align*}
\]

where \( y_i(t^+) = \lim_{s \to t^+} y_i(s) \), \( N(dt, du) \) is a real-valued Poisson counting measure with characteristic measure \( \lambda \) on a measurable subset \( \mathbb{Y} \) of \( \mathbb{R}_+ \) with \( \lambda(\mathbb{Y}) < +\infty \), \( \mu_i(dt, du) = N(dt, du) - \lambda(dt) dt \), \( \gamma(u) \) is bounded function, and \( \gamma(u) > -1 \), \( u \in \mathbb{Y} \); furthermore, we assume that \( B_i(t) \) is independent of \( N \). Let the initial data \( \xi(t) = (\xi_1(t), \xi_2(t)) \in C([-\tau, 0], R^2) \), where \( C([-\tau, 0]; R^2) \) represents the family of continuous functions from \([-\tau, 0]\) to \( R^2 \) with the norm \( \| \xi \| = \sup_{-\tau \leq t \leq 0} |\xi(t)|, i = 1, 2, r = \max \{\tau_1, \tau_2\} \). Other parameters are defined and required as before.

For convenience, we introduce the following notations:

\[
\begin{align*}
R^2 &= \{ g = (g_1, g_2) \in R^2 \mid g_i > 0, \ i = 1, 2 \}, \\
\langle f(t) \rangle &= t^{-1} \int_{0}^{t} f(s) ds, \\
\eta_i &= \int_{\mathbb{Y}} \ln \left( \frac{1}{1 + \gamma_i(u)} \right) \lambda(du), \ i = 1, 2, \\
\overline{\Psi}_1 &= 0.5 a_{22} \eta_2 - 0.5 a_{12} \eta_1 - a_{12} \eta_2, \\
\overline{\Psi}_2 &= 0.5 a_{11} \eta_1 - 0.5 a_{21} \eta_1 + a_{12} \eta_2 - a_{12} \eta_1.
\end{align*}
\]

Moreover, we impose the following assumptions in this paper.

**Assumption 1.** There exists a positive constant \( c \) such that \( |\ln(1 + \gamma(u))| \leq c \) for \( \gamma(u) > -1 \).

In this paper, we consider a stochastic competitive system with distributed delay and general Lévy jumps. Unlike the deterministic system, the stochastic system does not have an interior equilibrium. Therefore, we cannot investigate the stability of the stochastic system. In Section 2, we show that the solution to system (3) will tend to a point in time average. Furthermore, we establish almost sufficient and necessary conditions for stability in time average and extinction of each population. In Section 3, we present an example to illustrate our mathematical findings. Section 4 gives the conclusions and future directions of the research.

### 2. Main Content

**Lemma 2** (see Liu et al. [23]). Suppose that \( z(t) \in C(\Omega \times [0, +\infty), R^+) \).

(i) If there exist two positive constants \( T \) and \( \rho_0 \) such that

\[
\ln z(t) \leq pt - \rho_0 \int_{0}^{t} z(s) ds + \sum_{i=1}^{2} a_i B_i(t) \text{ for all } t \geq T,
\]

where \( a_i, i = 1, 2, \) are constants, then

\[
\limsup_{t \to +\infty} \frac{\ln z(t)}{t} \leq \frac{\rho}{\rho_0} \text{ a.s., if } \rho \geq 0;
\]

\[
\lim_{t \to +\infty} z(t) = 0 \text{ a.s., if } \rho < 0.
\]

(ii) If there exist three positive constants \( T, \rho, \) and \( \rho_0 \) such that

\[
\ln z(t) \geq \lambda t - \rho_0 \int_{0}^{t} z(s) ds + \sum_{i=1}^{2} a_i B_i(t) \text{ for all } t \geq T,
\]

then

\[
\liminf_{t \to +\infty} \frac{\ln z(t)}{t} \geq \frac{\rho}{\rho_0} \text{ a.s.}
\]

In order for the model to be significant, we shall show that the solution is global and nonnegative. However, theorem of existence and uniqueness ([24–28]) is not satisfied in system (3). By using method established by Mao et al. [8], we will show existence and uniqueness of the global positive solution of system (3).

**Lemma 3.** Let Assumption 1 hold. For any given initial value \( \xi(t) = (\xi_1(t), \xi_2(t)) \in C([-\tau, 0], \mathbb{R}^2) \); then system (3) has a unique positive solution \( x(t) = (x_1(t), x_2(t)) \) on \( t \geq -\tau \) a.s. and the solution satisfies

\[
\limsup_{t \to +\infty} \frac{\ln x_1(t)}{\ln t} \leq 1 \text{ a.s., } i = 1, 2.
\]

**Proof.** The proof is similar to Han et al. [29] by defining

\[
V(x) = V_1(x) + V_2(x),
\]

where

\[
V_1(x) = \sqrt{x_1} - 1 - 0.5 \ln x_1,
\]

\[
V_2(x) = \sqrt{x_2} - 1 - 0.5 \ln x_2.
\]

In addition, applying the inequality, for \( i = 1, 2, \)

\[
\int_{-\tau}^{t} x_i^2((s + \theta)^-) d\mu(\theta) ds \leq \int_{-\tau}^{t} x_i^2((s^-)^-) d\mu(\theta) ds \\
\leq \int_{-\tau}^{t} d\mu(\theta) \int_{0}^{t} x_i^2((s^-)^-) ds \\
\leq \int_{-\tau}^{t} x_i^2((s^-)^-) ds + \int_{0}^{t} x_i^2((s^-)^-) ds.
\]
So we omit it here. Now let us prove inequality (6).

Case 1 ($i = 1$). For any $t \geq 0$, applying the generalized Itô's formula [30] to (3) results in

$$
\begin{align*}
\frac{d}{dt} \ln y_1(t) &= e^t \ln y_1(0) + \int_0^t e^s \left[ \ln y_1(s) + b_1 - 0.5 \sigma_1^2 s \right] ds + \int_0^t e^s \sigma_1 dB_1(s) \\
&\quad + \int_0^t e^s (\ln (1 + y_1(u)) \lambda(du) - a_1 y_1(s)) ds
\end{align*}
$$

Thus

$$
\frac{d}{dt} \ln y_1(t) \leq \ln y_1(0) + \int_0^t e^s \left[ \ln y_1(s) + b_1 - 0.5 \sigma_1^2 s \right] ds + \int_0^t e^s \sigma_1 dB_1(s) + \int_0^t e^s \int_0^s \ln (1 + y_1(u)) \lambda(du) ds, du.
$$

The rest of the proof is analogous with Lemma 4.4 in [15]; we omitted it here.

Case 2 ($i = 2$). The proof is similar to Case 1; we left it out here. The proof is complete.

**Theorem 4.** For system (3), we suppose that Assumption 1, $\Psi_1 > 0$ and $\Psi_2 > 0$, holds.

(I) If $b_1 < 0.5 \sigma_1^2 + \eta_1$ and $b_2 < 0.5 \sigma_2^2 + \eta_2$, then both $y_1$ and $y_2$ are extinctive almost surely (a.s.); that is,  
$$
\lim_{t \to \infty} Y_1(t) = 0 \text{ a.s., } i = 1, 2.
$$

(II) If $b_1 > 0.5 \sigma_1^2 + \eta_1$ and $b_2 < 0.5 \sigma_2^2 + \eta_2$, then $y_2$ is extinctive a.s. and $y_1$ is stable in time average a.s.; that is,

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_1(s) ds = \frac{b_1 - 0.5 \sigma_1^2 - \eta_1}{a_1}, \text{ a.s.}
$$

(III) If $b_1 < 0.5 \sigma_1^2 + \eta_1$ and $b_2 > 0.5 \sigma_2^2 + \eta_2$, then $y_1$ is extinctive a.s. and $y_2$ is stable in time average a.s.; that is,

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_2 - 0.5 \sigma_2^2 - \eta_2}{a_2}, \text{ a.s.}
$$

(IV) If $b_1 > 0.5 \sigma_1^2 + \eta_1$, $b_2 > 0.5 \sigma_2^2 + \eta_2$, then $y_2$ is extinctive a.s. and $y_1$ is stable in time average a.s.:

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_1(s) ds = \frac{b_1 - 0.5 \sigma_1^2 - \eta_1}{a_1}, \text{ a.s.}
$$

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_2 - 0.5 \sigma_2^2 - \eta_2}{a_2}, \text{ a.s.}
$$

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_1(s) ds = \frac{b_1 - 0.5 \sigma_1^2 - \eta_1}{a_1}, \text{ a.s.}
$$

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_2 - 0.5 \sigma_2^2 - \eta_2}{a_2}, \text{ a.s.}
$$

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_1(s) ds = \frac{b_1 - 0.5 \sigma_1^2 - \eta_1}{a_1}, \text{ a.s.}
$$

$$
\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_2 - 0.5 \sigma_2^2 - \eta_2}{a_2}, \text{ a.s.}
$$

**Proof.** Applying Itô's formula [30] to the first equality in (3), we get

$$
\ln y_1(t) - \ln y_1(0) = \left[ b_1 - 0.5 \sigma_1^2 - \eta_1 \right] t - a_1 \int_0^t y_1(s) ds - a_2 \int_0^t \int_0^s \ln (1 + y_1(u)) \lambda(du) ds, du.
$$

Making use of the Fubini theorem and a substitution technique, we have

$$
\int_0^t \int_0^s y_2(s + \theta) d\mu_2(\theta) ds = \int_0^t d\mu_2(\theta)
$$

$$
\int_0^t y_2(s + \theta) ds = \int_0^t d\mu_2(\theta) \int_0^{s+\theta} y_2(s) ds
$$

$$
= \int_{-\tau_2}^t d\mu_2(\theta)
$$

$$
\int_0^t y_2(s) ds + \int_0^t y_2(s) ds + \int_0^{s+\theta} y_2(s) ds
$$

$$
= \int_{-\tau_2}^t d\mu_2(\theta)
$$

$$
\int_0^t y_2(s) ds + \int_{-\tau_2}^t y_2(s) ds
$$

$$
+ \int_{-\tau_2}^t d\mu_2(\theta) \int_0^{t+\theta} y_2(s) ds.
$$
Therefore, we derive
\[
\ln y_1(t) - \ln y_1(0) = \left( b_1 - 0.5 \sigma_1^2 - \eta_1 \right) t - a_{11} \int_0^t y_1(s) ds
\]

\[ - a_{12} \int_0^t y_2(s) ds - a_{12} \int_0^t \mu_2(\theta) \int_\theta^t y_2(s) ds \]

\[ - a_{12} \int_0^t \mu_2(\theta) \int_\theta^t y_2(s) ds + \sigma_1 B_1(t) \]

\[ + \int_0^t \int_y \ln(1 + y_1(u)) \, N(ds, du). \]

(19)

Similarly,
\[
\ln y_2(t) - \ln y_2(0) = \left( b_2 - 0.5 \sigma_2^2 - \eta_2 \right) t - a_{21} \int_0^t y_1(s) ds
\]

\[ - a_{21} \int_0^t y_1(s) ds - a_{21} \int_0^t \mu_1(\theta) \int_\theta^t y_1(s) ds \]

\[ - a_{21} \int_0^t \mu_1(\theta) \int_\theta^t y_1(s) ds + \sigma_2 B_2(t) \]

\[ + \int_0^t \int_y \ln(1 + y_2(u)) \, N(ds, du). \]

(20)

Making use of (9), for arbitrary \( \varepsilon > 0 \), there is \( T > 0 \) such that, for \( t \geq T \),
\[
\frac{-\varepsilon}{2} \leq a_{12} t^{-1} \int_0^t y_2(s + \theta) d\mu_2(\theta) ds \leq \frac{\varepsilon}{2}
\]

(24)

Substituting the above inequalities into (19), we can see that, for \( t \geq T \),
\[
\ln y_1(t) \leq (b_1 - 0.5 \sigma_1^2 - \eta_1 + \varepsilon) t - a_{11} \int_0^t y_1(s) ds
\]

\[ + \sigma_1 B_1(t) \]  

\[ + \int_0^t \int_y \ln(1 + y_1(u)) \, N(ds, du), \]

(25)

\[
\ln y_1(t) \geq (b_1 - 0.5 \sigma_1^2 - \eta_1 - \varepsilon) t - a_{11} \int_0^t y_1(s) ds
\]

\[ + \sigma_1 B_1(t) \]  

\[ + \int_0^t \int_y \ln(1 + y_1(u)) \, N(ds, du). \]

(26)

Since \( b_1 > 0.5 \sigma_1^2 + \eta_1 \), we can choose \( \varepsilon \) sufficiently small such that \( b_1 - 0.5 \sigma_1^2 - \eta_1 - \varepsilon > 0 \). Applying (i) and (ii) in Lemma 2 to (25) and (26), respectively, we derive
\[
\frac{b_1 - 0.5 \sigma_1^2 - \eta_1 - \varepsilon}{a_{11}} \leq \liminf_{t \to +\infty} \langle y_1(t) \rangle
\]

\[ \leq \limsup_{t \to +\infty} \langle y_1(t) \rangle \]

(27)

Let \( \varepsilon \to 0 \). Then, we have \( \lim_{t \to +\infty} y_1(t) = [b_1 - 0.5 \sigma_1^2 - \eta_1] / a_{11}, \) a.s.

The proof of (III) is homogeneous with (II) by symmetry; hence it is omitted.

Now let us prove (IV). For \( i = 1, 2 \), consider the following equation:
\[
d z_i(t) = z_i(t) \left[ b_i - a_i z_i(t) \right] dt + \sigma_i z_i(t) dB_i(t)
\]

\[ + z_i(t) \int_y y_i(u) \, N(dt, du), \]

(28)

\[ z_i(\theta) = y_i(\theta), \quad \theta \in [-\tau, 0]. \]

In virtue of the classic stochastic comparison theorem [32], we can find that
\[
y_1(t) \leq z_1(t), \]

\[ y_2(t) \leq z_2(t). \]

(29)
Since $b_i > 0.5\sigma_i^2 + \eta_i$, $i = 1, 2$, similar to the proof of (II), we can show that

$$
\lim_{t \to \infty} \left< z_i(t) \right> = \lim_{t \to \infty} t^{-1} \int_0^t z_i(s) \, ds = \left( b_i - 0.5\sigma_i^2 - \eta_i \right) \frac{1}{a_i}, \quad \text{a.s., } i = 1, 2,
$$

(30)

Thus

$$
\begin{align*}
\lim_{t \to \infty} t^{-1} \int_{t+\theta}^t z_1(s) \, ds &= \lim_{t \to \infty} \left( t^{-1} \int_0^t z_1(s) \, ds - t^{-1} \int_0^{t+\theta} z_1(s) \, ds \right) = 0, \\
\lim_{t \to \infty} t^{-1} \int_{t+\theta}^t z_2(s) \, ds &= 0, \quad \text{a.s.,}
\end{align*}
$$

(31)

which, together with (29), implies that

$$
\begin{align*}
\lim_{t \to \infty} t^{-1} \int_0^t y_1(s) \, ds &= 0, \\
\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) \, ds &= 0, \quad \text{a.s.}
\end{align*}
$$

(32)

On the other hand, calculating $(2.5) \times a_{11} - (2.4) \times a_{21}$ deduces

$$
\begin{align*}
a_{11} \ln \frac{y_2(t)}{y_2(0)} &= -a_{11}a_{21} \int_{-\nu_1}^0 d\mu_1(\theta) \int_{\theta_1}^0 y_1(s) \, ds \\
&\quad - a_{11}a_{21} \int_{-\nu_2}^0 d\mu_1(\theta) \int_{\theta_2}^{t+\theta} y_1(s) \, ds \\
&\quad + a_{21}a_{12} \int_{-\nu_2}^0 d\mu_2(\theta) \int_{\theta_2}^0 y_2(s) \, ds \\
&\quad + a_{21}a_{12} \int_{-\nu_2}^0 d\mu_2(\theta) \int_{\theta_2}^{t+\theta} y_2(s) \, ds \\
&\quad + a_{21} \ln \frac{y_1(t)}{y_1(0)} + \left( \Psi_2 - \Psi_2 \right) t \\
&\quad - \Psi \int_{0}^t y_2(s) \, ds - a_{21}\sigma_1 B_1(t) \\
&\quad - a_{21} \int_{0}^t \int_{\gamma} \ln(1 + y_1(u)) \, \mathcal{N}(ds, du) \\
&\quad + a_{11}\sigma_2 B_2(t) \\
&\quad + a_{11} \int_{0}^t \int_{\gamma} \ln(1 + y_2(u)) \, \mathcal{N}(ds, du).
\end{align*}
$$

(33)

By virtue of (6) and (32), for arbitrary $\varepsilon > 0$, there is $T > 0$ such that, for $t \geq T$,

$$
\begin{align*}
&- a_{11}a_{21} t^{-1} \int_{-\nu_1}^0 d\mu_1(\theta) \int_{\theta_1}^0 y_1(s) \, ds < \frac{\varepsilon}{3}, \\
&- a_{11}a_{21} t^{-1} \int_{-\nu_2}^0 d\mu_1(\theta) \int_{\theta_2}^{t+\theta} y_1(s) \, ds < \frac{\varepsilon}{3}, \\
&- a_{21}a_{12} t^{-1} \int_{-\nu_2}^0 d\mu_2(\theta) \int_{\theta_2}^0 y_2(s) \, ds < \frac{\varepsilon}{3}, \\
&- a_{21}a_{12} t^{-1} \int_{-\nu_2}^0 d\mu_2(\theta) \int_{\theta_2}^{t+\theta} y_2(s) \, ds < \frac{\varepsilon}{3},
\end{align*}
$$

(34)

Substituting the above inequalities into (33) results in

$$
\begin{align*}
a_{11} \ln y_2(t) &\leq \left( \Psi_2 - \Psi_2 + \varepsilon \right) t - \Psi \int_{0}^t y_2(s) \, ds \\
&\quad - a_{21}\sigma_1 B_1(t) + a_{11}\sigma_2 B_2 \\
&\quad + a_{11} \int_{0}^t \int_{\gamma} \ln(1 + y_2(u)) \, \mathcal{N}(ds, du),
\end{align*}
$$

(35)

for $t > T$. Meanwhile, calculating $(2.4) \times a_{22} - (2.5) \times a_{12}$ yields

$$
\begin{align*}
a_{22} \ln \frac{y_1(t)}{y_1(0)} &= -a_{11}a_{21} \int_{-\nu_1}^0 d\mu_1(\theta) \int_{\theta_1}^0 y_1(s) \, ds \\
&\quad - a_{11}a_{21} \int_{-\nu_2}^0 d\mu_1(\theta) \int_{\theta_2}^{t+\theta} y_1(s) \, ds \\
&\quad + a_{12}a_{21} \int_{-\nu_2}^0 d\mu_2(\theta) \int_{\theta_2}^0 y_2(s) \, ds \\
&\quad + a_{12}a_{21} \int_{-\nu_2}^0 d\mu_2(\theta) \int_{\theta_2}^{t+\theta} y_2(s) \, ds \\
&\quad + a_{21} \ln \frac{y_1(t)}{y_1(0)} + \left( \Psi_2 - \Psi_2 \right) t \\
&\quad - \Psi \int_{0}^t y_1(s) \, ds + a_{22}\sigma_2 B_1(t) \\
&\quad - a_{22} \int_{0}^t \int_{\gamma} \ln(1 + y_1(u)) \, \mathcal{N}(ds, du)
\end{align*}
$$

(36)
Using the same way, by (36) we can have that, for $t > T$,
\begin{align*}
    a_{22} \ln y_1(t) &\leq \left(\Psi_1 - \Psi_1 + \varepsilon\right) t - \Psi \int_0^t y_1(s) \, ds \\
    &+ a_{22} \sigma_1 B_1(t) \\
    &+ a_{22} \int_0^t \int_Y \ln (1 + y_1(u)) \, \tilde{N}(ds, du) \\
    &- a_{12} \sigma_2 B_2(t) \\
    &- a_{12} \int_0^t \int_Y \ln (1 + y_2(u)) \, \tilde{N}(ds, du).
\end{align*}
\begin{equation}
    (37)
\end{equation}

(A) Suppose $\Psi_1 > \Psi_1$ and $\Psi_2 < \Psi_2$. Note that $\Psi_2 < \Psi_2$, and then let $\varepsilon$ be sufficiently small such that $\Psi_2 - \Psi_2 + \varepsilon < 0$. Applying (i) in Lemma 2 to (35) gives $\lim_{t \to +\infty} y_2(t) = 0$, a.s. The proof of $\lim_{t \to +\infty} (y_1(t)) = (b_1 - 0.5\sigma_1^2 - \eta_1)/a_{11}$, a.s., is similar to (ii) and hence is omitted.

The proof of (B) is similar to (A) by symmetry and hence is left out.

(C) Suppose that $\Psi_1 > \Psi_1$ and $\Psi_2 > \Psi_2$. Since $\Psi_2 > \Psi_2$, it then follows from (33) and Lemma 2 that
\begin{equation}
    \limsup_{t \to +\infty} (y_2(t)) \leq \Psi_2 - \Psi_2, \quad \text{a.s.} \quad (38)
\end{equation}

Making use of the arbitrariness of $\varepsilon$, we can see that
\begin{equation}
    \limsup_{t \to +\infty} (y_2(t)) \leq \Psi_2 - \Psi_2, \quad \text{a.s.} \quad (39)
\end{equation}

It follows from (37), Lemma 2, and the arbitrariness of $\varepsilon$ that
\begin{equation}
    \limsup_{t \to +\infty} (y_1(t)) \leq \Psi_1 - \Psi_1, \quad \text{a.s.} \quad (40)
\end{equation}

likewise. Let $\varepsilon$ be sufficiently small such that $a_{11}(\Psi_2 - \Psi_2)/\Psi) - \varepsilon > 0$. When (32) and (39) are used in (19), we get
\begin{align*}
    t^{-1} \ln y_1(t) = t^{-1} \ln y_1(0) + b_1 - 0.5\sigma_1^2 - \eta_1 \\
    &- a_{11} \left(\frac{y_1(t)}{\Psi}ight) - a_{12} \left(\frac{y_2(t)}{\Psi}ight) + \frac{\sigma_1 B_1(t)}{t} \\
    &+ t^{-1} \int_0^t \int_Y \ln (1 + y_1(u)) \, \tilde{N}(ds, du) \\
    &- a_{12} \int_{-\tau_2}^0 d\mu_2(\theta) \int_0^\theta y_2(s) \, ds \\
    &+ \int_{-\tau_2}^0 d\mu_2(\theta) \int_0^{t+\theta} \int_Y \ln (1 + y_2(u)) \, \tilde{N}(ds, du) \geq b_1 - 0.5\sigma_1^2 - \eta_1 - \varepsilon \\
    &- a_{11} \left(\frac{y_1(t)}{\Psi}ight) - a_{12} \limsup_{t \to +\infty} \left(\frac{y_2(t)}{\Psi}\right) + \frac{\sigma_1 B_1(t)}{t}
\end{align*}
for sufficiently large $t$. In virtue of (ii) in Lemma 2 and the arbitrariness of $\varepsilon$, we get
\begin{equation}
    \liminf_{t \to +\infty} (y_1(t)) \geq \Psi_1 - \Psi_1, \quad \text{a.s.} \quad (42)
\end{equation}

Similarly, substituting (32) and (40) into (20) brings about $\liminf_{t \to +\infty} (y_2(t)) \geq (\Psi_2 - \Psi_2)/\Psi$, a.s. This, together with (39), (40), and (42), means $\lim_{t \to +\infty} (y_1(t)) = (\Psi_1 - \Psi_1)/\Psi$ and $\lim_{t \to +\infty} (y_2(t)) = (\Psi_2 - \Psi_2)/\Psi$.

Remark 5. It is important to designate that if $b_1 > 0.5\sigma_1^2 + \eta_1$, $b_2 > 0.5\sigma_2^2 + \eta_2$, and $\Psi > 0$, then $\Psi_1 < \Psi_1$ and $\Psi_2 < \Psi_2$ cannot hold simultaneously.

Remark 6. Theorem 4 implies an important fact that when $-1 < y_i(u) < 0$, $i = 1, 2$, the jump process can result in extinction of the population $y_i(t)$, for example, earthquakes and hurricanes, and when $y_i(u) > 0$, $i = 1, 2$, the jump process is always advantageous for the population $y_i(t)$, for example, ocean red tide.

Remark 7. From the perspective of the condition in Theorem 4, the distributed delay does not influence some of the properties including extinction and stability in time average.

3. Numerical Simulations

In this section, we employ the split-step $\theta$-method, whose approximate solution is mean-square convergent with order $p = 0.5$ (see [32, 33]), to discretize (3). Here, we choose the initial data $\xi(t) = (0.3e^t, 0.4e^t)$, $b_1 = 0.59$, $b_2 = 0.51$, $a_{11} = 0.8$, $a_{12} = 0.39$, $a_{21} = 0.51$, $a_{22} = 0.69$, $r_1 = r_2 = 0.3$, $\Psi = (0, +\infty)$, and $\lambda(\Psi) = 1$. Then $\Psi = 0.36$. The main difference between the conditions of the following Case 1–Case 4 is that the values of $y_1(u)$ and $y_2(u)$ are different. In Case 1, we let $y_1(u) = y_2(u) = 0$ and $\sigma_1^2 = \sigma_2^2 = 0$. Then by virtue of Kuang’s work [6], we have that the positive equilibrium $x^*_0 = (\Psi_1/\Psi, \Psi_2/\Psi) = (0.612, 0.277)$ is globally asymptotically stable. Figure 1(a) verifies this. In Case 2, we set $y_1(u) = -0.1, y_2(u) = -0.2$, and $\sigma_1^2 = 0.2, \sigma_2^2 = 0.4$. Then
$\Psi > \overline{\Psi}_1 = -0.03216$ and $\Psi_2 = 0.1 < \overline{\Psi}_2 = 0.2386$. In view of (A) in Theorem 4, $y_2$ goes to extinction and

$$\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) \, ds = \frac{b_2 - 0.5\sigma_2^2 - \eta_2}{a_{22}} = 0.41 \frac{0.7}{0.41} = 0.59.$$  

Figure 1(b) confirms this. In Case 3, we choose $\gamma_1(u) = -0.8$, $\gamma_2(u) = 0.01$, $\sigma_1^2 = 0.2$, and $\sigma_2^2 = 0.4$. Then $\Psi_1 = 0.22 < \overline{\Psi}_1 = 1.11$ and $\Psi_2 = 0.1 > \overline{\Psi}_2 = 0.03$. It follows from (B) in Theorem 4 that $y_1$ goes to extinction and

$$\lim_{t \to \infty} t^{-1} \int_0^t y_1(s) \, ds = \frac{b_1 - 0.5\sigma_1^2 - \eta_1}{a_{11}} = 0.595 \frac{0.8}{0.595} = 0.74.$$  

Figure 1(c) confirms this. In Case 4, we let $\gamma_1(u) = -0.03$, $\gamma_2(u) = 0.01$, $\sigma_1^2 = 0.2$, and $\sigma_2^2 = 0.4$, that is, $\Psi_1 = 0.22 > \overline{\Psi}_1 = -0.22$ and

$$\lim_{t \to \infty} t^{-1} \int_0^t y_2(s) \, ds = \frac{b_2 - 0.5\sigma_2^2 - \eta_2}{a_{22}} = 0.41 \frac{0.7}{0.41} = 0.59.$$  

Figure 1(d) confirms this.
\[ \Psi_2 = 0.1 > \bar{\Psi}_2 = 0.095. \] According to (C) in Theorem 4, we obtain
\[ \lim_{t \to +\infty} \frac{1}{t} \int_0^t y_1(s) ds = \frac{\Psi_1 - \bar{\Psi}_1}{\Psi} = 1.22, \]
\[ \lim_{t \to +\infty} \frac{1}{t} \int_0^t y_2(s) ds = \frac{\Psi_2 - \bar{\Psi}_2}{\Psi} = 0.0139. \] (45)

Figure 1(d) validates this.

**4. Conclusions and Remarks**

This paper investigates a stochastic competitive system with distributed delay and general Lévy jumps. Under the assumption \( \Psi > 0 \), the almost complete parameter analysis is fulfilled in detail. Our results imply that the general Lévy jumps can significantly change the properties of population models.

Some interesting and significant topics deserve our further engagement. One may put forward a more realistic and sophisticated model to integrate the colored noise into the model [10, 11, 34]. Another significant problem is devoted to stochastic model with infinite delays and general Lévy jumps. We will leave these for future investigation.

It should also be mentioned that "stability in time average" is not a good definition of persistence for stochastic population models. Some papers have introduced more appropriate definitions of permanence for stochastic population models, that is, stochastically persistent in probability or stochastic permanence (see, e.g., [35–37]). We will research these kinds of permanence of model (3) in detail in our following study.

**Competing Interests**

The authors declare that they have no competing interests.

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**References**


