Research Article

Periodic Event-Triggered Condition Design for the Consensus of Multiagent Systems with Communication Delays

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This paper focuses on the periodic event-triggered consensus problem for multiagent systems under undirected graphs. In the periodic event-triggered control, the event condition is tested only periodically to decide when to exert the control behaviors and when to broadcast the sampling data. Two sets of event-triggered strategies combining with sampled-data control are designed based on an exponential decay function and a quadratic Lyapunov function, respectively. Some useful theorems for these two periodic event-triggered strategies about the choice of period are derived to guarantee the asymptotic stability of the closed-loop multiagent systems. In addition, this paper gives a sufficient condition about the bounds of communication delay for the multiagent systems. It is rigorously proved that the overall system will achieve the consensus under the proposed strategy if the period and time delay satisfy the theorems. Finally, we extend these solutions to general linear dynamical systems. Simulation results are presented to show the effectiveness.

1. Introduction

Numerous contributions have been made towards cooperative control for multiagent systems, such as [1–3]. Most of these papers are considered in an ideal environment and have sufficient resources including communication, energy, and calculation. Recently, a hot issue proposed in multiagent systems is to design and implement distributed algorithms for control and communication of agents with limited resources [4–6].

Generally, each agent is equipped with a small embedded microprocessor to deal with information, which is a limitation of communication and energy resource. Event-triggered control is proposed to seek a trade-off between resource consumption and the desired level of performance. It attracted a great number of researchers [7, 8]. In event-triggered control, the decision for the execution of the control task is not fixed on a prespecified time instant but relies on a specific event. Some studies [9,10] compared traditional time-triggered control with event-triggered control and show that event-triggered control has a better performance and actuates more efficiency in many applications.

Motivated by this fact, event-triggered control has been widely used in the cooperative control of multiagent systems [11–15]. In these results, the event-triggered control strategies have to verify the event condition continuously to decide when the controller must be actuated. This not only implies that a special hardware device is needed but also may lead to a waste of communication and computation resources. These strategies could be called continuous event-triggered control. Recently, a new strategy called periodic event-triggered control is first presented by [16, 17] to achieve a balance between sampled-data control and event-triggered control. By combining both event-triggered control and periodic sampled-data control, the benefits of reduced resource consumption are achieved as the event condition is verified only periodically to decide whether the new sampling stated should be sent to neighbors, and all of the control behaviors are only executed at sampling time. Compared with continuous event-triggered control and sampled-data control, the periodic event-triggered control maintained the advantages of reduced resource utilisation as transmissions and controller updates are triggered by events, while the
event-triggering conditions are evaluated periodically instead of continuously. Then, the minimum interevent time of event-triggered control is bounded by the sampling period. Furthermore, the continuous measurement of state is hardly achieved; it is more realistic to approximate the continuous supervision by high fast rate sampling due to sensors’ intermittent working mechanism. Thus, periodic event-triggered control is better suited for practical implementation as it can be implemented in more standard time-sliced embedded software architectures.

On the basis of this method, [18] proposed a centralized periodic event-triggered strategy and control codesign for sampled-data control systems to determine whether or not to transmit the sampled data. In [19], a sampled-data event detection strategy is proposed for linear continuous-time systems; a discrete Lyapunov function is used directly as the threshold to characterize the stability property. The event condition is centralized and all agents are triggered synchronously in these two methods. Thus, [20] proposed a distributed event-triggering scheme which is designed based on a quadratic Lyapunov function. A sampled-data event sensor is used to obtain the states to their average.

However, it should be noted that all the above strategies do not consider the influence of network time delay, which is inevitable when communication resource is an important factor in cooperative control of multiagent systems. Each subsystem still must collect its neighbors’ states to calculate and judge whether the event is triggered or not at every period, which still represents big consumption of communication resources. In addition, a major concern in periodic event-triggered methods is how large the sampling period could be chosen to guarantee the stability and achieve good performance. And, in all the previous work, the plant is a single dynamic networked system [16–19] or multiagent system instead of single integrators [20].

Motivated by the above discussion, this paper proposes two sets of event-triggered strategies that take the benefits of sampled data for general linear multiagent systems in undirected graph. In previous work, most of the event conditions are designed as a time-dependent function or state-dependent function. In this paper, these two methods are both discussed in periodic event-triggered control. Particularly, a sufficient condition about the upper bound of the sampling period is derived in both cases to guarantee the asymptotic convergence, which is an important issue in periodic event-triggered control as well as sampled-data control. Finally, taking into account the case of time-varying delay, it is proved that the strategy is still available and obtains new bound conditions on the period and delay to guarantee the stability and performance. The proposed integrating event conditions in these methods do not continuously depend on the state of the system but on the error between the current and the latest broadcasted state, which results in that the number of generated events decreases when the system is close to the equilibrium state. The contributions of the proposed design are twofold: first, this paper gives the maximal allowable bounds of sampling period in periodic event-triggered design. Second, sufficient conditions are obtained in two sets of periodic event-triggered designs for the multiagent systems with time delay.

The preliminary work of the results in this paper was reported in [21]. The rest of this paper is organized as follows. Section 2 presents some mathematical preliminaries and the problem formulation. Section 3 has two parts: a periodic event-triggered control with time-dependent threshold is presented without delay and the next part discusses the case of time-varying delay. Section 4 has a similar structure to Section 3; a periodic event-triggered control with state-dependent threshold is proposed with/without delay. In Section 5, some simulation results are provided. Section 6 concludes the paper.

2. Preliminaries

For a graph $G$ with $N$ nodes $\mathcal{V} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E} = \{(i, j) \in \mathcal{V} \times \mathcal{V} | i, j$ adjacent$\}$, the Laplacian matrix $L$ is defined as $L = D - A$, where $D$ is the degree matrix and $A$ is the adjacency matrix of $G$. The set of neighbors of the node $i$ is denoted by $N_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. For connected graphs, $L$ has exactly one zero eigenvalue $\lambda_1(G)$ and the smallest nonzero eigenvalue $\lambda_2(G)$ is called algebraic connectivity [22]. The topology in this paper is assumed to be fixed and undirected.

The system model considered in this paper is of single-integrator dynamic

$$\dot{x}_i(t) = u_i(t),$$

where $x_i$ denotes the state of agent $i$ and $u_i$ denotes the control input for each agent. The consensus control law is given by

$$u_i = - \sum_{j \in N_i} a_{ij} (x_i - x_j).$$

In event-triggered consensus control, the operation of the controller is decided by a specified event condition, so the control law is exerted at discrete time instants where the event occurs, $t_{i0}, t_{i1}, \ldots$ is the sequence of event trigger times of agent $i$.

For the sequence of events $t_{i0}, t_{i1}, \ldots$, the state value of agent $i$ is a piecewise constant function $\tilde{x}_i(t) = x_i(t_k^i)$, $t \in [t_k^i, t_{k+1}^i)$, and $\tilde{x}_i(t)$ is the latest trigger value of the state and is sent to neighbors at every trigger time. The control law will be updated immediately with the state; thus, the control law could be redefined as

$$u(t) = - L \tilde{x}(t).$$

Event-triggered technique requires the continuous monitoring of trigger condition. Such continuous state detection and continuous trigger condition checking will consume much energy, which does not satisfy the original idea of using event-triggered control. In this part, the sampled-data control is integrated with event-triggered strategy in order to reduce the energy and communication consumption. The event detector is checked only at each sample time.

Each agent consists of the typical structure of an event-triggered control scheme with sampled-data sensor as shown
in Figure 1, where an event generator is located between period sampling sensor and network. At each sampling instant, agents send their sampled data to the event generator. The event generator detects the error between current sampled data and the latest transmission sampled data. Thus, the event generator only needs to work at each sampling instant. Whether or not the current sampled data should be sent to neighbors is determined by a predefined threshold. If an agent’s threshold is violated or its neighbor is triggered, the control signal is updated by the newest sampled data. Otherwise, the control signal is held by a zero-order hold (ZOH) operator. The overall goal is to propose a periodic event-triggered control mechanism to reduce communication and energy consumption while preserving asymptotic property of consensus. Therefore, the event generator is used to determine when the sampled local information should be sent. It is defined that $t^t_{ij}$ is the $k$th event instant for agent $i$ and $h$ is the sampling period for all systems.

In existing work of event-triggered control design, most of the event conditions are designed based on a time-dependent function or state-dependent function. Therefore, we consider these two main event condition designs in our proposed periodic event-triggered framework, so that a general and useful theorem for distributed periodic event-triggered application could be given.

### 3. Periodic Event-Triggered Strategy with Time-Dependent Threshold

In this section, the periodic event-triggered control with time-dependent threshold is proposed; some analysis and conclusions are obtained at the following part. First, the event condition for agent $i$ is

$$\|e_i(t_k^t + lh)\| = \|x_i(t_k^t) - x_i(t_k^t + lh)\| \leq ce^{-\alpha(t_k^t + lh)}; \quad (4)$$

$t_k^t = 0$ is the initial time, and all the measurements $x_i(t_k^t)$ are the sequence of the sample state $x_i(mh)$; that is to say, the event is only triggered at sampling time, $\{t_0^t, t_1^t, \ldots \} \subseteq \{0, h, 2h, \ldots \}$. This implies that the interevent time $|t_{k+1}^t - t_k^t|$ is lower bounded by sampling period $h$. Thus, the Zeno behavior does not exist.

The advantages of the event condition in (4) are obvious. First, different from centralized event detectors in [18, 19], every agent has to be aware of the global information and has to be updated synchronously. The event conditions in (4) are distributed; each agent only needs its own state to decide the trigger time; controllers can be triggered asynchronously. Second, different from the distributed event detector in [11], the event detector in (4) does not need to know the rendezvous location in advance and access to its global position. And also apart from [20] which still requires the neighbor state to verify the condition, the sampled event generator only needs to work at each sampling time and no neighbor real information is required. Finally, the sampled event detector guarantees that the next trigger time is at least one period later.

Taking the distributed event-triggered strategy into account, the closed-loop system for agent $i$ can be gotten as

$$\dot{x}_i(t) = - \sum_{j \in \mathcal{N}} (x_i(t_k^j) - x_j(t_k^j)), \quad (5)$$

where $t_k^j = \max\{t \mid t \in [t_k^j + lh, k = 0, 1, \ldots], t \leq t_0^j + lh\}$.

Notice that, for $t \in [t_k^j + lh, t_k^j + (l + 1)lh)$,

$$x_i(t_k^j) = x_i(t_k^j + lh) + e_i(t_k^j + lh),$$

$$x_j(t_k^j) = x_j(t_k^j + lh) + e_j(t_k^j + lh).$$

Combining the definition of $t_k^j$, (4) can be written in stack vector form as follows, for $t \in mh, (m + 1)h$:

$$\dot{x}(t) = -L \cdot (x(mh) + e(mh)). \quad (6)$$

In the case when the time delay is zero, we will have the following theorem.

**Theorem 1.** Consider a multiagent system (1) with the protocol in (2) and triggered by event condition (4) under undirected graph. Then, all agents converge together asymptotically if

$$|L - hI| < 1 \implies 0 < h < \frac{2}{\lambda_n}. \quad (7)$$

**Proof.** Following [23], since $G$ is strongly connected, the state vector $x$ can be decomposed as $x(t) = 1 \cdot a + \delta(t)$, where $a$ is an invariant quantity and $\delta$ is called the disagreement vector and $1^T \delta = 0, 1$ is the vector of ones. Then, it can be gotten that

$$\dot{\delta}(t) = \dot{x}(t) = -L(\delta(mh) + e(mh))$$

$$= -L(1a + \delta(mh) + e(mh))$$

$$= -L(\delta(mh) + e(mh)). \quad (8)$$

Then, solving this equation, it can be gotten that

$$\int_{0}^{t} \dot{\delta}(s) \, ds = \int_{0}^{h} -L(\delta(0) + e(0)) \, ds$$

$$+ \int_{h}^{2h} -L(\delta(h) + e(h)) \, ds + \cdots$$

$$+ \int_{mh}^{t} -L(\delta(mh) + e(mh)) \, ds,$$

where $mh < t < (m + 1)h$. 

**Figure 1:** Conceptual framework of the proposed event-triggered control with period sampling.
The solution of (10) is
\[
\delta (t) = \delta (mh) + \int_{mh}^{t} -L (\delta (mh) + e (mh)) \, ds
= (I - (t - mh) L) \delta (mh) - (t - mh) Le (mh),
\]
(11)
\[
\delta (mh) = (I - hL) \delta ((m - 1) h) - hLe ((m - 1) h).
\]
To solve the sequence, the equation of \(\delta(mh)\) only depends on its initial value and measurement errors of every sampling time. And all of these are known constants. Thus, \(\delta(mh)\) could be obtained as
\[
\delta (mh) = (1 - hL)^m \delta (0) - hL e ((m - 1) h) - \cdots
- (I - hL)^{m-1} hL e (0),
\]
(12)
where \(0 < l < m\). The bound of \(\delta(mh)\) could be calculated as follows:
\[
\| \delta (mh) \| \leq \left\| (I - hL)^m \delta (0) \right\| + \| hLe ((m - 1) h) \|
+ \left\| (1 - hL) hLe (h) \right\| + \cdots
+ \left\| (1 - hL)^{m-1} hL \right\| e (0) \|
\leq \left\| (I - hL)^m \right\| \| \delta (0) \| + \| hL \| e ((m - 1) h) \|
+ \cdots + \left\| (1 - hL)^{m-1} hL \right\| e (h) \|
+ \left\| (I - hL)^{m-1} hL \right\| e (0) \|.
\]
(13)
The purpose is to prove the asymptotic convergence of all agents; that is to say, \(\lim_{t \to \infty} \| \delta(t) \| = 0\) should be guaranteed. So
\[
\| \delta (t) \| \to 0 \iff \| \delta (mh) \| \to 0
\]
(14)
As \(t \to \infty\), we have \(m \to \infty\), \(\| e(mh) \| \leq \sqrt{Ne^{-\gamma mh}} \to 0\), and \(\| \delta(0) \|, \| e(mh) \|\) are constants. The only solution that can guarantee condition (14) is
\[
| I - hL|^m \to 0 \iff | I - hL | < 1.
\]
(15)
Observing this analysis, we get that protocol (2) with event condition (4) globally asymptotically solves the consensus problem if (8) holds.

Considering the practical situation, the transmission delay which affects the stability and convergence stability needs to be analyzed in detail. The event-triggered consensus problem with time-varying delays is solved in the following part. It is shown that the proposed event-triggered consensus strategy with sampled data is still available and effective in this case, and the closed-loop system is still stable with some constraints of delay and sample period. It is assumed that the time delay is varying at [0, \(h\)]. For the nonzero delay situation, the following theorem is obtained.

**Theorem 2.** Consider a multiagent system (1) with the protocol in (2) and triggered by time-dependent event condition (4); the communication suffers a time delay; then the disagreement vector \(\delta\) converges to zero asymptotically, if (16) is satisfied:
\[
\tau < \frac{1}{\lambda_{\text{max}}},
\]
(16)
\[
h < 2\tau + \frac{2}{\lambda_{\text{max}}}.
\]
Proof. In this situation, the dynamics of system (7) become
\[
\dot{x} (t) = \begin{cases}
-L (x ((m - 1) h) + e ((m - 1) h)), & t \in [mh, mh + \tau_m), \\
-L (x (mh) + e (mh)), & t \in [mh + \tau_m, (m + 1) h);
\end{cases}
\]
(17)
\[
\tau_m \text{ varies from } 0 \text{ to } h \text{ at every period.}
\]
Then, applying (9), the dynamics of the disagreement vector are given as follows:
\[
\begin{bmatrix}
\delta ((m + 1) h) \\
\delta (mh)
\end{bmatrix} = A \begin{bmatrix}
\delta (mh) \\
\delta ((m - 1) h)
\end{bmatrix} + B \begin{bmatrix}
e (mh) \\
e ((m - 1) h)
\end{bmatrix},
\]
\]
(18)
where \(A = \begin{bmatrix}
1 - (h - \tau_m) L & -\tau_m L \\
\tau_m L & 0
\end{bmatrix}, B = \begin{bmatrix}
1 - (h - \tau_m) L & -\tau_m L \\
\tau_m L & 0
\end{bmatrix}.
\]
And, solving this sequence, we can get
\[
\begin{bmatrix}
\delta (mh + h) \\
\delta (mh)
\end{bmatrix} = \prod_{i=1}^{m} A_i \begin{bmatrix}
\delta (h) \\
\delta (0)
\end{bmatrix} + B \begin{bmatrix}
e (mh + h) \\
e (h)
\end{bmatrix} + \cdots + \sum_{i=1}^{m-1} A_i B \begin{bmatrix}
e (mh - h) \\
e (h)
\end{bmatrix} + \cdots + \sum_{i=1}^{m-1} A_i B \begin{bmatrix}
e (0) \\
e (0)
\end{bmatrix}.
\]
(19)
As $t, m$ tend to infinity, $e(mh)$ tends to zero. Similar to (14), the condition of guaranteeing stability is $\|\prod_{i=m}^{\infty} A_i\| \rightarrow 0$. Notice that $|A_i| = \tau_i L > 0$; thus $\|\prod_{i=m}^{\infty} A_i\| \leq |A_{\text{max}}|^m$; then matrix $A_{\text{max}}$ is analyzed to determine the convergence and stability of (17). Applying the property of $L$, it can be given that $A$ is similar to

$$\text{triag}\left\{ \begin{array}{ccc}
1 & 0 & 0 \\
1 - (h - \tau_{\text{max}}) \lambda & -\tau_{\text{max}} \lambda & 0 \\
1 & 0 & \lambda
\end{array} \right\}, \quad (20)$$

Let diag{$A_1', A_2', \ldots, A_n'$} represent this matrix; it can be easily found that 0, 1 are two eigenvalues of $A_1'$ and $A$. To guarantee $\lim_{t \to \infty} \|\delta(t)\| = 0$ (i.e., dynamics of (17) are convergent and stable at 0), the only condition is that the norm of all the other eigenvalues of $A$ is less than 1. Notice that $A_2', \ldots, A_n'$ have the same form; thus, they can be analyzed together as

$$\left[ 1 - (h - \tau_{\text{max}}) \lambda, \quad -\tau_{\text{max}} \lambda \right]. \quad (21)$$

The characteristic polynomial of (21) is

$$a(z) = z^2 + ((h - \tau_{\text{max}}) \lambda - 1) z + \tau_{\text{max}} \lambda. \quad (22)$$

By using the bilinear transformation $w = (z + 1)/(z - 1)$, (22) becomes

$$a(w) = hw^2 + 2(1 - \tau_{\text{max}}) w + 2(2\tau_{\text{max}} - h) \lambda. \quad (23)$$

Then, we can use the famous theorem of Routh Criterion directly, which requires

$$\begin{align*}
(h\lambda > 0, \quad 2 \left(1 - \tau_{\text{max}} \lambda\right) > 0, \quad 2 + (2\tau_{\text{max}} - h) \lambda > 0) \implies \\
\tau_{\text{max}} < \frac{1}{\lambda_{\text{max}}} \\
h < 2\tau_{\text{max}} + \frac{2}{\lambda_{\text{max}}}.
\end{align*} \quad (24)$$

So we can conclude Theorem 2.

4. Periodic Event-Triggered Strategy with State-Dependent Threshold

In this section, a sampled-data event condition with state-dependent threshold is discussed. The event condition for agent $i$ has the following form:

$$\|e_i(t_k + nh)\|^2 = \|x_i(t_k) - x_i(t_k' - nh)\|^2 \leq \sigma_i \|x_i(t_k' + nh)\|^2. \quad (25)$$

Similar to condition (4), this state-dependent condition is also distributed and triggered asynchronously. The threshold is decreased with the convergence of system state. And no neighbor information is required to judge whether the event of each agent is triggered or not. The closed-loop system also can be obtained as (5). And a useful theorem is obtained for periodic event-triggered strategy with state-dependent threshold without time delays.

**Theorem 3.** Consider a multiagent system (1) with the protocol in (2) and triggered by a state-dependent event condition (25) under undirected graph; then, the system converges asymptotically, if equations

$$\begin{align*}
\frac{1}{2\lambda_n} &\leq h \leq \frac{3}{4\lambda_n}, \\
\sigma_{\text{max}} &\leq \frac{3/2 - 2h\lambda_n}{2h\lambda_n - 1/2}, \\
or 0 &\leq h \leq \frac{1}{2\lambda_n}, \\
\sigma_{\text{max}} &\leq 1
\end{align*} \quad (26)$$

are satisfied.

**Proof.** In order to prove the asymptotic convergence of all agents, a Lyapunov functional candidate is proposed as follows:

$$V(t) = \frac{1}{2} x^T(t) x(t). \quad (27)$$

Taking the derivative of $V(t)$ along trajectory (8) for any $t \in [mh, (m+1)h)$ yields

$$\dot{V}(t) = -x^T(t) L (x(mh) + e(mh))$$

$$= (t - mh) (x(mh) + e(mh))^T L_x (x(mh) + e(mh))$$

$$- x^T(mh) L x(mh) + e(mh))$$

$$\leq h\lambda_n (x(mh) + e(mh))^T L (x(mh) + e(mh))$$

$$- x^T(mh) L x(mh) - x^T(mh) Le(mh)$$

$$= - (1 - h\lambda_n) x^T(mh) L x(mh)$$

$$+ h\lambda_n e^T(mh) Le(mh)$$

$$+ (2h\lambda_n - 1) x^T(mh) Le(mh).$$

Then, using the inequality $x^T(mh)Le(mh) \leq (1/2)x^T(mh)Ly(mh) + (1/2)e^Te(mh)Le(mh)$, $\dot{V}(t)$ can be bound as

$$\dot{V}(t) \leq - (1 - h\lambda_n) x^T(mh) L x(mh) + h\lambda_n e^T(mh)$$

$$\cdot Le(mh) + \left| h\lambda_n - \frac{1}{2} \right|$$

$$\cdot \left( x^T(mh) L x(mh) + e^T(mh) Le(mh) \right). \quad (29)$$
If $h\lambda_n - 1/2 > 0$ and applying the event condition in (25), we get
\[
\dot{V}(t) \leq -\left(\frac{3}{2} - 2h\lambda_n\right)x^T(mh)Lx(mh) + \left(2h\lambda_n - \frac{1}{2}\right)e^T(mh)Le(mh)
\leq \left(2h\lambda_n - \frac{3}{2} + (2h\lambda_n - \frac{1}{2})\sigma_{\max}\right)x^T(mh)
\cdot Lx(mh).
\] (30)

In order to guarantee $\dot{V}(t) \leq 0$, we must have $2h\lambda_n - 3/2 < 0$, $2h\lambda_n - 3/2 + (2h\lambda_n - 1/2)\sigma_{\max} < 0$; then, a condition is obtained as $1/2\lambda_n \leq h \leq 3/4\lambda_n, \sigma_{\max} \leq (3/2 - 2h\lambda_n)/(2h\lambda_n - 1/2)$.

If $h\lambda_n - 1/2 < 0$, we get
\[
\dot{V}(t) \leq -\frac{1}{2}x^T(mh)Lx(mh) + \frac{1}{2}e^T(mh)Le(mh)
\leq -\frac{1}{2}(1 - \sigma_{\max})x^T(mh)Lx(mh);
\] (31)

thereby $\dot{V}(t) \leq 0$, if $0 \leq h \leq 1/2\lambda_n, \sigma_{\max} \leq 1$.

Combining the previous analyses, a sufficient condition is concluded as when $1/2\lambda_n \leq h \leq 3/4\lambda_n, \sigma_{\max} \leq (3/2 - 2h\lambda_n)/(2h\lambda_n - 1/2)$, when $0 \leq h \leq 1/2\lambda_n, \sigma_{\max} \leq 1$. So, from LaSalle’s invariance principle, $\dot{V}(t) \leq 0$ for $t \geq 0$ implies consensus for agents.

When the communication delay is nonzero, the periodic event-triggered control of multiagent systems becomes more complicated. The following part will give a detailed analysis of periodic event-triggered control with state-dependent threshold when time delay $\tau \neq 0$. A sufficient condition about the bounds of delay is derived to guarantee the consensus and stability.

**Theorem 4.** Consider a multiagent system consisting of first-order integrators with the protocol in (3) and triggered by a state-dependent event condition (25) under undirected graph; then, the system converges asymptotically, if equations
\[
(c + ((h - \tau)\lambda - 1)\sqrt{c} + r\lambda) > 0,
\]
\[
(2c - 2r\lambda) > 0,
\] (32)
\[
(c + \sqrt{c}) + \sqrt{c}(r - h)\lambda + r\lambda > 0
\]
are satisfied for $c = 1 - \sqrt{\sigma_{\max}}(h - \tau_1)\lambda_1$.

**Proof.** First, to rewrite (18) into a standard linear system form, we set $\Phi(m) = (\delta((m + 1)h), \delta(mh))^T$ and $E(m) = (e((m + 1)h), e(mh))^T$ to represent the system disagreement vector and measurement error, respectively; then, the system dynamics can be obtained as
\[
\Phi(m + 1) = A\Phi(m) + BE(m).
\] (33)

As proposed in Section 3, applying the property of $L$, we can always compute a set of vectors, such that $A = V\Delta V^{-1}, B = V\Xi V^{-1}$, where $\Delta$ is diagonal $[A_1', A_2', \ldots, A_n']$ and $\Xi$ is diagonal $[B_1', B_2', \ldots, B_n']$, $A_i' = \begin{bmatrix} 1 - (h - \tau_i)\lambda_i & r_i \\ r_i & 0 \end{bmatrix}$, $B_i' = \begin{bmatrix} -(h - \tau_i)\lambda_i & r_i \\ 0 & 0 \end{bmatrix}$. Let $\Delta = \langle \Delta_1, \ldots, \Delta_n \rangle^T, \Xi = \langle \Xi_1, \ldots, \Xi_n \rangle^T$; then, the system dynamics are rewritten as
\[
\Phi(m + 1) = V \cdot \langle \Delta_1V^{-1}\Phi(m) \\
+ \Xi_1V^{-1}E(m), \ldots, \Delta_nV^{-1}\Phi(m) + \Xi_nV^{-1}E(m) \rangle^T.
\] (34)

According to the definition of the norm of matrix and the event condition (25), we can get
\[
\|\Phi(m + 1)\| = \|V \cdot \langle \Delta_1V^{-1}\Phi(m) \\
+ \Xi_1V^{-1}E(m), \ldots, \Delta_nV^{-1}\Phi(m) + \Xi_nV^{-1}E(m) \rangle^T \|
\leq \|V\| \|\langle \Delta_11\rangle V^{-1}\Phi(m) \|
\leq \|V\| \|\langle \Delta_11\rangle V^{-1}\Phi(m) \| + \|\Xi\| \|\langle \Delta_11\rangle V^{-1}\Xi(m) \| \|\Phi(m)\|
= \|\Psi\| \|\Phi(m)\|,
\]
where $\Psi = \begin{bmatrix} \psi_1 & \cdots & \psi_n \end{bmatrix}$, $\psi_i = \begin{bmatrix} 1 - (h - \tau_i)\lambda_i & r_i \\ r_i & 0 \end{bmatrix} + \sqrt{\sigma_{\max}} \begin{bmatrix} -(h - \tau_i)\lambda_i & r_i \\ 0 & 0 \end{bmatrix}$. It can be easily concluded that the system is convergent if all the eigenvalues of $\Psi$ are less than 1. As $\sqrt{\sigma_{\max}} \begin{bmatrix} -(h - \tau_i)\lambda_i & r_i \\ 0 & 0 \end{bmatrix} \leq \sqrt{\sigma_{\max}}(h - \tau_1)\lambda_1$, we can obtain that all the eigenvalues of $A_i'$ must be less than $c = 1 - \sqrt{\sigma_{\max}}(h - \tau_1)\lambda_1$. The characteristic polynomial of $A_i'$ is the same as (22). Using a new transformation $w = (z + \sqrt{c})/(z - \sqrt{c})$, the characteristic polynomial becomes
\[
a(w) = (c - ((h - \tau)\lambda - 1)\sqrt{c} + r\lambda)w^2
\]
\[
+ (2c - 2r\lambda)w + (c + \sqrt{c}) + \sqrt{c}(r - h)\lambda + r\lambda > 0
\]
and then we can use the famous theorem of Routh Criterion directly, which requires $(c + ((h - \tau)\lambda - 1)\sqrt{c} + r\lambda) > 0, (2c - 2r\lambda) > 0, (c + \sqrt{c}) + \sqrt{c}(r - h)\lambda + r\lambda > 0$. Hence, Theorem 4 can be concluded.

**Remark 5.** The previous solutions could be easily extended to general linear dynamical systems. If the system model considered in this paper consists of $n$ agents with general linear dynamics,
\[
x_i(t) = A_i x_i(t) + B_i u_i(t), \quad i = 1, 2, \ldots, n
\]
where $x_i \in \mathbb{R}^m$ denotes the state of agent $i$ and $u_i \in \mathbb{R}^m$ denotes the control input for each agent. $A_i \in \mathbb{R}^{m \times m}$ and $B_i \in \mathbb{R}^{m \times m}$ are constant matrices. The consensus control laws are given by
\[
u_i = -K_i \sum_{j \in N_i} a_{ij} (x_i - x_j).
\] (38)
Then, the control law could be redefined as

\[ u(t) = -L \otimes K \hat{x}(t). \]  

(39)

For the assumption that \( \text{rank}(AB) = \text{rank}(A) \), there always exists \( K \in R^{m \times n} \), such that \( ABK = A \). This assumption is not a strong restriction on the system. Obviously, the multiagent systems with the first-order integrator or double integrator are special cases of (37). And this assumption is a weak condition since these two typical multiagent systems satisfy this assumption. Then, using the transformation function of \( y(t) = e^{-A t} x(t) \), \( \tilde{y}(t) = e^{-A t} \hat{x}(t) \), system (1) can be transferred as

\[ \dot{\tilde{y}}(t) = -L \otimes I_n \cdot \tilde{y}(t). \]  

(40)

According to [24], we can obtain that system (37) has the same solution of single-integrator system (40) if all the eigenvalues of \( A_i \) belong to the imaginary axis or all the states \( x_i(t) \) asymptotically converge to zero if \( A_i \) possesses eigenvalues belonging to the left-half complex plane.

5. Simulation Results

In this section, numerical simulations will be given to illustrate the effectiveness and stability of the proposed periodic event-triggered control. Suppose that all initial parameters of the simulation are the same. Consider a network consisting of ten linear agents. The dynamics of each agent are described by

\[ \dot{x}_i(t) = \begin{bmatrix} -4 & 1 \\ 4 & -2 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} u_i(t), \]  

(41)

\[ i = 1, 2, 3, \ldots, 10. \]

The Laplacian matrix is given by

\[ L = \begin{bmatrix} 4 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 7 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 6 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}. \]

(42)

Without loss of generality, we apply the proposed strategy into a network of ten agents. The initial conditions are chosen as \( x_i(0) = [3; 8] \), \( x_j(0) = [2; 3] \), \( x_k(0) = [1; 6] \), \( x_4(0) = [7; 5] \), \( x_5(0) = [4; 6] \), \( x_6(0) = [6; 7] \), \( x_7(0) = [6; 9] \), \( x_8(0) = [2; 7] \), \( x_9(0) = [2; 1] \), and \( x_{10}(0) = [6; 5] \). The sampling period \( h \) for all agents is chosen as \( h = 0.05 \). The simulations of consensus problem with the proposed algorithms are presented. In time-dependent algorithm, the threshold for measurement error is set to be \( e^{-0.3 h} \). In state-dependent algorithm, the threshold for measurement error is set to be \( 0.1 ||x_i(t)||^2 \).

Due to the large numbers of agents, we choose agent \( x_2 \) as a typical example in Figures 2 and 3 to show the norm of measurement error. It can be seen in Figures 2 and 3 that the event is not triggered even though the measurement error \( ||e(t)|| \) surpasses the threshold until the next sampling time. For instance, in Figure 2, though the measurement error \( ||e_2||^2 \) at about 0.98 s is beyond the threshold, the control action is not triggered until 1.00 s. This is because the trigger condition is only verified at sampling time with the periodic event-triggered methods. The detector does not work between periods even though the threshold is violated and the event is not triggered until the next sampling time 1.00 s, as \( h = 0.05 \). Similarly, in Figure 3, the event is triggered at sampling time 0.8 s, though the threshold has been violated at 0.78 s. Thus, it can concluded that an event is triggered only when the norm of error signal is beyond the threshold at any sampling time, and the error signal is reset to zero immediately. Therefore, the number of control updates and communication transmissions is reduced.

Figures 4 and 5 show the state evolution of each agent to reach consensus. The ten agents asymptotically reach consensus to zero as time goes on, which illustrates the effectiveness of the proposed design methods. It can be seen in Figure 4 that the convergence time is faster than that in Figure 3, and the oscillations in Figure 4 at the final states are larger than in Figure 5. The reason is also illustrated by Figures 2 and 3. In Figure 2, the threshold at 6 s is nearly 0.2 while in Figure 3 it is nearly zero. So, in Figure 4, the proposed method has a larger measurement error than in state-dependent periodic event-triggered method at 6 s, whose threshold is decreased with the convergence of system state. Another reason is that the evolution of thresholds is the same for each agent in time-dependent method, while the threshold in state-dependent method is different even
Figure 3: The evolution of measurement error for $x_2$ agent with state-dependent event threshold $0.1\|x_i(t)\|^2$ when $h = 0.05$ s.

Figure 4: The state evolution of each agent with time-dependent threshold $e^{-0.3t}$ in the proposed periodic event-triggered control when $h = 0.05$ s.

Figure 5: The state evolution of each agent with state-dependent threshold $0.1\|x_i(t)\|^2$ in the proposed periodic event-triggered control when $h = 0.05$ s.

6. Conclusion

In this paper, two sets of distributed event-triggered control designs integrated with sampled-data control have been proposed for the consensus of multiagent systems under undirected graph. An upper bound of sampling period is derived according to the stability analysis in two cases of event conditions design. Furthermore, the result is extended to the case with communication delay; a sufficient condition about sampling period and time delay is obtained to guarantee the asymptotical stability of the closed-loop multiagent system. It is rigorously proved that the overall system will achieve the consensus with the two proposed periodic event-triggered control designs.

Competing Interests

The authors declare that they have no competing interests.

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