Research Article

Coupling Systems for a New Type of Phase Synchronization

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Using the usual phase in plane, we propose a general method to design coupling between systems that will exhibit phase synchronization. Numerical results are shown for Lorenz systems. Phase synchronization and antiphase synchronization are equally probable depending on initial conditions. A new network with Lorenz phase synchronized system is obtained.

1. Introduction

Nonlinear dynamics brought to light the rich dynamics of nonlinear systems including chaotic behavior. By coupling two nonlinear systems a new nonlinear system with an increased dimension and richer dynamics is obtained. The couplings can be suggested by reality (like diffusive coupling) or can result from the design in order to achieve a certain task. Depending on the coupling at least two distinctive cases can occur after the coupling is on: synchronization [1, 2] and Amplitude Death (AD) [3, 4]. Synchronization is a fundamental phenomenon observed in nature and science and used in engineering and medicine [1, 2]. There are several types of synchronization (complete synchronization, lag synchronization, projective synchronization, and generalized synchronization including amplification/attenuation). We mention here just some results that show how to design the couplings for synchronization [5–13]. A weaker type of synchronization is phase synchronization (PS) as locking of the phase of the coupled systems, while their amplitudes remain uncorrelated. The great interest for this kind of synchronization has been triggered by the seminal paper [14]. For chaotic systems it is difficult to introduce a phase. In [14], the phase was introduced by using the analytic signal concept. Many results on PS are observed for chaotic attractors with rather coherent phase dynamics [1, 2]. Phase coherent dynamics is a strong limitation for applications. A step forward is the study of the effect of noncoherence on the onset PS [15] by using a phase based on the general idea of curvature of an arbitrary curve.

Also PS was obtained by using active control and by nonlinear state observer algorithm [16, 17] when the difference of two homologous variables is constant. However, a concern has arisen recently related to the use of the analytic signal concept [18].

In this paper we adopt a new strategy. By using the usual definition of phase in plane we deduce a system that governs the dynamics of the phase differences. From this dynamics we search for coupling terms that assure that the phase differences become asymptotically zero. The method is general but it does not assure that we can find coupling terms for any systems. In addition, it is not assured that the dynamics of the coupled system is bounded. For Lorenz system, the method gives good results. Coupling for two Lorenz systems suggests an all-to-all coupling for a network of PS and antiphase synchronized (APS) systems. Such a network can have different outputs depending on initial conditions.

The organization of this paper is as follows. In Section 2, the general theory is presented, where the system that governs the dynamics of the phase differences is deduced. Section 3 gives the results of the theory applied to Lorenz systems with numerical results. Section 4 contains several comments. Finally, conclusions are presented in Section 5.

2. Theory

Without losing generality, we consider 3 dimensional systems $ds/dt = F(s)$ with $s = (x, y, z)^T$ like Lorenz, Chen, Rössler,

Sprott, and so forth. $F(s) = Ls + N(s)$, where L is a 3×3 constant matrix and $N(s)$ is a 3×1 vector which contains nonlinear terms. Two systems that can be coupled by terms u_1 and u_2 have equations:

$$\frac{ds_1}{dt} = Ls_1 + N(s_1) + u_1(s_1, s_2), \quad (1a)$$

$$\frac{ds_2}{dt} = Ls_2 + N(s_2) + u_2(s_1, s_2). \quad (1b)$$

We need to determine u_1 and u_2 in such a manner that the two systems will reach PS in at least two pairs of variables (let us say $x_i, y_i, i = 1, 2$). As phase we adopt the usual phase in plane Ox_iy_i : $\theta_i = \tan^{-1}(y_i/x_i), i = 1, 2$. The cross product $P = s_1 \times s_2$ has the following components:

$$(P_1, P_2, P_3) = (z_2y_1 - z_1y_2, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1). \quad (2)$$

The component P_3 depends on phase difference:

$$P_3 = (x_1^2 + y_1^2)^{1/2} (x_2^2 + y_2^2)^{1/2} \sin(\theta_2 - \theta_1). \quad (3)$$

$P_3 = 0$ means $x_1^2 + y_1^2 = 0$ (Amplitude Death) or $x_2^2 + y_2^2 = 0$ (Amplitude Death) or $\sin(\theta_2 - \theta_1) = 0 \Rightarrow \theta_2 = \theta_1$; that is, PS or $\theta_2 = \theta_1 + \pi$, that is, APS.

The dynamics of P_i can be obtained by using the original dynamics (1a) and (1b) and the components P_i given by (2):

$$\frac{dP_1}{dt} = (L_{22} + L_{33})P_1 - L_{21}P_2 - L_{31}P_3 + M_1, \quad (4a)$$

$$\frac{dP_2}{dt} = -L_{12}P_1 + (L_{11} + L_{33})P_2 - L_{32}P_3 + M_2, \quad (4b)$$

$$\frac{dP_3}{dt} = -L_{13}P_1 - L_{23}P_2 + (L_{11} + L_{22})P_3 + M_3, \quad (4c)$$

or

$$\frac{dP^T}{dt} = L^*P^T + M, \quad (5)$$

where $M = (M_1, M_2, M_3)^T$ with M containing all nonlinear terms, $N(s)$, and coupling terms of u_1 and u_2 .

Now we can give a partial answer to the question [15] "It is still unknown which type of chaotic oscillators can be synchronized in phase"? The general and partial answer is that system (4a), system (4b), and system (4c) or system (5) contains information on the PS (with phase $\theta_i = \tan^{-1}(y_i/x_i), i = 1, 2$). This is the main result of this research and is general because (4a), (4b), and (4c) are general. The method of finding u_1 and u_2 depends on the system and can be more or less complicated (see Section 4).

If we find coupling terms u_1 and u_2 in such a manner that $M_i = 0$ and $dP_i/dt = -kP_i$ ($k > 0$), then we have PS or AD as above. If one line in L^* (in (5)) contains just zeros, then we can add and subtract $-kP_i$ and look for coupling terms to have $kP_i + M_i = 0$.

If by choosing u_1 and u_2 we can have $M_i = M_j = 0$ and the system is $dP_i/dt = a_1P_i + a_2P_j$ and $dP_j/dt = a_3P_i + a_4P_j$ and if

$a_i, i = 1-4$, assures that $P_i \rightarrow 0$ and $P_j \rightarrow 0$, then all variables are PS (or APS).

If we find u_1 and u_2 , where all $M_i = 0, i = 1, 2, 3$, and L^* (in (5)) is a Hurwitz matrix (a matrix with negative real part eigenvalues), then $P_i \rightarrow 0, i = 1, 2, 3$, and all variables will be PS and amplitudes are uncorrelated.

At the eye inspection of the temporal pattern of two systems PS or APS (let us say x_1 and x_2 and y_1 and y_2) we will observe (see Figures 1 and 2; numerical results) that the homologous variables cross the time axis at the same time as expected from (2). This conclusion can have practical importance at the investigation of real data.

To check numerically PS we can use coefficients (c_1, c_2 , and c_3) with

$$c_3 = \frac{x_1x_2 + y_1y_2}{(x_1^2 + y_1^2)^{1/2} (x_2^2 + y_2^2)^{1/2}} \quad (6)$$

and similar expressions for c_1 and c_2 . For c_3 , we have

$$c_3 = \cos(\theta_2 - \theta_1). \quad (7)$$

For PS, $c_3 = 1$ and $c_3 = -1$ means APS. Again, this result can be of practical use. There can be identified two limitations of this general strategy. Firstly, the coupling terms do not assure that the dynamics of the coupled systems are bounded. Secondly, the coupling terms are not unique, but this can be an advantage when the first limitation occurs.

3. Phase Synchronized Lorenz Systems

We apply the above strategy to Lorenz systems ($dx/dt = -\sigma x + \sigma y$, $dy/dt = rx - y - xz$, and $dz/dt = -bz + xy$). In this case,

$$L = ((-\sigma, \sigma, 0), (r, -1, 0), (0, 0 - b))^T, \quad (8)$$

$$L^* = ((-b - 1, -r, 0), (\sigma, -\sigma - b, 0), (0, 0, -\sigma - 1))^T, \quad (9)$$

$$N(s) = (0, -xz, xy)^T. \quad (10)$$

Equation (4c) with (9) gives

$$\frac{dP_3}{dt} = -(\sigma + 1)P_3 + M_3 \quad (11)$$

with

$$M_3 = y_2u_{11} - y_1u_{21} + x_1u_{22} - x_2u_{12} + x_1x_2(z_1 - z_2). \quad (12)$$

We need to find coupling terms to have $M_3 = 0$.

FIGURE 1: Lorenz systems coupled in Case 1 (see text; $dx_1/dt = -\sigma x_1 + \sigma y_1$, $dy_1/dt = rx_1 - y_1 - x_1 z_1$, $dz_1/dt = -bz_1 + x_1 y_1 + x_2 y_2$, $dx_2/dt = -\sigma x_2 + \sigma y_2$, $dy_2/dt = rx_2 - y_2 - x_2 z_2$, and $dz_2/dt = -bz_2 + x_1 y_1 + x_2 y_2$). Initial conditions. (a) $(x_1, y_1, z_1, x_2, y_2, z_2)(0) = (1, 2, 3, 4, 5, 6)$, $c_3 = 1$, and $P_3 = 0$, PS. (b) $(x_1, y_1, z_1, x_2, y_2, z_2)(0) = (1, 2, 3, 4, -5, 6)$, $c_3 = -1$, and $P_3 = 0$, APS. In both cases, x_1 and x_2 (resp. y_1 and y_2) cross the time axis at the same moment. $z_1 = z_2$ asymptotically.

FIGURE 2: Lorenz systems coupled in Case 2 ($dx_1/dt = -\sigma x_1 + \sigma y_1$, $dy_1/dt = rx_1 - y_1 - x_1 z_1 - x_1 z_2$, $dz_1/dt = -bz_1 + x_1 y_1$, $dx_2/dt = -\sigma x_2 + \sigma y_2$, $dy_2/dt = rx_2 - y_2 - x_2 z_1 - x_2 z_2$, and $dz_2/dt = -bz_2 + x_2 y_2$). Initial conditions. (a) $(x_1, y_1, z_1, x_2, y_2, z_2)(0) = (1, 2, 3, 4, 5, 6)$, $c_3 = 1$, and $P_3 = 0$, PS. (b) $(x_1, y_1, z_1, x_2, y_2, z_2)(0) = (1, 2, 3, 4, -5, 6)$, $c_3 = -1$, and $P_3 = 0$, APS. In both cases $z_i, i = 1, 2$, are uncorrelated; similar variables cross the time axis at the same moment.

Three cases can be identified.

Case 1. One has

$$\begin{aligned} u_{11} &= u_{12} = u_{21} = u_{22} = 0, \\ u_{13} &= x_2 y_2, \\ u_{23} &= x_1 y_1. \end{aligned} \quad (13)$$

In this case, subtracting the 3rd equations of (1a) and (1b), we have $d(z_2 - z_1)/dt = -b(z_2 - z_1)$ so $z_2 \rightarrow z_1$ and, from (12), $M_3 = 0$ and, from (11), $P_3 \rightarrow 0$. The final state will be PS or APS depending on initial conditions like in Figure 1. Numerical results are given in Figure 1 with $(\sigma, r, b) = (10, 28, 8/3)$. Two examples are given: one PS and one APS from different initial conditions. $P_3 = 0$ means PS or APS; $c_3 = 1$ means PS and $c_3 = -1$ means APS.

Case 2. One has

$$\begin{aligned} u_{11} &= u_{21} = 0, \\ u_{22} &= -x_2 z_1, \\ u_{12} &= -x_1 z_2, \\ u_{13} &= u_{23} = 0. \end{aligned} \quad (14)$$

Numerical results are given in Figure 2.

Case 3. One has

$$\begin{aligned} u_{11} &= y_1, \\ u_{21} &= y_2, \\ u_{12} &= -x_1 z_2, \\ u_{22} &= -x_2 z_1, \\ u_{13} &= u_{23} = 0. \end{aligned} \quad (15)$$

Numerical results (not shown) are qualitatively similar to Case 2.

4. Discussions

In our numerical experiments, we did not find any AD and we do not have any indications how or if it can be found. In some cases one oscillation is much smaller than the other but not AD. This is worthy to be deeply investigated.

For Chen systems, the results are very similar to Lorenz system.

Unfortunately, we did not find reasonable coupling terms for Rössler system using a direct method like we used for Lorenz system. Other methods should be tried. Here we need to clarify this problem. We report here that we did not find coupling terms for Rössler by simple eye inspection as we obtained for Lorenz system. In fact, in general, we need to find coupling terms in (4a), (4b), and (4c) or (5) which assures that

components of P go to zero. As an ultimate method the OPCL (open-plus-closed-loop) method [19] can be used which can offer driving for any system to reach any desired dynamics. The coupling terms can be complicated but for engineering applications it is important that they can be possible. This will be elaborated (with examples) elsewhere.

For system A from Sprott collection [20] ($dx/dt = y$, $dy/dt = -x + yz$, and $dz/dt = 1 - y^2$), $L = ((0, 1, 0), (-1, 0, 0), (0, 0, 0))^T$, $N(s) = (0, yz, 1 - y^2)^T$, and $L^* = ((0, 1, 0), (-1, 0, 0), (0, 0, 0))^T$. Equation (4c) in this case is $dP_3/dt = M_3$; $M_3 = y_2 z_2 x_1 + u_{22} x_1 + y_2 u_{11} - u_{21} y_1 - x_2 y_1 z_1 - x_2 u_{12}$. So we subtract and add kP_3 and find the coupling terms from $kP_3 + M_3 = 0$ as

$$\begin{aligned} u_{11} &= -z_2 x_1, \\ u_{21} &= -x_2 z_1, \\ u_{12} &= -k y_1, \\ u_{22} &= -k y_2, \\ u_{13} &= u_{23} = 0. \end{aligned} \quad (16)$$

Numerical results (not shown) for initial condition

$$(x_1, y_1, z_1, x_2, y_2, z_2)(0) = (1, 2, 3, 2, -3, 4) \quad (17)$$

give bounded dynamics just for $k > 2.1$.

Initial conditions in connection with PS and APS were considered also in [21]. Antiphase synchronization (for diffusive coupling) was investigated in [22].

Case 1 for Lorenz systems (13) suggests a network of PS and APS systems given by

$$\begin{aligned} \frac{dx_i}{dt} &= -\sigma x_i + \sigma y_i, \\ \frac{dy_i}{dt} &= r x_i - y_i - x_i z_i, \\ \frac{dz_i}{dt} &= -b z_i + \sum_{j=1}^N x_j y_j. \end{aligned} \quad (18)$$

The coupling here is a nonlinear global coupling used recently [23]. Here N is the number of systems coupled all-to-all. In this case all systems of the network organize in 2 PS clusters and APS between them. Such a network can have a very interesting output. If the output is $X = \sum_{j=1}^N x_j$, $Y = \sum_{j=1}^N y_j$, and $Z = \sum_{j=1}^N z_j$, then the output will depend on the initial conditions. If all systems are PS, then the output will be maximum, and if the two PS clusters (that are APS one to another) are with nearly equal numbers of systems, then the output will be very small in comparison to the previous case. Figure 3 gives numerical results for $N = 3$ Lorenz systems starting from different initial conditions. For similar initial conditions the output is very different. As a common characteristic, all variables cross the time axis at the same moment.

This property of crossing time axis at the same moment can be searched by eye inspection in real signal (like EKG,

FIGURE 3: Three Lorenz systems coupled as in (18) for different initial conditions. (a) $(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)(0) = (1, 2, 3, 4, 5, 6, 7, 8, 9)$, 2 clusters (Systems 2 and 3 are PS (1st cluster) in antiphase with system 1 (2nd cluster)); (b) $(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)(0) = (1, 2, 3, 4, 5, 6, 8, 5, 4)$, all PS. (c) $(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)(0) = (1, 2, 3, 4, 5, 6, 3, 2, 1)$, all PS.

EEG, EMG, or others) and if it happens then a model with this type of coupling can be used.

Based on this general strategy other results will be presented elsewhere.

5. Conclusions

In this paper new strategy for phase synchronization is used a. Phase is not calculated effectively but a coupling is searched which assures that the two current phases are equal. Phase is the usual definition in plane: $(\theta_i = \tan^{-1}(y_i/x_i), i = 1, 2)$. The main result with possible practical importance is that the PS or APS signals pass zero value at the same moment at an eye inspection. The present phase synchronized Lorenz systems can be used as a tool for signal processing of real data as complete synchronized Lorenz systems have been used recently [24]. Deviation from $c_3 = 1$ measures the difference between two signals; results can be much better because c_3 is adimensional.

Competing Interests

The authors declare that they have no competing interests.

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