Research Article

A New Soliton Hierarchy Associated with $\mathfrak{so}(3, \mathbb{R})$ and Its Conservation Laws

Hanyu Wei,¹ Tiecheng Xia,² and Guoliang He³

¹College of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China

²Department of Mathematics, Shanghai University, Shanghai 200444, China

³Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China

Correspondence should be addressed to Hanyu Wei; weihanyu8207@163.com

Received 28 April 2016; Accepted 12 June 2016

Academic Editor: Maria L. Gandarias

Copyright © 2016 Hanyu Wei et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Based on the three-dimensional real special orthogonal Lie algebra $\mathfrak{so}(3, \mathbb{R})$, we construct a new hierarchy of soliton equations by zero curvature equations and show that each equation in the resulting hierarchy has a bi-Hamiltonian structure and thus integrable in the Liouville sense. Furthermore, we present the infinitely many conservation laws for the new soliton hierarchy.

1. Introduction

Conservation laws are widespread in applied mathematics, which reflect a phenomenon that some physical quantities do not change with time. In soliton theory, they play an important role in the study of integrability of soliton equations. An infinite number of conservation laws for KdV equation was first discovered by Miura in 1968 [1]; then many methods have been developed to find them. This may be mainly due to the contributions of Wadati et al. [2–4]. Conservation laws also play an important role in mathematics and engineering. Many papers dealing with symmetries and conservation laws have been presented. The direct construction method of multipliers for the conservation laws was presented in [5]; the Lagrangian approach for evolution equations was considered in [6, 7].

It is known that soliton hierarchies such as the Ablowitz-Kaup-Newell-Segur hierarchy and the Kaup-Newell hierarchy are generated from spectral problems associated with matrix Lie algebras [8]. The trace identity is used to find Hamiltonian structures of soliton hierarchies [9]. When associated matrix Lie algebras are non-semisimple, we can obtain integrable couplings [10, 11] and generate their Hamiltonian structures by the variational identity [12, 13].

We first recall a standard procedure for building soliton hierarchies. Starting from a spectral problem

$$\begin{aligned}\varphi_x &= U\varphi, \\ U &= U(u, \lambda) \in \tilde{\mathfrak{g}},\end{aligned}\tag{1}$$

where λ is the spectral parameter and u is a dependent variable, based on a matrix loop algebra $\tilde{\mathfrak{g}}$ associated with a given matrix Lie algebra $\mathfrak{g}$, often being simple. We take solution

$$W = W(u, \lambda) = \sum_{n \geq 0} W_{0n} \lambda^{-n}, \quad W_{0n} \in \mathfrak{g}, \quad n \geq 0\tag{2}$$

to the stationary zero curvature equation

$$W_x = [U, W].\tag{3}$$

We introduce the Lax matrices

$$V^{[n]} = V^{[n]}(u, \lambda) = (\lambda^n W)_+ + \Delta_n \in \tilde{\mathfrak{g}}, \quad n \geq 0,\tag{4}$$

where P_+ denotes the polynomial part of P in λ and Δ_n is the modification terms, to formulate the temporal spectral problems

$$\varphi_{t_n} = V^{[n]} \varphi, \quad n \geq 0.\tag{5}$$

The zero curvature equations

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad n \geq 0 \quad (6)$$

can engender a hierarchy of soliton equations with Hamiltonian structures

$$U_{t_n} = K_n(u) = J \frac{\delta H_n}{\delta u}, \quad n \geq 0. \quad (7)$$

Those Hamiltonian functionals H_n can often be generated by applying the trace identity [9, 14]

$$\frac{\delta}{\delta u} \int \text{tr} \left(V \frac{\partial U}{\partial \lambda} \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(V \frac{\partial U}{\partial u} \right), \quad (8)$$

where the constant γ is determined by

$$\gamma = \frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(VV)|. \quad (9)$$

We will make use of the three-dimensional real special orthogonal Lie algebra $\text{so}(3, \mathbb{R})$; this Lie algebra is simple and has a basis [15]

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ e_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\ e_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (10)$$

along with the communicative operation

$$\begin{aligned} [e_1, e_2] &= e_3, \\ [e_2, e_3] &= e_1, \\ [e_1, e_3] &= -e_2. \end{aligned} \quad (11)$$

The derived algebra $[\text{so}(3, \mathbb{R}), \text{so}(3, \mathbb{R})]$ is $\text{so}(3, \mathbb{R})$ itself, and $\text{so}(3, \mathbb{R})$ is one of the only two three-dimensional real Lie algebras with a three-dimensional derived algebra.

The matrix loop algebra we will adopt in what follows is

$$\begin{aligned} \widetilde{\text{so}}(3, \mathbb{R}) &= \{M \in \text{so}(3, \mathbb{R}) \mid \text{entries of } M \\ &\quad - \text{Laurent series in } \lambda\}. \end{aligned} \quad (12)$$

The algebra $\text{so}(3, \mathbb{R})$ contains matrices of the form $\lambda^m e_1 + \lambda^n e_2 + \lambda^l e_3$ with arbitrary integers m, n, l . This matrix loop algebra lays a foundation for our study of soliton equations.

In this paper, we would like to from a spectral problem, based on the matrix loop algebra $\text{so}(3, \mathbb{R})$, and construct a

new soliton hierarchy from associated zero curvature equations. The trace identity is used to furnish the corresponding Hamiltonian structures and so all equations in the resulting soliton hierarchy are Liouville integrable. Additionally, we introduce two variables E and F to construct conservation laws of the equation hierarchy and the first two conserved densities and fluxes are listed.

2. A New Spectral Problem and Soliton Hierarchy

To construct a soliton hierarchy from the matrix loop algebra $\widetilde{\text{so}}(3, \mathbb{R})$, we consider an isospectral problem

$$\begin{aligned} \varphi_x &= U\varphi = U(u, \lambda)\varphi, \\ u &= \begin{pmatrix} q \\ r \end{pmatrix}, \\ \varphi &= \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} U(u, \lambda) &= e_1 + qe_1 + re_2 + \lambda e_3 \\ &= \begin{pmatrix} 0 & -\lambda & -1 - q \\ \lambda & 0 & -r \\ 1 + q & r & 0 \end{pmatrix}. \end{aligned} \quad (14)$$

This isospectral problem is of the same type as the Broer-Kaup-Kupersmidt one [16], but its underlying loop algebra is different.

Starting from the stationary zero curvature equation

$$V_x = [U, V], \quad (15)$$

we have

$$\begin{aligned} a_x &= rc - \lambda b, \\ b_x &= \lambda a - c - qc, \\ c_x &= b + qb - ra. \end{aligned} \quad (16)$$

Setting

$$V = ae_1 + be_2 + ce_3 = \begin{pmatrix} 0 & -c & -a \\ c & 0 & -b \\ a & b & 0 \end{pmatrix} \in \widetilde{\text{so}}(3, \mathbb{R}), \quad (17)$$

further letting

$$\begin{aligned} a &= \sum_{m \geq 0} a_m \lambda^{-m}, \\ b &= \sum_{m \geq 0} b_m \lambda^{-m}, \\ c &= \sum_{m \geq 0} c_m \lambda^{-m}, \end{aligned} \quad (18)$$

and choosing the initial data

$$\begin{aligned} a_0 &= b_0 = 0, \\ c_0 &= 1, \end{aligned} \quad (19)$$

thus, substituting (18) into system (16), we obtain

$$\begin{aligned} a_{mx} &= rc_m - b_{m+1}, \\ b_{mx} &= a_{m+1} - c_m - qc_m, \quad m \geq 0, \\ c_{mx} &= b_m + qb_m - ra_m. \end{aligned} \quad (20)$$

We impose the following conditions on constants of integration

$$a_m|_{u=0} = b_m|_{u=0} = c_m|_{u=0} = 0, \quad m \geq 0. \quad (21)$$

From the recursion relations in (20), the first few results can be obtained as follows:

$$\begin{aligned} a_1 &= 1 + q, \\ b_1 &= r, \\ c_1 &= 0, \\ a_2 &= r_x, \\ b_2 &= -q_x, \\ c_2 &= -q - \frac{1}{2}q^2 - \frac{1}{2}r^2, \\ a_3 &= -q_{xx} - q - \frac{1}{2}r^2 - \frac{3}{2}q^2 - \frac{1}{2}q^3 - \frac{1}{2}qr^2, \\ b_3 &= -r_{xx} - qr - \frac{1}{2}q^2r - \frac{1}{2}r^3, \\ c_3 &= -r_x + rq_x - r_xq, \\ a_4 &= -r_{xxx} - 3r_xq - \frac{3}{2}q^2r_x - \frac{3}{2}r^2r_x - r_x, \\ b_4 &= q_{xxx} + q_x + 3qq_x + \frac{3}{2}q^2q_x + \frac{3}{2}r^2q_x, \\ c_4 &= q_{xx} + qq_{xx} + rr_{xx} - \frac{1}{2}q_x^2 - \frac{1}{2}r_x^2 + q + 2q^2 + \frac{3}{2}q^3 \\ &\quad + \frac{1}{2}r^2 + \frac{3}{2}qr^2 + \frac{3}{4}q^2r^2 + \frac{3}{8}q^4 + \frac{3}{8}r^4, \dots \end{aligned} \quad (22)$$

Then, we consider the auxiliary spectral problem

$$\varphi_{t_n} = V^{(n)} \varphi = (\lambda^n V)_+ \varphi, \quad n \geq 0, \quad (23)$$

where

$$V^{(n)} = \sum_{m=0}^n \begin{pmatrix} 0 & -c_m & -a_m \\ c_m & 0 & -b_m \\ a_m & b_m & 0 \end{pmatrix} \lambda^{n-m}. \quad (24)$$

Substituting (23) into the zero curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \quad (25)$$

we get a new soliton hierarchy

$$u_{t_n} = K_n = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = J \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}. \quad (26)$$

When $n = 2$, hierarchy (26) can be reduced to the second-order nonlinear integrable equations

$$\begin{aligned} q_{t_2} &= r_{xx} + qr + \frac{1}{2}q^2r + \frac{1}{2}r^3, \\ r_{t_2} &= -q_{xx} - q - \frac{1}{2}r^2 - \frac{3}{2}q^2 - \frac{1}{2}q^3 - \frac{1}{2}qr^2. \end{aligned} \quad (27)$$

As $n = 3$, hierarchy (26) is reduced to the third-order nonlinear integrable equations

$$\begin{aligned} q_{t_3} &= -q_{xxx} - q_x - 3qq_x - \frac{3}{2}q^2q_x - \frac{3}{2}r^2q_x, \\ r_{t_3} &= -r_{xxx} - 3r_xq - \frac{3}{2}q^2r_x - \frac{3}{2}r^2r_x - r_x. \end{aligned} \quad (28)$$

3. Hamiltonian Structures of the New Soliton Hierarchy

In this section, we will establish Hamiltonian structures of the new soliton hierarchy by trace identity [9, 14]

$$\frac{\delta}{\delta u} \int \text{tr} \left(V \frac{\partial U}{\partial \lambda} \right) dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \text{tr} \left(V \frac{\partial U}{\partial u} \right), \quad (29)$$

where the constant γ is determined as in (9).

A direct calculation reads

$$\begin{aligned} \text{tr} \left(V \frac{\partial U}{\partial \lambda} \right) &= -2c, \\ \text{tr} \left(V \frac{\partial U}{\partial q} \right) &= -2a, \\ \text{tr} \left(V \frac{\partial U}{\partial r} \right) &= -2b. \end{aligned} \quad (30)$$

Substitute the above formula into the trace identity (29) yields

$$\frac{\delta}{\delta u} \int c dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \begin{pmatrix} a \\ b \end{pmatrix}. \quad (31)$$

Comparing the coefficients of λ on both sides of (31) gives rise to

$$\frac{\delta}{\delta u} \int c_{n+1} dx = (\gamma - n) \begin{pmatrix} a_n \\ b_n \end{pmatrix}. \quad (32)$$

By employing the computing formula (9) on the constant γ , we obtain $\gamma = 0$. Therefore, we conclude that

$$\begin{aligned} \frac{\delta H_n}{\delta u} &= \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix}, \\ H_n &= \int -\frac{c_{n+2}}{n+1} dx, \\ n &\geq 0. \end{aligned} \quad (33)$$

So, the new soliton hierarchy (26) possesses the following Hamiltonian structures:

$$u_{t_n} = K_n(u) = J \frac{\delta H_n}{\delta u}, \quad n \geq 0. \quad (34)$$

From the recursion relations in (20), we can obtain the hereditary recursion operator Φ which satisfies that

$$\frac{\delta H_n}{\delta u} = \Phi \frac{\delta H_{n-1}}{\delta u}, \quad (35)$$

where

$$\Phi = \begin{pmatrix} -(1+q)\partial^{-1}r & \partial + (1+q)\partial^{-1}(1+q) \\ -\partial - r\partial^{-1}r & r\partial^{-1}(1+q) \end{pmatrix}. \quad (36)$$

4. Liouville Integrability

It is a direct but lengthy computation to verify that J defined by (26) and

$$M = \Phi J = \begin{pmatrix} \partial + (1+q)\partial^{-1}(1+q) & (1+q)\partial^{-1}r \\ r\partial^{-1}(1+q) & \partial + r\partial^{-1}r \end{pmatrix} \quad (37)$$

constitute a Hamiltonian pair [17, 18]; that is, any linear combination N of J and M satisfies

$$\int K^T N'(u) [NS] T dx + \text{cycle}(K, S, T) = 0, \quad (38)$$

for all vector fields K, S , and T . This implies that operator Φ defined by (36) is hereditary [19], which satisfies

$$\begin{aligned} \Phi'(u) [\Phi K] S - \Phi \Phi'(u) [K] S \\ = \Phi'(u) [\Phi S] K - \Phi \Phi'(u) [S] K, \end{aligned} \quad (39)$$

for all vector fields K and S .

The hereditary property (39) is equivalent to

$$L_{\Phi K} \Phi = \Phi L_K \Phi, \quad (40)$$

where K is an arbitrary vector field. The Lie derivative $L_K \Phi$ above is defined by

$$(L_K \Phi) S = \Phi [K, S] - [K, \Phi S], \quad (41)$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields. It is known that an autonomous operator $\Phi = \Phi(u, u_x, \dots)$ is a recursion

operator of an evolution equation $u_t = K(u)$ if operator Φ needs to satisfy

$$L_K \Phi = 0. \quad (42)$$

Because operator Φ (36) satisfies

$$L_{K_0} \Phi = 0, \quad K_0 = \begin{pmatrix} -r \\ 1+q \end{pmatrix}, \quad (43)$$

thus

$$L_{K_m} \Phi = L_{\Phi K_{m-1}} \Phi = \Phi L_{K_{m-1}} \Phi = 0, \quad m \geq 1, \quad (44)$$

where K'_m s are defined by (26). This implies that operator Φ defined by (36) is a common hereditary recursion operator for the soliton hierarchy (26).

The new soliton hierarchy (26) has the following bi-Hamiltonian structures [17, 20]:

$$u_{t_n} = K_n = J \frac{\delta H_n}{\delta u} = M \frac{\delta H_{n-1}}{\delta u}, \quad n \geq 1, \quad (45)$$

where J, M , and H_m are defined by (26), (37), and (33), respectively, and thus, the hierarchy is Liouville integrable; that is, it possesses infinitely many commuting symmetries and conservation laws. In particular, we have the Abelian symmetry algebra

$$[K_k, K_l] = K'_K(u) [K_l] - K'_l(u) [K_k] = 0, \quad k, l \geq 0 \quad (46)$$

and the Abelian algebras of conserved functionals

$$\begin{aligned} \{H_k, H_l\}_J &= \int \left(\frac{\delta H_k}{\delta u} \right)^T J \frac{\delta H_l}{\delta u} dx = 0, \quad k, l \geq 0, \\ \{H_k, H_l\}_M &= \int \left(\frac{\delta H_k}{\delta u} \right)^T M \frac{\delta H_l}{\delta u} dx = 0, \quad k, l \geq 0. \end{aligned} \quad (47)$$

The first two nonlinear integrable systems in hierarchy (26) are as follows:

$$\begin{aligned} u_{t_1} = K_1 &= \begin{pmatrix} q \\ r \end{pmatrix}_{t_1} = \begin{pmatrix} q_x \\ r_x \end{pmatrix} = J \frac{\delta H_1}{\delta u} = M \frac{\delta H_0}{\delta u}, \\ u_{t_2} = K_2 &= \begin{pmatrix} q \\ r \end{pmatrix}_{t_2} \\ &= \begin{pmatrix} r_{xx} + qr + \frac{1}{2}q^2r + \frac{1}{2}r^3 \\ -q_{xx} - q - \frac{1}{2}r^2 - \frac{3}{2}q^2 - \frac{1}{2}q^3 - \frac{1}{2}qr^2 \end{pmatrix} \\ &= J \frac{\delta H_2}{\delta u} = M \frac{\delta H_1}{\delta u}, \end{aligned} \quad (48)$$

where the Hamiltonian functionals H_0 , H_1 , and H_2 are given by

$$\begin{aligned} H_0 &= \frac{1}{2} \int (2q + q^2 + r^2) dx, \\ H_1 &= -\frac{1}{2} \int (-r_x + rq_x - qr_x) dx, \\ H_2 &= -\frac{1}{3} \int \left(q_{xx} + qq_{xx} + rr_{xx} - \frac{1}{2}q_x^2 - \frac{1}{2}r_x^2 + q \right. \\ &\quad \left. + 2q^2 + \frac{3}{2}q^3 + \frac{1}{2}r^2 + \frac{3}{2}qr^2 + \frac{3}{4}q^2r^2 + \frac{3}{8}q^4 \right. \\ &\quad \left. + \frac{3}{8}r^4 \right) dx. \end{aligned} \quad (49)$$

5. Conservation Laws of the New Soliton Hierarchy

In the following, we will construct conservation laws of the new soliton hierarchy. We introduce the variables

$$\begin{aligned} E &= \frac{\varphi_1}{\varphi_3}, \\ F &= \frac{\varphi_2}{\varphi_3}. \end{aligned} \quad (50)$$

From (13), we have

$$\begin{aligned} E_x &= -(1+q) - \lambda F - (1+q)E^2 - rEF, \\ F_x &= -r - \lambda E - (1+q)EF - rF^2. \end{aligned} \quad (51)$$

Expand E, F in the power of λ :

$$\begin{aligned} E &= \sum_{j=1}^{\infty} e_j \lambda^{-j}, \\ F &= \sum_{j=1}^{\infty} f_j \lambda^{-j}. \end{aligned} \quad (52)$$

Substituting (52) into (51) and comparing the coefficients of the same power of λ , we obtain

$$\begin{aligned} e_1 &= -r, \\ f_1 &= -1 - q, \\ e_2 &= q_x, \\ f_2 &= r_x, \\ e_3 &= -r_{xx} - 2r(1+q)^2, \\ f_3 &= -q_{xx} - 2(1+q)r^2, \dots \end{aligned} \quad (53)$$

and a recursion formula for e_j and f_j :

$$e_{j+1} = -f_{jx} - (1+q) \sum_{l=1}^{j-1} e_l f_{j-l} - r \sum_{l=1}^{j-1} f_l f_{j-l}, \quad (54)$$

$$f_{j+1} = -e_{jx} - (1+q) \sum_{l=1}^{j-1} e_l e_{j-l} - r \sum_{l=1}^{j-1} e_l f_{j-l}, \quad j \geq 2.$$

Because

$$\frac{\partial}{\partial t} (E + qE + rF) = \frac{\partial}{\partial x} (aE + bF), \quad (55)$$

where

$$\begin{aligned} a &= (1+q)\lambda + r_x, \\ b &= r\lambda - q_x, \end{aligned} \quad (56)$$

in order to obtain the conservation laws for integrable hierarchy, we define

$$\begin{aligned} \theta &= E + qE + rF, \\ \delta &= aE + bF. \end{aligned} \quad (57)$$

Then (55) can be written as $\theta_t = \delta_x$, which is just the formal definition of conservation laws. We expand θ and δ as series in powers of λ with the coefficients, which are called conserved densities and fluxes, respectively:

$$\begin{aligned} \theta &= \sum_{j=1}^{\infty} \theta_j \lambda^{-j}, \\ \delta &= \sum_{j=1}^{\infty} \delta_j \lambda^{-j}; \end{aligned} \quad (58)$$

the first two conserved densities and fluxes are read as follows:

$$\begin{aligned} \theta_1 &= -2r(1+q), \\ \delta_1 &= 2(1+q)q_x - 2rr_x, \\ \theta_2 &= (1+q)q_x + rr_x, \\ \delta_2 &= rq_{xx} - (1+q)r_{xx} - 2r(1+q)^3 + 2(1+q)r^3. \end{aligned} \quad (59)$$

The recursion relation for θ_j and δ_j is

$$\begin{aligned} \theta_j &= (1+q)e_j + rf_j, \\ \delta_j &= (1+q)e_{j+1} + r_x e_j - rf_{j+1} - q_x f_j, \end{aligned} \quad (60)$$

where e_j and f_j can be calculated from (54). We can display the first two conservation laws as

$$\begin{aligned} \theta_{1t} &= \delta_{1x}, \\ \theta_{2t} &= \delta_{2x}, \end{aligned} \quad (61)$$

where $\theta_1, \delta_1, \theta_2$, and δ_2 are defined in (59). Then the infinitely many conservation laws of (26) can be easily obtained from (50)–(61), respectively.

6. Conclusions

Based on the real loop algebra $\mathfrak{so}(3, \mathbb{R})$, we introduced a spectral problem and generated a new hierarchy of soliton equations from the associated zero curvature equations. The Liouville integrability of the resulting soliton equations has been shown upon furnishing a bi-Hamiltonian formulation by the trace identity. Finally, the conservation laws of the new equation hierarchy are also obtained. As its reduction, we gain the nonlinear integrable equations. The solution of reduced equations is a very important and difficult work, we will plan to study and discuss this problem in the near future.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (11547175, 11271008, and 11501526) and the Key Scientific Research Projects of Henan Province (16A110026).

References

- [1] R. M. Miura, C. S. Gardner, and M. D. Kruskal, "Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion," *Journal of Mathematical Physics*, vol. 9, no. 8, pp. 1204–1209, 1968.
- [2] W. X. Ma and B. Fuchssteiner, "Integrable theory of the perturbation equations," *Chaos, Solitons and Fractals*, vol. 7, no. 8, pp. 1227–1250, 1996.
- [3] W.-X. Ma and W. Strampp, "An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems," *Physics Letters A*, vol. 185, no. 3, pp. 277–286, 1994.
- [4] M. Wadati, H. Sanuki, and K. Konno, "Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws," *Progress of Theoretical Physics*, vol. 53, no. 2, pp. 419–436, 1975.
- [5] S. C. Anco and G. W. Bluman, "Examples of conservation law classifications," *European Journal of Applied Mathematics*, vol. 9, pp. 567–585, 2002.
- [6] A. H. Kara and F. M. Mahomed, "Relationship between symmetries and conservation laws," *International Journal of Theoretical Physics*, vol. 39, no. 1, pp. 23–40, 2000.
- [7] N. H. Ibragimov and T. Kolsrud, "Lagrangian approach to evolution equations: symmetries and conservation laws," *Nonlinear Dynamics*, vol. 36, no. 1, pp. 29–40, 2004.
- [8] M. A. Ablowitz and P. A. Clarkson, "Introduction," in *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, J. W. S. Cassels, Ed., chapter 1, pp. 1–68, Cambridge University Press, Cambridge, UK, 1991.
- [9] G. Z. Tu, "On Liouville integrability of zero-curvature equations and the Yang hierarchy," *Journal of Physics A: Mathematical and General*, vol. 22, no. 13, article 2375, 1989.
- [10] W.-X. Ma, X.-X. Xu, and Y. F. Zhang, "Semi-direct sums of Lie algebras and continuous integrable couplings," *Physics Letters A: General, Atomic and Solid State Physics*, vol. 351, no. 3, pp. 125–130, 2006.
- [11] W.-X. Ma, X.-X. Xu, and Y. F. Zhang, "Semidirect sums of Lie algebras and discrete integrable couplings," *Journal of Mathematical Physics*, vol. 47, no. 5, Article ID 053501, 2006.
- [12] W.-X. Ma and M. Chen, "Hamiltonian and quasi-Hamiltonian structures associated with semi-direct sums of Lie algebras," *Journal of Physics A: Mathematical and General*, vol. 39, no. 34, 2006.
- [13] W. X. Ma, "Variational identities and Hamiltonian structures," *AIP Conference Proceedings*, vol. 1212, pp. 1–27, 2010.
- [14] W. X. Ma, "A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction," *Chinese Annals of Mathematics, Series A*, vol. 13, pp. 115–123, 1992.
- [15] W. X. Ma, "A spectral problem based on $\mathfrak{so}(3, \mathbb{R})$ and its associated commuting soliton equations," *Journal of Mathematical Physics*, vol. 54, Article ID 103509, 2013.
- [16] H.-X. Yang and Y.-P. Sun, "Hamiltonian and super-hamiltonian extensions related to Broer-Kaup-Kupershmidt system," *International Journal of Theoretical Physics*, vol. 49, no. 2, pp. 349–364, 2010.
- [17] F. Magri, "A simple model of the integrable Hamiltonian equation," *Journal of Mathematical Physics*, vol. 19, no. 5, pp. 1156–1162, 1978.
- [18] B. Fuchssteiner and A. S. Fokas, "Symplectic structures, their Bäcklund transformations and hereditary symmetries," *Physica D: Nonlinear Phenomena*, vol. 4, no. 1, pp. 47–66, 1981.
- [19] B. Fuchssteiner, "Application of hereditary symmetries to nonlinear evolution equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 3, no. 6, pp. 849–862, 1979.
- [20] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Edited by P. R. Halmos, chapter 7, Springer, Berlin, Germany, 1986.