Research Article

On Characterization of Rough Type-2 Fuzzy Sets

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Received 19 October 2015; Accepted 31 December 2015

Academic Editor: Rosana Rodriguez-Lopez

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Rough sets theory and fuzzy sets theory are important mathematical tools to deal with uncertainties. Rough fuzzy sets and fuzzy rough sets as generalizations of rough sets have been introduced. Type-2 fuzzy set provides additional degree of freedom, which makes it possible to directly handle high uncertainties. In this paper, the rough type-2 fuzzy set model is proposed by combining the rough set theory with the type-2 fuzzy set theory. The rough type-2 fuzzy approximation operators induced from the Pawlak approximation space are defined. The rough approximations of a type-2 fuzzy set in the generalized Pawlak approximation space are also introduced. Some basic properties of the rough type-2 fuzzy approximation operators and the generalized rough type-2 fuzzy approximation operators are discussed. The connections between special crisp binary relations and generalized rough type-2 fuzzy approximation operators are further examined. The axiomatic characterization of generalized rough type-2 fuzzy approximation operators is also presented. Finally, the attribute reduction of type-2 fuzzy information systems is investigated.

1. Introduction

Rough sets theory [1, 2] proposed by Pawlak is a mathematical tool to deal with uncertainties and imprecision. Currently, rough set theory has been found to have very successful applications in many fields such as expert systems, machine learning, pattern recognition, decision analysis, and knowledge discovery. In Pawlak rough set model, the equivalence relation plays an important role and is a very restrictive requirement, which may limit its applications in real applications. Hence, the generalization of rough set model is an important branch of rough set theory study. Afterward, several authors proposed generalized rough set model, variable precision rough set model, covering rough set model, rough fuzzy set model and fuzzy rough set model, and so forth.

Fuzzy set theory (type-1 fuzzy sets) was proposed by Zadeh in 1965 [3]. The type-1 fuzzy set theory has been widely applied in various fields [4–6]. Since rough set theory is highly complementary with fuzzy set theory, it is desirable to study the combination of fuzzy sets and rough sets. Dubois and Prade proposed rough fuzzy set model and fuzzy rough set model in [7]. Afterwards, many interesting achievements are obtained on the basis of [7]. Currently, the knowledge representation of fuzzy rough sets and the attribute

reduction using fuzzy rough sets are two main branches on the researches of fuzzy rough sets. On the knowledge representation of fuzzy rough sets, researchers mainly focus on the construction of fuzzy rough approximation operators and the axiomatic characterization of fuzzy rough sets. At present, there exist many valuable results on the knowledge representation of fuzzy rough sets [8–12]. The definitions for generalized fuzzy lower and upper approximation operators determined by a residual implication were introduced in [8]. The lower and upper approximations of rough sets and fuzzy rough sets were, respectively, characterized by using outer and inner products in [9]. A general framework for the study of fuzzy rough sets was presented in [10]. The lattice and topological structures of fuzzy rough sets were proposed in [11]. The axiomatic approach of fuzzy rough sets was shown in [12]. On the attribute reduction using fuzzy rough sets, researchers mainly focus on the development of fast reduction algorithms. Jensen and Shen did a pioneering work on attribute reduction using fuzzy rough sets in [13]. A fuzzy rough QuickReduct algorithm was presented in [13], and this novel approach has been applied to aid classification of web content, with very promising results. However, the QuickReduct algorithm has been proved to have some shortcomings in [14]. Hence, some improved algorithms were proposed in [14, 15].

As a generalization of type-1 fuzzy sets, intuitionistic fuzzy sets were proposed by Atanassov [16, 17]. By setting the degree of membership, the degree of nonmembership, and the degree of hesitation, intuitionistic fuzzy sets depict the nature of ambiguity and solve the hesitant information in judging problems. Cornelis et al. [18] proposed intuitionistic fuzzy rough sets by combining intuitionistic fuzzy sets and rough sets. Afterwards, the theory and application studies of intuitionistic fuzzy rough sets have been expanded in [19–22]. The concept of interval-valued intuitionistic fuzzy sets was firstly proposed in [23]. Interval-valued intuitionistic fuzzy sets provide more flexible and effective way to deal with uncertainties because both membership degree and nonmembership degree of interval-valued intuitionistic fuzzy sets are denoted by interval numbers of $[0, 1]$. In [24], an interval-valued intuitionistic fuzzy rough model has been proposed by combining the classical Pawlak rough set theory with the interval-valued intuitionistic fuzzy set theory. Furthermore, in [25], the rough approximations of an interval-valued intuitionistic fuzzy set in the classical Pawlak approximation space and the generalized Pawlak approximation space have been presented, respectively.

Zadeh in 1975 proposed type-2 fuzzy sets which can enhance the system's ability to deal with uncertainties [26]. The applications of type-2 fuzzy sets were limited due to the computational complexity. However, interval type-2 fuzzy sets, as a special case of general type-2 fuzzy sets, have been extensively studied in practical applications. There are some achievements about interval type-2 fuzzy rough sets by combining the characteristics of rough sets and interval type-2 fuzzy sets [27–29]. General type-2 fuzzy sets may be better than the interval type-2 fuzzy sets to deal with uncertainties because general type-2 fuzzy sets can obtain more degree of freedom [30]. In order to simplify the calculation for general type-2 fuzzy sets, the representation of general type-2 fuzzy sets is also an important research issue. Liu in [31] firstly proposed the α -plane representation of the general type-2 fuzzy sets and claimed that the α -plane representation can greatly reduce the computational workload. Currently, α -plane method has been extensively studied [32–35] because it can take advantage of the interval type-2 fuzzy sets theory to study the general type-2 fuzzy sets.

Although general type-2 fuzzy sets and rough sets have important applications, a few results about the combination of general type-2 fuzzy set theory and rough set theory were presented [36–38]. In [36], we proposed general type-2 fuzzy rough sets based on general type-2 fuzzy relations. However, there are no contributions on the definitions of approximation operators of general type-2 fuzzy sets in the Pawlak approximation space and the generalized Pawlak approximation space. Moreover, it can be seen that the attribute reduction of information systems is an important application of rough set theory. The traditional rough set model can effectively handle these information systems in which both condition attributes and decision attributes are clear. In some cases, condition attributes may be clear and decision attributes may be interval type-2 fuzzy sets on objects set. That is to say, every decision attribute value is an interval on $[0, 1]$. At this time, the rough approximations

of an interval type-2 fuzzy set in the generalized Pawlak approximation space were also introduced in [39]. Therefore, the interval type-2 rough fuzzy set model proposed in [39] may be suitable to deal with these information systems with interval values. However, in highly uncertain situations, we would encounter these information systems where all condition attributes are clear and all decision attributes are general type-2 fuzzy sets on objects set; that is, every decision attribute value is a type-1 fuzzy set on $[0, 1]$. To provide certain theoretical basis for dealing with such problem, we propose rough type-2 fuzzy set model by combining the classical Pawlak rough set theory with the general type-2 fuzzy set theory in this paper. The rest of our work is organized as follows. In Section 2, the basic definitions and terminologies on type-2 fuzzy sets are reviewed briefly. In Section 3, the rough type-2 fuzzy approximation operators based on the clear equivalence relation are firstly defined. The generalized rough type-2 fuzzy approximation operators based on the clear generalized binary relation are also derived in Section 3. In Section 4, the generalized rough type-2 fuzzy approximation operators are characterized by axioms. The attribute reduction method of type-2 fuzzy information systems is presented in Section 5. The last section concludes this paper.

2. Preliminaries

In this section, the basic definitions and terminologies on type-2 fuzzy sets with some modified notations are recalled.

Definition 1 (see [40]). A type-2 fuzzy set, denoted by $\tilde{A}$, is expressed as

$$A = \int_{x \in U} \frac{u_A(x)}{x} = \int_{x \in U} \frac{\left[\int_{u \in J_x} f_x(u)/u \right]}{x} \quad (1)$$

$$J_x \subseteq [0, 1],$$

where J_x is the primary membership of x ; $u_A(x) = \int_{u \in J_x} f_x(u)/u$ is the secondary membership function; $f_x(u)$ is a secondary membership grade.

In the following sections, the class of all type-2 fuzzy sets of the universe of discourse U is denoted as $F_2(U)$, and the class of all crisp sets of the universe of discourse U is denoted as $P(U)$. Currently, it is difficult to deal with the type-2 fuzzy sets whose secondary membership functions are not normal and convex. For simplicity, we only study these type-2 fuzzy sets whose secondary membership function is normal and convex in this paper.

Definition 2 (see [32]). An α -plane for type-2 fuzzy set A , which is denoted by A_α , is defined as follows:

$$A_\alpha = \int_{x \in U} \int_{u \in J_x} \{(x, u) \mid f_x(u) \geq \alpha\}$$

$$= \int_{x \in U} \frac{\left[\int_{u \in [S_L^\alpha(x|\alpha), S_U^\alpha(x|\alpha)]} u \right]}{x} \quad \alpha \in [0, 1], \quad (2)$$

where $[S_L^A(x | \alpha), S_U^A(x | \alpha)]$ denote an α -cut of the secondary membership function $u_A(x)$.

Definition 3 (see [32]). The α -plane representation (theorem) for type-2 fuzzy set A is

$$A = \bigcup_{\alpha \in [0,1]} \frac{\alpha}{A_\alpha}. \quad (3)$$

Theorem 4 (see [32]). Let $(A \cup B)_\alpha$ and $(A \cap B)_\alpha$ be α -plane of $A \cup B$ and $A \cap B$, respectively; one has

$$\begin{aligned} A \cup B &= \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(A_\alpha \cup B_\alpha)}; \\ A \cap B &= \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(A_\alpha \cap B_\alpha)}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} A_\alpha \cup B_\alpha &= \int_{x \in U} \frac{\left[\int_{u \in [S_L^A(x|\alpha) \vee S_L^B(x|\alpha), S_U^A(x|\alpha) \vee S_U^B(x|\alpha)]} u \right]}{x}; \\ A_\alpha \cap B_\alpha &= \int_{x \in U} \frac{\left[\int_{u \in [S_L^A(x|\alpha) \wedge S_L^B(x|\alpha), S_U^A(x|\alpha) \wedge S_U^B(x|\alpha)]} u \right]}{x}. \end{aligned} \quad (5)$$

Obviously, $S_L^{A \cup B}(x | \alpha) = S_L^A(x | \alpha) \vee S_L^B(x | \alpha)$, $S_U^{A \cup B}(x | \alpha) = S_U^A(x | \alpha) \vee S_U^B(x | \alpha)$, $S_L^{A \cap B}(x | \alpha) = S_L^A(x | \alpha) \wedge S_L^B(x | \alpha)$, and $S_U^{A \cap B}(x | \alpha) = S_U^A(x | \alpha) \wedge S_U^B(x | \alpha)$ hold.

Theorem 5 (see [36]). Let $(A^c)_\alpha$ be α -plane of A^c ; one has

$$A^c = \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(A^c)_\alpha} = \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(A_\alpha)^c}, \quad (6)$$

where $A_\alpha = \int_{x \in U} \left[\int_{u \in [S_L^A(x|\alpha), S_U^A(x|\alpha)]} u \right] / x$.

Definition 6 (see [36]). Let $A, B \in F_2(U)$, and define $A \subseteq B$ if $S_L^A(x | \alpha) \leq S_L^B(x | \alpha)$ and $S_U^A(x | \alpha) \leq S_U^B(x | \alpha)$ hold for any $\alpha \in [0, 1]$ and $x \in U$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

3. Rough Type-2 Fuzzy Sets

In this section, we introduce rough type-2 fuzzy approximation operators and generalized rough type-2 fuzzy approximation operators induced from the Pawlak approximation space and the generalized Pawlak approximation space, respectively, and discuss their properties.

3.1. Rough Type-2 Fuzzy Approximation Operators

Based on the Equivalence Relation

Definition 7. Let U be a nonempty universe of discourse and let $R \in P(U \times U)$ be the clear equivalence relation on U . $[x]_R$ denotes the equivalence class. Then (U, R) is called the Pawlak approximation space. For any $A \in F_2(U)$, define the upper

and lower rough type-2 fuzzy approximation operators $\bar{R}(A)$ and $\underline{R}(A)$ about (U, R) by

$$\bar{R}(A) \equiv \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(\bar{R}(A))_\alpha}, \quad (7)$$

$$\underline{R}(A) \equiv \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(\underline{R}(A))_\alpha},$$

where $(\underline{R}(A))_\alpha = \int_{x \in U} \left[\int_{u \in [S_L^{\bar{R}(A)}(x|\alpha), S_U^{\bar{R}(A)}(x|\alpha)]} u \right] / x$, $(\bar{R}(A))_\alpha = \int_{x \in U} \left[\int_{u \in [S_L^{\underline{R}(A)}(x|\alpha), S_U^{\underline{R}(A)}(x|\alpha)]} u \right] / x$, and

$$\begin{aligned} S_L^{\bar{R}(A)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_L^A(y | \alpha), \\ S_U^{\bar{R}(A)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_U^A(y | \alpha), \\ S_L^{\underline{R}(A)}(x | \alpha) &= \bigvee_{y \in [x]_R} S_L^A(y | \alpha), \\ S_U^{\underline{R}(A)}(x | \alpha) &= \bigvee_{y \in [x]_R} S_U^A(y | \alpha). \end{aligned} \quad (8)$$

Clearly, if A is degraded to type-1 fuzzy set, then the rough type-2 fuzzy approximation operators defined in Definition 7 reduce to the rough type-1 fuzzy approximation operators. In the following, we define type-2 fuzzy universe set $U = \int_{x \in U} \left[\int_{u \in [1,1]} 1/u \right] / x$ and type-2 fuzzy empty set $\emptyset = \int_{x \in U} \left[\int_{u \in [0,0]} 1/u \right] / x$. Obviously, $S_L^U(x | \alpha) = S_U^U(x | \alpha) = 1$ and $S_L^\emptyset(x | \alpha) = S_U^\emptyset(x | \alpha) = 0$ for any $x \in U$ and $\alpha \in [0, 1]$.

Theorem 8. Let (U, R) be a Pawlak approximation space and let $\underline{R}$ and $\bar{R}$ be the rough type-2 fuzzy lower and upper approximation operators about (U, R) ; for any $A \in F_2(U)$, the following properties hold:

- (1) $\underline{R}(A) \subseteq A \subseteq \bar{R}(A)$;
- (2) $\underline{R}(A^c) = (\bar{R}(A))^c$, $\bar{R}(A^c) = (\underline{R}(A))^c$;
- (3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\bar{R}(A \cup B) = \bar{R}(A) \cup \bar{R}(B)$;
- (4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $A \subseteq B \Rightarrow \bar{R}(A) \subseteq \bar{R}(B)$;
- (5) $\underline{R}(\underline{R}(A)) = \underline{R}(A)$, $\bar{R}(\bar{R}(A)) = \bar{R}(A)$;
- (6) $\underline{R}(U) = \bar{R}(U) = U$, $\underline{R}(\emptyset) = \bar{R}(\emptyset) = \emptyset$.

Proof. (1) For any $x \in U$ and $\alpha \in [0, 1]$, we have

$$S_L^{\bar{R}(A)}(x | \alpha) = \bigwedge_{y \in [x]_R} S_L^A(y | \alpha) \leq S_L^A(x | \alpha), \quad (9)$$

$$S_U^{\bar{R}(A)}(x | \alpha) = \bigwedge_{y \in [x]_R} S_U^A(y | \alpha) \leq S_U^A(x | \alpha).$$

Thus, $\underline{R}(A) \subseteq A$. Similarly, $A \subseteq \bar{R}(A)$. That is, $\underline{R}(A) \subseteq A \subseteq \bar{R}(A)$.

(2) For any $x \in U$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
 S_L^{R(A^c)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_L^A(y | \alpha) \\
 &= \bigwedge_{y \in [x]_R} [1 - S_U^A(y | \alpha)] \\
 &= 1 - \bigvee_{y \in [x]_R} S_U^A(y | \alpha) \\
 &= 1 - \bigvee_{y \in [x]_R} S_U^A(y | \alpha) \\
 &= 1 - S_U^{\bar{R}(A)}(x | \alpha) = S_L^{(\bar{R}(A))^c}(x | \alpha).
 \end{aligned} \tag{10}$$

Furthermore,

$$\begin{aligned}
 S_U^{R(A^c)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_U^A(y | \alpha) \\
 &= \bigwedge_{y \in [x]_R} [1 - S_L^A(y | \alpha)] \\
 &= 1 - \bigvee_{y \in [x]_R} S_L^A(y | \alpha) \\
 &= 1 - \bigvee_{y \in [x]_R} S_L^A(y | \alpha) = 1 - S_L^{\bar{R}(A)}(x | \alpha) \\
 &= S_U^{(\bar{R}(A))^c}(x | \alpha).
 \end{aligned} \tag{11}$$

Thus, $\underline{R}(A^c) = (\bar{R}(A))^c$. Similarly, we can prove $\bar{R}(A^c) = (\underline{R}(A))^c$.

(3) For any $x \in U$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
 S_L^{R(A \cap B)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_L^{A \cap B}(y | \alpha) \\
 &= \bigwedge_{y \in [x]_R} [S_L^A(y | \alpha) \wedge S_L^B(y | \alpha)] \\
 &= \left[\bigwedge_{y \in [x]_R} S_L^A(y | \alpha) \right] \\
 &\quad \wedge \left[\bigwedge_{y \in [x]_R} S_L^B(y | \alpha) \right] \\
 &= S_L^{R(A)}(x | \alpha) \wedge S_L^{R(B)}(x | \alpha) \\
 &= S_L^{R(A) \cap R(B)}(x | \alpha).
 \end{aligned} \tag{12}$$

Similarly, we have $S_U^{R(A \cap B)}(x | \alpha) = S_U^{R(A) \cap R(B)}(x | \alpha)$. Thus, $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$. Similarly, we can prove $\bar{R}(A \cup B) = \bar{R}(A) \cup \bar{R}(B)$.

(4) For any $x \in U$ and $\alpha \in [0, 1]$, since $A \subseteq B$, we have $S_L^A(x | \alpha) \leq S_L^B(x | \alpha)$ and $S_U^A(x | \alpha) \leq S_U^B(x | \alpha)$. Thus,

$S_L^{R(A)}(x | \alpha) = \bigwedge_{y \in [x]_R} S_L^A(y | \alpha) \leq \bigwedge_{y \in [x]_R} S_L^B(y | \alpha) = S_L^{R(B)}(x | \alpha)$. Furthermore, $S_U^{R(A)}(x | \alpha) = \bigwedge_{y \in [x]_R} S_U^A(y | \alpha) \leq \bigwedge_{y \in [x]_R} S_U^B(y | \alpha) = S_U^{R(B)}(x | \alpha)$. Hence, $\underline{R}(A) \subseteq \underline{R}(B)$. That is, $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$. Similarly, we can prove $A \subseteq B \Rightarrow \bar{R}(A) \subseteq \bar{R}(B)$.

(5) For any $x \in U$ and $\alpha \in [0, 1]$,

$$\begin{aligned}
 S_L^{R(R(A))}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_L^{R(A)}(y | \alpha) \\
 &= \bigwedge_{y \in [x]_R} \left[\bigwedge_{z \in [y]_R} S_L^A(z | \alpha) \right] \\
 &= \bigwedge_{z \in [x]_R} S_L^A(z | \alpha) = S_L^{R(A)}(x | \alpha).
 \end{aligned} \tag{13}$$

Similarly, we have $S_U^{R(R(A))}(x | \alpha) = S_U^{R(A)}(x | \alpha)$. Thus, $\underline{R}(\underline{R}(A)) = \underline{R}(A)$. In addition

$$\begin{aligned}
 S_L^{\bar{R}(R(A))}(x | \alpha) &= \bigvee_{y \in [x]_R} S_L^{R(A)}(y | \alpha) \\
 &= \bigvee_{y \in [x]_R} \left[\bigwedge_{z \in [y]_R} S_L^A(z | \alpha) \right] \\
 &= \bigvee_{y \in [x]_R} \left[\bigwedge_{z \in [x]_R} S_L^A(z | \alpha) \right] \\
 &= \bigwedge_{z \in [x]_R} S_L^A(z | \alpha) = S_L^{R(A)}(x | \alpha).
 \end{aligned} \tag{14}$$

Similarly, we can obtain $S_U^{\bar{R}(R(A))}(x | \alpha) = S_U^{R(A)}(x | \alpha)$. Hence, $\bar{R}(\bar{R}(A)) = \bar{R}(A)$. That is, $\underline{R}(\bar{R}(A)) = \bar{R}(\underline{R}(A)) = \underline{R}(A)$. Similarly, we can prove $\bar{R}(\bar{R}(A)) = \bar{R}(\bar{R}(A)) = \bar{R}(A)$.

(6) For any $x \in U$ and $\alpha \in [0, 1]$, we have

$$\begin{aligned}
 S_L^{R(U)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_L^U(y | \alpha) = 1, \\
 S_U^{R(U)}(x | \alpha) &= \bigwedge_{y \in [x]_R} S_U^U(y | \alpha) = 1.
 \end{aligned} \tag{15}$$

Thus, $\underline{R}(U) = U$. Similarly, $\bar{R}(U) = U$. That is to say, $\underline{R}(U) = \bar{R}(U) = U$. The second equation can be proved in a similar way. $\square$

Definition 9. Let (U, R) be a Pawlak approximation space and let A and B be the type-2 fuzzy sets on U .

- (1) A and B are called lower rough equal denoted by $A \approx B$ if $\underline{R}(A) = \underline{R}(B)$.
- (2) A and B are called upper rough equal denoted by $A \approx B$ if $\bar{R}(A) = \bar{R}(B)$.

- (3) A and B are called rough equal denoted by $A \approx B$ if A and B are both lower rough equal and upper rough equal.

Theorem 10. Let (U, R) be a Pawlak approximation space. Then the following properties hold for any $A, B, A', B' \in F_2(U)$:

- (1) $A \approx B \Leftrightarrow A \cap B \approx A$ and $A \cap B \approx B$;
- (2) $A \approx B \Leftrightarrow A \cup B \approx A$ and $A \cup B \approx B$;
- (3) if $A \approx A'$ and $B \approx B'$, then $(A \cap B) \approx (A' \cap B')$;
- (4) if $A \approx A'$ and $B \approx B'$, then $(A \cup B) \approx (A' \cup B')$;
- (5) if $A \approx \emptyset$ or $B \approx \emptyset$, then $(A \cap B) \approx \emptyset$;
- (6) if $A \approx U$ or $B \approx U$, then $(A \cup B) \approx U$;
- (7) if $A \subseteq B$ and $B \approx \emptyset$, then $A \approx \emptyset$;
- (8) if $A \subseteq B$ and $A \approx U$, then $B \approx U$;
- (9) $A \approx U \Leftrightarrow A = U$;
- (10) $A \approx \emptyset \Leftrightarrow A = \emptyset$.

Proof. By Definition 9, the proof procedure is trivial. $\square$

Theorem 11. Let (U, R) be a Pawlak approximation space and let A be type-2 fuzzy set on U . Then, one has

- (1) $\underline{R}(A) = \bigcap \{B \in F_2(U); B \approx A\}$;
- (2) $\overline{R}(A) = \bigcup \{B \in F_2(U); B \approx A\}$.

Proof. By Theorem 8(5), this theorem can be easily proved. $\square$

3.2. Generalized Rough Type-2 Fuzzy Approximation Operators Based on the Generalized Binary Relation

Definition 12. Let U be a nonempty finite universe of discourse and let $R \in P(U \times U)$ be an arbitrary binary relation on U ; $R_s(x) = \{y \in U; (x, y) \in R\}$; then (U, R) is called the generalized Pawlak approximation space. For any $A \in F_2(U)$, the upper and lower generalized rough type-2 fuzzy approximation operators $\overline{\text{app}}(A)$ and $\underline{\text{app}}(A)$ about (U, R) are, respectively, defined as follows:

$$\begin{aligned}\overline{\text{app}}(A) &\equiv \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(\overline{\text{app}}(A))_\alpha}, \\ \underline{\text{app}}(A) &\equiv \bigcup_{\alpha \in [0,1]} \frac{\alpha}{(\underline{\text{app}}(A))_\alpha},\end{aligned}\quad (16)$$

where $(\underline{\text{app}}(A))_\alpha = \int_{x \in U} [\int_{u \in [S_L^{\text{app}(A)}(x|\alpha), S_U^{\text{app}(A)}(x|\alpha)]} u] / x$, $(\overline{\text{app}}(A))_\alpha = \int_{x \in U} [\int_{u \in [S_L^{\overline{\text{app}}(A)}(x|\alpha), S_U^{\overline{\text{app}}(A)}(x|\alpha)]} u] / x$, and

$$\begin{aligned}S_L^{\text{app}(A)}(x|\alpha) &= \bigwedge_{y \in R_s(x)} S_L^A(y|\alpha), \\ S_U^{\text{app}(A)}(x|\alpha) &= \bigwedge_{y \in R_s(x)} S_U^A(y|\alpha), \\ S_L^{\overline{\text{app}}(A)}(x|\alpha) &= \bigvee_{y \in R_s(x)} S_L^A(y|\alpha), \\ S_U^{\overline{\text{app}}(A)}(x|\alpha) &= \bigvee_{y \in R_s(x)} S_U^A(y|\alpha).\end{aligned}\quad (17)$$

In particular, if R is an equivalence relation of the universe U , then $R_s(x)$ degrades to the equivalence class $[x]_R$. The lower approximation $\underline{\text{app}}(A)$ and the upper approximation $\overline{\text{app}}(A)$ reduce to the lower approximation $\underline{R}(A)$ and the upper approximation $\overline{R}(A)$ in the classical Pawlak approximation space, respectively.

In [36], a type-2 fuzzy singleton set 1_y and its complement $1_{U-\{y\}}$ are, respectively, defined as follows:

$$\begin{aligned}u_{1_y}(x) &= \begin{cases} \frac{1}{1} & x = y \\ \frac{1}{0} & x \neq y, \end{cases} \\ u_{1_{U-\{y\}}}(x) &= \begin{cases} \frac{1}{0} & x = y \\ \frac{1}{1} & x \neq y. \end{cases}\end{aligned}\quad (18)$$

Based on the above definition, we can obtain that $S_L^{1_y}(x|\alpha) = S_U^{1_y}(x|\alpha) = \begin{cases} 1 & x=y \\ 0 & x \neq y \end{cases}$ and $S_L^{1_{U-\{y\}}}(x|\alpha) = S_U^{1_{U-\{y\}}}(x|\alpha) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$ hold for any $\alpha \in [0, 1]$ and $x \in U$.

Theorem 13. Let (U, R) be a generalized Pawlak approximation space and let $\underline{\text{app}}$ and $\overline{\text{app}}$ be the generalized rough type-2 fuzzy lower and upper approximation operators about (U, R) ; for any $A \in F_2(U)$, the following properties hold:

- (1) $\underline{\text{app}}(A^c) = (\overline{\text{app}}(A))^c$, $\overline{\text{app}}(A^c) = (\underline{\text{app}}(A))^c$;
- (2) $\underline{\text{app}}(A \cap B) = \underline{\text{app}}(A) \cap \underline{\text{app}}(B)$, $\overline{\text{app}}(A \cup B) = \overline{\text{app}}(A) \cup \overline{\text{app}}(B)$;
- (3) $A \subseteq B \Rightarrow \underline{\text{app}}(A) \subseteq \underline{\text{app}}(B)$, $A \subseteq B \Rightarrow \overline{\text{app}}(A) \subseteq \overline{\text{app}}(B)$;
- (4) $\underline{\text{app}}(U) = \overline{\text{app}}(U) = U$, $\underline{\text{app}}(\emptyset) = \overline{\text{app}}(\emptyset) = \emptyset$;
- (5) $S_L^{\text{app}(1_{U-\{y\}})}(x|\alpha) = S_U^{\text{app}(1_{U-\{y\}})}(x|\alpha) = 1 - R(x, y)$ for any $\alpha \in [0, 1]$ and $x, y \in U$;
- (6) $S_L^{\overline{\text{app}}(1_y)}(x|\alpha) = S_U^{\overline{\text{app}}(1_y)}(x|\alpha) = R(x, y)$ for any $\alpha \in [0, 1]$ and $x, y \in U$.

Proof. The proofs of (1)–(4) are analogous to Theorem 8, and therefore we omit them.

(5) For any $\alpha \in [0, 1]$ and $x, y \in U$, we have

$$S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = \bigwedge_{z \in R_s(x)} S_L^{1_{U-\{y\}}}(z | \alpha). \quad (19)$$

If $R(x, y) = 1$, then $y \in R_s(x)$. Thus, $\bigwedge_{z \in R_s(x)} S_L^{1_{U-\{y\}}}(z | \alpha) \leq S_L^{1_{U-\{y\}}}(y | \alpha) = 0$. That is, $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = 0$. On the other hand, if $R(x, y) = 0$, then $y \notin R_s(x)$. Thus, $\bigwedge_{z \in R_s(x)} S_L^{1_{U-\{y\}}}(z | \alpha) = 1$. Therefore, we have $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = 1 - R(x, y)$.

Similarly, we can obtain $S_U^{\text{app}(1_{U-\{y\}})}(x | \alpha) = 1 - R(x, y)$. That is to say, $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = S_U^{\text{app}(1_{U-\{y\}})}(x | \alpha) = 1 - R(x, y)$.

(6) The proof procedure is similar to (5). $\square$

Theorem 14. Let (U, R) be a generalized Pawlak approximation space and let app and $\overline{\text{app}}$ be the generalized rough type-2 fuzzy lower and upper approximation operators about (U, R) ; for any $A \in F_2(U)$, the following statements are equivalent:

- (1) R is reflexive;
- (2) $\text{app}(A) \subseteq A$;
- (3) $A \subseteq \overline{\text{app}}(A)$.

Proof.

(1) $\Rightarrow$ (2). If R is reflexive, then $x \in R_s(x)$ for any $x \in U$. For any $\alpha \in [0, 1]$,

$$S_L^{\text{app}(A)}(x | \alpha) = \bigwedge_{y \in R_s(x)} S_L^A(y | \alpha) \leq S_L^A(x | \alpha). \quad (20)$$

Similarly, we can derive $S_U^{\text{app}(A)}(x | \alpha) \leq S_U^A(x | \alpha)$. Thus, $\text{app}(A) \subseteq A$.

(2) $\Rightarrow$ (3). We can obtain the conclusion according to Theorem 13(1).

(3) $\Rightarrow$ (1). For any $y \in U$, let $A = 1_y$; thus $S_L^A(x | \alpha) = S_U^A(x | \alpha) = \begin{cases} 1 & x=y \\ 0 & x \neq y \end{cases}$.

If $A \subseteq \overline{\text{app}}(A)$, then $S_L^A(x | \alpha) \leq S_L^{\overline{\text{app}}(A)}(x | \alpha)$ and $S_U^A(x | \alpha) \leq S_U^{\overline{\text{app}}(A)}(x | \alpha)$ hold for any $\alpha \in [0, 1]$ and $x \in U$. Hence, $1 \leq S_L^{\overline{\text{app}}(A)}(y | \alpha)$ and $1 \leq S_U^{\overline{\text{app}}(A)}(y | \alpha)$. We can obtain $1 = S_L^{\overline{\text{app}}(A)}(y | \alpha)$ and $1 = S_U^{\overline{\text{app}}(A)}(y | \alpha)$. Therefore, $\bigvee_{z \in R_s(y)} S_L^A(z | \alpha) = \bigvee_{z \in R_s(y)} S_U^A(z | \alpha) = 1$. That is to say, there exists $z \in R_s(y)$ such that $S_L^A(z | \alpha) = S_U^A(z | \alpha) = 1$. Thus, $y \in R_s(y)$.

We can conclude that R is reflexive. $\square$

Theorem 15. Let (U, R) be a generalized Pawlak approximation space and let app and $\overline{\text{app}}$ be the generalized rough type-2 fuzzy lower and upper approximation operators about (U, R) ; the following statements are equivalent:

- (1) R is symmetric.

(2) For any $\alpha \in [0, 1]$ and $x, y \in U$, $S_L^{\overline{\text{app}}(1_x)}(y | \alpha) = S_L^{\overline{\text{app}}(1_y)}(x | \alpha)$ and $S_U^{\overline{\text{app}}(1_x)}(y | \alpha) = S_U^{\overline{\text{app}}(1_y)}(x | \alpha)$ hold.

(3) For any $\alpha \in [0, 1]$ and $x, y \in U$, $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = S_L^{\text{app}(1_{U-\{x\}})}(y | \alpha)$ and $S_U^{\text{app}(1_{U-\{y\}})}(x | \alpha) = S_U^{\text{app}(1_{U-\{x\}})}(y | \alpha)$ hold.

Proof.

(1) $\Rightarrow$ (2). For any $\alpha \in [0, 1]$ and $x, y \in U$

$$\begin{aligned} S_L^{\overline{\text{app}}(1_y)}(x | \alpha) &= \bigvee_{z \in R_s(x)} S_L^{1_y}(z | \alpha), \\ S_L^{\overline{\text{app}}(1_x)}(y | \alpha) &= \bigvee_{z \in R_s(y)} S_L^{1_x}(z | \alpha). \end{aligned} \quad (21)$$

If $x \in R_s(y)$, then $S_L^{\overline{\text{app}}(1_x)}(y | \alpha) = 1$.

Since R is symmetric, we have $y \in R_s(x)$. Hence, $S_L^{\overline{\text{app}}(1_y)}(x | \alpha) = 1$. That is, $S_L^{\overline{\text{app}}(1_x)}(y | \alpha) = S_L^{\overline{\text{app}}(1_y)}(x | \alpha)$.

Similarly, we can prove $S_U^{\overline{\text{app}}(1_x)}(y | \alpha) = S_U^{\overline{\text{app}}(1_y)}(x | \alpha)$.

(2) $\Rightarrow$ (3). For any $\alpha \in [0, 1]$ and $x, y \in U$

$$\begin{aligned} S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) &= \bigwedge_{z \in R_s(x)} S_L^{1_{U-\{y\}}}(z | \alpha) \\ &= \bigwedge_{z \in R_s(x)} S_L^{(1_y)^c}(z | \alpha) \\ &= \bigwedge_{z \in R_s(x)} [1 - S_U^{1_y}(z | \alpha)] \\ &= 1 - \bigvee_{z \in R_s(x)} S_U^{1_y}(z | \alpha) \\ &= 1 - S_U^{\overline{\text{app}}(1_y)}(x | \alpha), \\ S_L^{\text{app}(1_{U-\{x\}})}(y | \alpha) &= \bigwedge_{z \in R_s(y)} S_L^{1_{U-\{x\}}}(z | \alpha) \\ &= \bigwedge_{z \in R_s(y)} S_L^{(1_x)^c}(z | \alpha) \\ &= \bigwedge_{z \in R_s(y)} [1 - S_U^{1_x}(z | \alpha)] \\ &= 1 - \bigvee_{z \in R_s(y)} S_U^{1_x}(z | \alpha) \\ &= 1 - S_U^{\overline{\text{app}}(1_x)}(y | \alpha). \end{aligned} \quad (22)$$

Since $S_U^{\overline{\text{app}}(1_x)}(y | \alpha) = S_U^{\overline{\text{app}}(1_y)}(x | \alpha)$, thus $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = S_L^{\text{app}(1_{U-\{x\}})}(y | \alpha)$.

Similarly, we can prove $S_U^{\text{app}(1_{U-\{y\}})}(x | \alpha) = S_U^{\text{app}(1_{U-\{x\}})}(y | \alpha)$.

(3) $\Rightarrow$ (1). For any $\alpha \in [0, 1]$ and $x, y \in U$, if $y \in R_s(x)$, in the following, we only should prove $x \in R_s(y)$. Thus, $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = \bigwedge_{z \in R_s(x)} S_L^{1_{U-\{y\}}}(z | \alpha) = 0$.

We know that $S_L^{\text{app}(1_{U-\{y\}})}(x | \alpha) = S_L^{\text{app}(1_{U-\{x\}})}(y | \alpha)$. Hence, $S_L^{\text{app}(1_{U-\{x\}})}(y | \alpha) = \bigwedge_{z \in R_s(y)} S_L^{1_{U-\{x\}}}(z | \alpha) = 0$. That is to say, there exists $z \in R_s(y)$ such that $S_L^{1_{U-\{x\}}}(z | \alpha) = 0$.

We can obtain $x \in R_s(y)$. Therefore, R is symmetric. $\square$

Theorem 16. Let (U, R) be a generalized Pawlak approximation space and let $\underline{\text{app}}$ and $\overline{\text{app}}$ be the generalized rough type-2 fuzzy lower and upper approximation operators about (U, R) ; for any $A \in F_2(U)$, the following statements are equivalent:

- (1) R is transitive;
- (2) $\overline{\text{app}}(\overline{\text{app}}(A)) \subseteq \overline{\text{app}}(A)$;
- (3) $\underline{\text{app}}(A) \subseteq \underline{\text{app}}(\underline{\text{app}}(A))$.

Proof.

(1) $\Rightarrow$ (2). For any $\alpha \in [0, 1]$ and $x \in U$

$$\begin{aligned} S_L^{\overline{\text{app}}(\overline{\text{app}}(A))}(x | \alpha) &= \bigvee_{y \in R_s(x)} S_L^{\overline{\text{app}}(A)}(y | \alpha) \\ &= \bigvee_{y \in R_s(x)} \left[\bigvee_{z \in R_s(y)} S_L^A(z | \alpha) \right] \\ &= \bigvee_{y \in R_s(x) \wedge z \in R_s(y)} S_L^A(z | \alpha). \end{aligned} \quad (23)$$

Since R is transitive, we have $y \in R_s(x) \wedge z \in R_s(y) \Rightarrow z \in R_s(x)$. Thus, $S_L^{\overline{\text{app}}(\overline{\text{app}}(A))}(x | \alpha) = \bigvee_{y \in R_s(x) \wedge z \in R_s(y)} S_L^A(z | \alpha) \leq \bigvee_{z \in R_s(x)} S_L^A(z | \alpha) = S_L^{\overline{\text{app}}(A)}(x | \alpha)$. Similarly, $S_U^{\overline{\text{app}}(\overline{\text{app}}(A))}(x | \alpha) \leq S_U^{\overline{\text{app}}(A)}(x | \alpha)$. Hence, $\overline{\text{app}}(\overline{\text{app}}(A)) \subseteq \overline{\text{app}}(A)$.

(2) $\Rightarrow$ (1). For any $\alpha \in [0, 1]$ and $x, y, z \in U$, if $y \in R_s(x)$ and $z \in R_s(y)$, then

$$\begin{aligned} S_L^{\overline{\text{app}}(\overline{\text{app}}(1_z))}(x | \alpha) &= \bigvee_{t \in R_s(x)} S_L^{\overline{\text{app}}(1_z)}(t | \alpha) \\ &= \bigvee_{t \in R_s(x)} \left[\bigvee_{p \in R_s(t)} S_L^{1_z}(p | \alpha) \right] \\ &= \left\{ \bigvee_{t \in R_s(x) \wedge t \neq y} \left[\bigvee_{p \in R_s(t)} S_L^{1_z}(p | \alpha) \right] \right\} \end{aligned}$$

$$\begin{aligned} &\vee \left\{ \bigvee_{p \in R_s(y)} S_L^{1_z}(p | \alpha) \right\} \\ &= \left\{ \bigvee_{t \in R_s(x) \wedge t \neq y} \left[\bigvee_{p \in R_s(t)} S_L^{1_z}(p | \alpha) \right] \right\} \\ &\vee \left\{ \left[\bigvee_{p \in R_s(y) \wedge p \neq z} S_L^{1_z}(p | \alpha) \right] \vee S_L^{1_z}(z | \alpha) \right\} \\ &= \left\{ \bigvee_{t \in R_s(x) \wedge t \neq y} \left[\bigvee_{p \in R_s(t)} S_L^{1_z}(p | \alpha) \right] \right\} \\ &\vee \left\{ \left[\bigvee_{p \in R_s(y) \wedge p \neq z} S_L^{1_z}(p | \alpha) \right] \vee 1 \right\} = 1. \end{aligned} \quad (24)$$

Furthermore, $S_L^{\overline{\text{app}}(1_z)}(x | \alpha) = \bigvee_{t \in R_s(x)} S_L^{1_z}(t | \alpha)$. Since $\overline{\text{app}}(\overline{\text{app}}(A)) \subseteq \overline{\text{app}}(A)$, we have $S_L^{\overline{\text{app}}(\overline{\text{app}}(1_z))}(x | \alpha) \leq S_L^{\overline{\text{app}}(1_z)}(x | \alpha)$. That is, $1 \leq \bigvee_{t \in R_s(x)} S_L^{1_z}(t | \alpha)$. Thus, there exists $t \in R_s(x)$ such that $S_L^{1_z}(t | \alpha) = 1$. That is to say, $z \in R_s(x)$.

We can conclude that $y \in R_s(x) \wedge z \in R_s(y) \Rightarrow z \in R_s(x)$. Therefore, R is transitive.

(2) $\Leftrightarrow$ (3). We can directly obtain the statement according to Theorem 13(1). $\square$

4. Axiomatic Characterization of Generalized Rough Type-2 Fuzzy Sets

In this section, the axiomatic characterization of generalized rough type-2 fuzzy approximation operators is presented. To characterize the generalized rough type-2 fuzzy approximation operators by axioms, we define a constant type-2 fuzzy set $\vec{\beta} = \int_{x \in U} u_{\vec{\beta}}(x)/x = \int_{x \in U} \beta/x$, where β is secondary membership function and is not related to x . Clearly, $S_L^{\vec{\beta}}(x | \alpha) = S_L^{\vec{\beta}}(y | \alpha) = l(\alpha)$ and $S_U^{\vec{\beta}}(x | \alpha) = S_U^{\vec{\beta}}(y | \alpha) = r(\alpha)$ hold for any $\alpha \in [0, 1]$ and $x, y \in U$, where $l(\alpha)$ and $r(\alpha)$ are functions related to α .

Similar to [36], we define two special type-2 fuzzy sets denoted by $S_L^A(x | \alpha)$ and $S_U^A(x | \alpha)$ for any $\alpha \in [0, 1]$, $x \in U$, and $A \in F_2(U)$, respectively. The two special type-2 fuzzy sets satisfy the following properties: $S_L^{\overleftarrow{S_L^A(x|\alpha)}}(y | \eta) = S_L^A(x | \eta)$, $S_U^{\overleftarrow{S_L^A(x|\alpha)}}(y | \eta) = S_U^A(x | \eta)$, $S_L^{\overleftarrow{S_U^A(x|\alpha)}}(y | \eta) = S_L^A(x | \eta)$, and $S_U^{\overleftarrow{S_U^A(x|\alpha)}}(y | \eta) = S_U^A(x | \eta)$ for any $\eta \in [0, 1]$ and $y \in U$. Obviously, $S_L^A(x | \alpha)$ and $S_U^A(x | \alpha)$ are constant type-2 fuzzy sets.

Definition 17. Let $L, H : F_2(U) \rightarrow F_2(U)$ be two operators. They are called dual operators if for all $A \in F_2(U)$

$$(L1) (L(A^c))^c = H(A);$$

$$(H1) (H(A^c))^c = L(A).$$

Lemma 18. Let (U, R) be a generalized Pawlak approximation space and let $\underline{\text{app}}$ and $\overline{\text{app}}$ be the generalized rough type-2 fuzzy lower and upper approximation operators about (U, R) ; for any $A \in F_2(U)$ and constant type-2 fuzzy set $\overleftrightarrow{\beta}$, the following statements hold:

$$(1) \overline{\text{app}}(A \cap \overleftrightarrow{\beta}) = \overline{\text{app}}(A) \cap \overleftrightarrow{\beta};$$

$$(2) \underline{\text{app}}(A \cup \overleftrightarrow{\beta}) = \underline{\text{app}}(A) \cup \overleftrightarrow{\beta}.$$

Proof. (1) For any $\alpha \in [0, 1]$ and $x \in U$,

$$\begin{aligned} S_L^{\overline{\text{app}}(A \cap \overleftrightarrow{\beta})}(x | \alpha) &= \bigvee_{y \in R_s(x)} S_L^{A \cap \overleftrightarrow{\beta}}(y | \alpha) \\ &= \bigvee_{y \in R_s(x)} [S_L^A(y | \alpha) \wedge S_L^{\overleftrightarrow{\beta}}(y | \alpha)] \\ &= \bigvee_{y \in R_s(x)} [S_L^A(y | \alpha) \wedge I(\alpha)] \\ &= \left[\bigvee_{y \in R_s(x)} S_L^A(y | \alpha) \right] \wedge I(\alpha) \\ &= S_L^{\overline{\text{app}}(A)}(x | \alpha) \wedge S_L^{\overleftrightarrow{\beta}}(x | \alpha) \\ &= S_L^{\overline{\text{app}}(A) \cap \overleftrightarrow{\beta}}(x | \alpha). \end{aligned} \quad (25)$$

Similarly, we can obtain $S_U^{\overline{\text{app}}(A \cap \overleftrightarrow{\beta})}(x | \alpha) = S_U^{\overline{\text{app}}(A) \cap \overleftrightarrow{\beta}}(x | \alpha)$.

Thus, $\overline{\text{app}}(A \cap \overleftrightarrow{\beta}) = \overline{\text{app}}(A) \cap \overleftrightarrow{\beta}$.

(2) The proof procedure is similar to (1). $\square$

Lemma 19 (see [36]). For any $A \in F_2(U)$, $\alpha \in [0, 1]$, and $y \in U$, the following statements hold:

$$(1) A = \bigcap_{y \in U} [1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y | \alpha)}];$$

$$(2) A = \bigcap_{y \in U} [1_{U-\{y\}} \cup \overleftrightarrow{S_U^A(y | \alpha)}];$$

$$(3) A = \bigcup_{y \in U} [1_y \cap \overleftrightarrow{S_L^A(y | \alpha)}];$$

$$(4) A = \bigcup_{y \in U} [1_y \cap \overleftrightarrow{S_U^A(y | \alpha)}].$$

Theorem 20. Let $L, H : F_2(U) \rightarrow F_2(U)$ be two dual operators. Then there exists a crisp binary relation R on U such that, for all $A \in F_2(U)$, $L(A) = \underline{\text{app}}(A)$ and $H(A) = \overline{\text{app}}(A)$ hold if and only if L and H satisfy the axioms: for all $A, B \in F_2(U)$ and any constant type-2 fuzzy set $\overleftrightarrow{\beta}$,

$$(L2) L(A \cap B) = L(A) \cap L(B);$$

$$(L3) L(A \cup \overleftrightarrow{\beta}) = L(A) \cup \overleftrightarrow{\beta};$$

$$(L4) L(1_{U-\{y\}}) \in P(U) \text{ for any } y \in U.$$

Proof. “ $\Rightarrow$ ” It follows immediately from Theorem 13 and Lemma 18.

“ $\Leftarrow$ ” By employing L and axiom (L4), we can define a crisp relation R on U :

$$\begin{aligned} y \in R_s(x) &\iff R(x, y) = 1 \iff S_L^{L(1_{U-\{y\}})}(x | \alpha) \\ &= S_U^{L(1_{U-\{y\}})}(x | \alpha) = 0 \quad \text{for any } \alpha \in [0, 1], \\ y \notin R_s(x) &\iff R(x, y) = 0 \iff S_L^{L(1_{U-\{y\}})}(x | \alpha) \\ &= S_U^{L(1_{U-\{y\}})}(x | \alpha) = 1 \quad \text{for any } \alpha \in [0, 1]. \end{aligned} \quad (26)$$

Thus, $1 - R(x, y) = S_L^{L(1_{U-\{y\}})}(x | \alpha) = S_U^{L(1_{U-\{y\}})}(x | \alpha)$. Then, we can obtain

$$\begin{aligned} S_L^{\underline{\text{app}}(A)}(x | \alpha) &= S_L^{\underline{\text{app}}(\bigcap_{y \in U} [1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y | \alpha)}])}(x | \alpha) \\ &\quad \text{(According to Lemma 19 (1))} \\ &= S_L^{\bigcap_{y \in U} \underline{\text{app}}([1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y | \alpha)}])}(x | \alpha) \\ &\quad \text{(According to Theorem 13 (2))} \\ &= S_L^{\bigcap_{y \in U} [\underline{\text{app}}(1_{U-\{y\}}) \cup \overleftrightarrow{S_L^A(y | \alpha)}]}(x | \alpha) \\ &\quad \text{(According to Lemma 18 (2))} \\ &= \bigwedge_{y \in U} \left[S_L^{\underline{\text{app}}(1_{U-\{y\}})}(x | \alpha) \vee \overleftrightarrow{S_L^A(y | \alpha)}(x | \alpha) \right] \\ &= \bigwedge_{y \in U} \left\{ [1 - R(x, y)] \vee \overleftrightarrow{S_L^A(y | \alpha)}(x | \alpha) \right\} \\ &\quad \text{(According to Theorem 13 (5))} \end{aligned} \quad (27)$$

$$\begin{aligned} &= \bigwedge_{y \in U} \left[S_L^{L(1_{U-\{y\}})}(x | \alpha) \vee \overleftrightarrow{S_L^A(y | \alpha)}(x | \alpha) \right] \\ &= S_L^{\bigcap_{y \in U} [L(1_{U-\{y\}}) \cup \overleftrightarrow{S_L^A(y | \alpha)}]}(x | \alpha) \\ &= S_L^{\bigcap_{y \in U} [L(1_{U-\{y\}}) \cup \overleftrightarrow{S_L^A(y | \alpha)}]}(x | \alpha) \\ &\quad \text{(According to (L3))} \\ &= S_L^{L(\bigcap_{y \in U} [1_{U-\{y\}} \cup \overleftrightarrow{S_L^A(y | \alpha)}])}(x | \alpha) \\ &\quad \text{(According to (L2))} \end{aligned}$$

$$= S_L^{L(A)}(x | \alpha) \quad \text{(According to Lemma 19 (1)).}$$

Similarly, $S_U^{\overline{\text{app}}(A)}(x | \alpha) = S_U^{L(A)}(x | \alpha)$. Thus, $L(A) = \underline{\text{app}}(A)$.

We have $H(A) = \overline{\text{app}}(A)$ according to Theorem 13(1). $\square$

Theorem 21. Let $L, H : F_2(U) \rightarrow F_2(U)$ be two dual operators. Then there exists a crisp binary relation R on U such that, for all $A \in F_2(U)$, $L(A) = \underline{f}(A)$ and $H(A) = \overline{f}(A)$ hold if and only if L and H satisfy the axioms: for all $A, B \in F_2(U)$ and any constant type-2 fuzzy set $\overleftrightarrow{\beta}$,

$$(H2) \ H(A \cup B) = H(A) \cup H(B);$$

$$(H3) \ H(A \cap \overleftrightarrow{\beta}) = H(A) \cap \overleftrightarrow{\beta};$$

$$(H4) \ H(1_y) \in P(U) \text{ for any } y \in U.$$

Proof. “ $\Rightarrow$ ” It follows immediately from Theorem 13 and Lemma 18.

“ $\Leftarrow$ ” By employing H and axiom (H4), we can define a crisp relation R on U :

$$\begin{aligned} y \in R_s(x) &\iff R(x, y) = 1 \iff S_L^{H(1_y)}(x | \alpha) \\ &= S_U^{H(1_y)}(x | \alpha) = 1 \quad \text{for any } \alpha \in [0, 1], \\ y \notin R_s(x) &\iff R(x, y) = 0 \iff S_L^{H(1_y)}(x | \alpha) \\ &= S_U^{H(1_y)}(x | \alpha) = 0 \quad \text{for any } \alpha \in [0, 1]. \end{aligned} \quad (28)$$

Thus, $R(x, y) = S_L^{H(1_y)}(x | \alpha) = S_U^{H(1_y)}(x | \alpha)$. Then, we can obtain

$$\begin{aligned} S_L^{\overline{\text{app}}(A)}(x | \alpha) &= S_L^{\overline{\text{app}}(\bigcup_{y \in U} [1_y \cap \overleftrightarrow{S_L^A(y|\alpha)}])}(x | \alpha) \\ &\quad \text{(According to Lemma 19 (3))} \\ &= S_L^{\bigcup_{y \in U} \overline{\text{app}}(1_y \cap \overleftrightarrow{S_L^A(y|\alpha)})}(x | \alpha) \\ &\quad \text{(According to Theorem 13 (2))} \\ &= S_L^{\bigcup_{y \in U} [\overline{\text{app}}(1_y) \cap \overleftrightarrow{S_L^A(y|\alpha)}]}(x | \alpha) \\ &\quad \text{(According to Lemma 18 (1))} \\ &= \bigvee_{y \in U} \left[S_L^{\overline{\text{app}}(1_y)}(x | \alpha) \wedge S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x | \alpha) \right] \\ &= \bigvee_{y \in U} \left[R(x, y) \wedge S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x | \alpha) \right] \\ &\quad \text{(According to Theorem 13 (6))} \\ &= \bigvee_{y \in U} \left[S_L^{H(1_y)}(x | \alpha) \wedge S_L^{\overleftrightarrow{S_L^A(y|\alpha)}}(x | \alpha) \right] \\ &= S_L^{\bigcup_{y \in U} [H(1_y) \cap \overleftrightarrow{S_L^A(y|\alpha)}]}(x | \alpha) \\ &= S_L^{\bigcup_{y \in U} H(1_y \cap \overleftrightarrow{S_L^A(y|\alpha)})}(x | \alpha) \quad \text{(According to (H3))} \end{aligned}$$

$$\begin{aligned} &= S_L^{H(\bigcup_{y \in U} [1_y \cap \overleftrightarrow{S_L^A(y|\alpha)}])}(x | \alpha) \quad \text{(According to (H2))} \\ &= S_L^{H(A)}(x | \alpha) \quad \text{(According to Lemma 19 (3))}. \end{aligned} \quad (29)$$

Similarly, $S_U^{\overline{\text{app}}(A)}(x | \alpha) = S_U^{H(A)}(x | \alpha)$. Thus, $H(A) = \overline{\text{app}}(A)$.

We have $L(A) = \underline{\text{app}}(A)$ according to Theorem 13(1). $\square$

Definition 22. Suppose that $L, H : F_2(U) \rightarrow F_2(U)$ are two dual operators. If L satisfies axioms (L2), (L3), and (L4) or equivalently H satisfies axioms (H2), (H3), and (H4), then the system $(F_2(U), \cap, \cup, c, L, H)$ is called a generalized rough type-2 fuzzy set algebra, and L and H are called generalized type-2 fuzzy approximation operators.

Theorem 23. Let $L, H : F_2(U) \rightarrow F_2(U)$ be a pair of dual generalized type-2 fuzzy approximation operators. Then there exists a reflexive relation R on U such that, for all $A \in F_2(U)$, $L(A) = \underline{\text{app}}(A)$ and $H(A) = \overline{\text{app}}(A)$ hold if and only if L and H satisfy the following axioms:

$$(L5) \ L(A) \subseteq A;$$

$$(H5) \ A \subseteq H(A).$$

Proof. “ $\Rightarrow$ ” It follows immediately from Theorem 14.

“ $\Leftarrow$ ” It follows immediately from Theorems 14, 20, and 21. $\square$

Theorem 24. Let $L, H : F_2(U) \rightarrow F_2(U)$ be a pair of dual generalized type-2 fuzzy approximation operators. Then there exists a symmetric relation R on U such that, for all $A \in F_2(U)$, $L(A) = \underline{\text{app}}(A)$ and $H(A) = \overline{\text{app}}(A)$ hold if and only if L and H satisfy the following axioms:

$$(L6) \ \text{For any } \alpha \in [0, 1] \text{ and } x, y \in U, S_L^{L(1_{U-\{y\}})}(x | \alpha) = S_L^{L(1_{U-\{x\}})}(y | \alpha) \text{ and } S_U^{L(1_{U-\{y\}})}(x | \alpha) = S_U^{L(1_{U-\{x\}})}(y | \alpha) \text{ hold.}$$

$$(H6) \ \text{For any } \alpha \in [0, 1] \text{ and } x, y \in U, S_L^{H(1_x)}(y | \alpha) = S_L^{H(1_y)}(x | \alpha) \text{ and } S_U^{H(1_x)}(y | \alpha) = S_U^{H(1_y)}(x | \alpha) \text{ hold.}$$

Proof. “ $\Rightarrow$ ” It follows immediately from Theorem 15.

“ $\Leftarrow$ ” It follows immediately from Theorems 15, 20, and 21. $\square$

Theorem 25. Let $L, H : F_2(U) \rightarrow F_2(U)$ be a pair of dual generalized type-2 fuzzy approximation operators. Then there exists a transitive relation R on U such that, for all $A \in F_2(U)$, $L(A) = \underline{\text{app}}(A)$ and $H(A) = \overline{\text{app}}(A)$ hold if and only if L and H satisfy the following axioms:

$$(L7) \ L(A) \subseteq L(L(A));$$

$$(H7) \ H(H(A)) \subseteq H(A).$$

Proof. “ $\Rightarrow$ ” It follows immediately from Theorem 16.

“ $\Leftarrow$ ” It follows immediately from Theorems 16, 20, and 21. $\square$

Theorem 26. Let $L, H : F_2(U) \rightarrow F_2(U)$ be a pair of dual generalized type-2 fuzzy approximation operators. Then there exists an equivalence relation R on U such that, for all $A \in F_2(U)$, $L(A) = \underline{\text{app}}(A)$ and $H(A) = \overline{\text{app}}(A)$ hold if and only if L satisfies the axioms (L5)–(L7) and H satisfies the axioms (H5)–(H7).

Proof. “ $\Rightarrow$ ” It follows immediately from Theorems 14, 15, and 16.

“ $\Leftarrow$ ” It follows immediately from Theorems 14, 15, 16, 20, and 21. $\square$

5. The Attribute Reduction of Type-2 Fuzzy Information System

Let (U, A, F) be an information system. Here, U is the set of objects; that is, $U = \{x_1, x_2, \dots, x_n\}$. A is the attribute set; that is, $A = \{a_1, a_2, \dots, a_m\}$. F is the relation set of U and A ; that is, $F = \{f_j; 1 \leq j \leq m\}$, $(f_j : U \rightarrow V_j)$, and V_j is the domain of the attribute a_j .

We call (U, A, F, D, G) a decision information system, where (U, A, F) is the classical information system, A is the condition attribute set, and D is the decision attribute set; that is, $D = \{d_1, d_2, \dots, d_p\}$. G is the relation set of U and D , $G = \{g_j; 1 \leq j \leq p\}$, $(g_j : U \rightarrow V_j')$, and V_j' is the domain of the attribute d_j .

(U, A, F, D, G) is called a type-2 fuzzy information system, where (U, A, F) is the classical information system, and d_j ($1 \leq j \leq p$) is the type-2 fuzzy sets on U .

Definition 27. Let (U, A, F, D, G) be a type-2 fuzzy information system, and

$$L_B(x | \alpha) = [S_L^{R_B(d_1)}(x | \alpha), S_L^{R_B(d_2)}(x | \alpha), \dots, S_L^{R_B(d_p)}(x | \alpha), S_U^{R_B(d_1)}(x | \alpha), S_U^{R_B(d_2)}(x | \alpha), \dots, S_U^{R_B(d_p)}(x | \alpha)], \quad (30)$$

where R_B is a crisp relation determined by attribute set B . If $L_B(x | \alpha) = L_A(x | \alpha)$ for any $\alpha \in [0, 1]$ and $x \in U$, then B is called lower approximation consistent set of A .

If $B \subseteq A$ is the lower approximation consistent set of A and any subset of B are not the lower approximation consistent sets of A , then B is called the lower approximation reduction of A .

Definition 28. Let (U, A, F, D, G) be a type-2 fuzzy information system; for any $\alpha \in [0, 1]$ and $x, y \in U$,

$$D_{xy} = \begin{cases} \{a_i \in A; f_i(x) \neq f_i(y)\} & L_A(x | \alpha) \neq L_A(y | \alpha) \\ \emptyset & L_A(x | \alpha) = L_A(y | \alpha) \end{cases} \quad (31)$$

is called discernibility matrix of (U, A, F, D, G) .

Theorem 29. Let (U, A, F, D, G) be the type-2 fuzzy information system. For any $x, y \in U$ and $\alpha \in [0, 1]$, B is the lower approximation consistent set of A if and only if $L_A(x | \alpha) \neq L_A(y | \alpha) \Rightarrow [x]_{R_B} \cap [y]_{R_B} = \emptyset$.

Proof. “ $\Rightarrow$ ” If $[x]_{R_B} \cap [y]_{R_B} \neq \emptyset$, then $[x]_{R_B} = [y]_{R_B}$.

In the following, we only should prove $L_A(x | \alpha) = L_A(y | \alpha)$.

For any $j \leq p$, $S_L^{R_B(d_j)}(x | \alpha) = \bigwedge_{t \in [x]_{R_B}} S_L^{d_j}(t | \alpha) = \bigwedge_{t \in [y]_{R_B}} S_L^{d_j}(t | \alpha) = S_L^{R_B(d_j)}(y | \alpha)$. Similarly, we have $S_U^{R_B(d_j)}(x | \alpha) = S_U^{R_B(d_j)}(y | \alpha)$.

Since B is the lower approximation consistent set of A , we can obtain

$$\begin{aligned} S_L^{R_B(d_j)}(x | \alpha) &= S_L^{R_A(d_j)}(x | \alpha), \\ S_U^{R_B(d_j)}(x | \alpha) &= S_U^{R_A(d_j)}(x | \alpha), \\ S_L^{R_B(d_j)}(y | \alpha) &= S_L^{R_A(d_j)}(y | \alpha), \\ S_U^{R_B(d_j)}(y | \alpha) &= S_U^{R_A(d_j)}(y | \alpha). \end{aligned} \quad (32)$$

Hence, $S_L^{R_A(d_j)}(x | \alpha) = S_L^{R_A(d_j)}(y | \alpha)$ and $S_U^{R_A(d_j)}(x | \alpha) = S_U^{R_A(d_j)}(y | \alpha)$. That is, $L_A(x | \alpha) = L_A(y | \alpha)$. That is to say, there must be $[x]_{R_B} \cap [y]_{R_B} = \emptyset$ which hold for $L_A(x | \alpha) \neq L_A(y | \alpha)$.

“ $\Leftarrow$ ” Since $J([x]_{R_B}) = \{[z]_{R_A}; [z]_{R_A} \subseteq [x]_{R_B}\}$ forms a partition of $[x]_{R_B}$, we have

$$\begin{aligned} [x]_{R_B} &= \bigcup \{[z]_{R_A}; z \in [x]_{R_B}\} \\ &= \bigcup \{[z]_{R_A}; [z]_{R_A} \in J([x]_{R_B})\}. \end{aligned} \quad (33)$$

If $L_A(x | \alpha) \neq L_A(y | \alpha)$, then $[x]_{R_B} \cap [y]_{R_B} = \emptyset$. If $[x]_{R_B} \cap [y]_{R_B} \neq \emptyset$, then $L_A(x | \alpha) = L_A(y | \alpha)$.

Furthermore, we can obtain $[x]_{R_B} = [y]_{R_B}$ from $[x]_{R_B} \cap [y]_{R_B} \neq \emptyset$.

Thus,

$$\begin{aligned} S_L^{R_B(d_j)}(x | \alpha) &= \bigwedge_{t \in [x]_{R_B}} S_L^{d_j}(t | \alpha) \\ &= \bigwedge_{t \in \bigcup \{[z]_{R_A}; [z]_{R_A} \in J([x]_{R_B})\}} S_L^{d_j}(t | \alpha) \\ &= \bigwedge_{t \in [z]_{R_A} \wedge [z]_{R_A} \in J([x]_{R_B})} S_L^{d_j}(t | \alpha) \\ &= \bigwedge_{[z]_{R_A} \in J([x]_{R_B})} \left[\bigwedge_{t \in [z]_{R_A}} S_L^{d_j}(t | \alpha) \right] \\ &= \bigwedge_{[z]_{R_A} \in J([x]_{R_B})} S_L^{R_A(d_j)}(z | \alpha) \\ &= S_L^{R_A(d_j)}(x | \alpha). \end{aligned} \quad (34)$$

TABLE 1: A type-2 fuzzy information system.

Object	a_1	a_2	a_3	a_4	d_1	d_2
x_1	1	2	3	1	$T(0.7, 0.75, 0.8, 0.85)$	$T(0.2, 0.3, 0.4, 0.6)$
x_2	1	2	1	3	$S(0.5, 0.6, 0.8)$	$T(0.2, 0.7, 0.8, 0.96)$
x_3	1	2	3	1	$T(0.57, 0.7, 0.8, 0.85)$	$S(0.58, 0.7, 0.8)$
x_4	1	2	1	3	$T(0.8, 0.85, 0.9, 1)$	$S(0.86, 0.9, 0.96)$
x_5	1	3	1	2	$S(0.8, 0.9, 0.95)$	$T(0.6, 0.75, 0.85, 0.9)$
x_6	1	2	1	3	$S(0.4, 0.8, 0.95)$	$S(0.7, 0.86, 0.9)$
x_7	1	3	1	2	$T(0.75, 0.85, 0.9, 1)$	$S(0.75, 0.8, 0.96)$
x_8	1	2	3	1	$S(0.8, 0.9, 1)$	$T(0.47, 0.65, 0.8, 1)$

Similarly, $S_U^{R_B(d_j)}(x | \alpha) = S_U^{R_A(d_j)}(x | \alpha)$. That is, $L_B(x | \alpha) = L_A(x | \alpha)$. Hence, B is the lower approximation consistent set of A . $\square$

Theorem 30. Let (U, A, F, D, G) be the type-2 fuzzy information system. For any $x, y \in U$ and $\alpha \in [0, 1]$, B is the lower approximation consistent set of A if and only if $D_{xy} \neq \emptyset \Rightarrow B \cap D_{xy} \neq \emptyset$.

Proof. “ $\Rightarrow$ ” If B is the lower approximation consistent set of A and $D_{xy} \neq \emptyset$, then we have $L_A(x | \alpha) \neq L_A(y | \alpha)$. Thus, we can obtain $[x]_{R_B} \cap [y]_{R_B} = \emptyset$ from Theorem 29. That is, there exists $a_i \in B$ such that $f_i(x) \neq f_i(y)$. Hence, $B \cap D_{xy} \neq \emptyset$.

“ $\Leftarrow$ ” For any $\alpha \in [0, 1]$ and $x, y \in U$, if $L_A(x | \alpha) \neq L_A(y | \alpha)$ holds, in the following, we only should prove $[x]_{R_B} \cap [y]_{R_B} = \emptyset$.

If $D_{xy} \neq \emptyset$, then $B \cap D_{xy} \neq \emptyset$. That is, there exists $a_i \in B$ such that $f_i(x) \neq f_i(y)$. Thus, $[x]_{R_B} \neq [y]_{R_B}$. That is to say, $[x]_{R_B} \cap [y]_{R_B} = \emptyset$.

We have that B is the lower approximation consistent set of A from Theorem 29.

A discernibility function $f(S)$ for $S = (U, A, F, D, G)$ is a Boolean function of m Boolean variables $\bar{a}_1, \dots, \bar{a}_m$ corresponding to the attributes $a_1, \dots, a_m$, respectively, and is defined as follows:

$$f(S)(\bar{a}_1, \dots, \bar{a}_m) = \bigwedge \{ \bigvee (D_{xy}) : D_{xy} \neq \emptyset \}, \quad (35)$$

where $\bigvee (D_{xy})$ is the disjunction of all variables $\bar{a}$ such that $a \in D_{xy}$. In the sequel, we write a_i instead of $\bar{a}_i$ when no confusion can arise. $\square$

Theorem 31. Suppose $B \subseteq A$; then B is a lower approximation reduction of A if and only if B is the minimal set satisfying $D_{xy} \neq \emptyset \Rightarrow B \cap D_{xy} \neq \emptyset$.

Proof. The proof is straightforward by Theorem 30. $\square$

Let $g(S)$ be the reduced disjunctive form of $f(S)$ obtained from $f(S)$ by applying the multiplication and absorption laws as many times as possible. Then, there exist l and $A_k \subseteq A$ for $k = 1, \dots, l$ such that $g(S) = (\bigwedge A_1) \vee \dots \vee (\bigwedge A_l)$, where every element in A_1 only appears one time. We have the following theorem.

Theorem 32. If $\text{Red}(S)$ is the collection of all lower approximation reductions of A , then $\text{Red}(S) = \{A_1, \dots, A_l\}$.

Proof. The proof is omitted since this theorem is similar to the one in [14]. $\square$

The all lower approximation reductions of type-2 fuzzy information systems can be computed by discernibility matrix approach. In the following, we develop an algorithm to find all lower approximation reductions of type-2 fuzzy information system (U, A, F, D, G) .

Step 1. Decide on how many α -planes will be used. Call that number $\Delta + 1$. Regardless of $\Delta + 1$, $\alpha = 0$ and $\alpha = 1$ must always be used.

Step 2. For every α , x_i ($1 \leq i \leq n$), and d_j ($1 \leq j \leq p$), compute $S_L^{R_A(d_j)}(x_i | \alpha)$ and $S_U^{R_A(d_j)}(x_i | \alpha)$.

Step 3. For every α , x_i ($1 \leq i \leq n$), compute $L_A(x_i | \alpha)$.

Step 4. Compute D_{xy} using Definition 28.

Step 5. Define $f(S) = \bigwedge \{ \bigvee (D_{xy}) : D_{xy} \neq \emptyset \}$.

Step 6. Compute $g(S) = (\bigwedge A_1) \vee \dots \vee (\bigwedge A_l)$ by $f(S)$.

Step 7. Output all lower approximation reductions $\{A_1, \dots, A_l\}$.

Example 33. Table 1 gives a type-2 fuzzy information system, where the universe is $U = \{x_1, x_2, \dots, x_8\}$, the condition attribute set is $A = \{a_1, a_2, a_3, a_4\}$, and the decision attribute set is $D = \{d_1, d_2\}$, where $d_i \in F_2(U)$ ($i = 1, 2$). $T(\cdot, \cdot, \cdot, \cdot)$ denotes trapezoid function, the first parameter and fourth parameter of $()$ denote bottom left and right endpoint, respectively, and the second parameter and third parameter of $()$ denote top left and right endpoint, respectively. Furthermore, $S(\cdot, \cdot, \cdot)$ denotes triangular function, the first parameter and third parameter of $()$ denote bottom left and right endpoint, respectively, and the second parameter of $()$ denotes apex.

Obviously, the universe U can be divided into three basic equivalence classes in accordance with the condition attribute set $A = \{a_1, a_2, a_3, a_4\}$:

$$\begin{aligned} \frac{U}{R_A} &= \{X_1, X_2, X_3\} \\ &= \{\{x_1, x_3, x_8\}, \{x_2, x_4, x_6\}, \{x_5, x_7\}\}. \end{aligned} \quad (36)$$

TABLE 2: The approximation results of the type-2 fuzzy object.

U/R_A	$S_L^{R_A(d_1)}(x 0.5)$	$S_U^{R_A(d_1)}(x 0.5)$	$S_L^{R_A(d_2)}(x 0.5)$	$S_U^{R_A(d_2)}(x 0.5)$
X_1	0.6350	0.8250	0.2500	0.5000
X_2	0.5500	0.7000	0.4500	0.8800
X_3	0.7800	0.8800	0.5600	0.8750

TABLE 3: The discernibility matrix.

Object	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	$\emptyset$	$\{a_3, a_4\}$	$\emptyset$	$\{a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\emptyset$
x_2	$\{a_3, a_4\}$	$\emptyset$	$\{a_3, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\{a_3, a_4\}$
x_3	$\emptyset$	$\{a_3, a_4\}$	$\emptyset$	$\{a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\emptyset$
x_4	$\{a_3, a_4\}$	$\emptyset$	$\{a_3, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\{a_3, a_4\}$
x_5	$\{a_2, a_3, a_4\}$	$\{a_2, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_3, a_4\}$
x_6	$\{a_3, a_4\}$	$\emptyset$	$\{a_3, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\{a_3, a_4\}$
x_7	$\{a_2, a_3, a_4\}$	$\{a_2, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_4\}$	$\emptyset$	$\{a_2, a_3, a_4\}$
x_8	$\emptyset$	$\{a_3, a_4\}$	$\emptyset$	$\{a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\{a_3, a_4\}$	$\{a_2, a_3, a_4\}$	$\emptyset$

In the following, we should decide on how many α -planes will be used, where $\alpha \in [0, 1]$. Call that number $\Delta + 1$. Regardless of $\Delta + 1$, $\alpha = 0$ and $\alpha = 1$ must always be used. If $\Delta + 1 = 11$, then $\alpha = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$. When $\alpha = 0.5$, the approximation results of the type-2 fuzzy object are summarized in Table 2.

Similarly, we can compute the other approximation results for $\alpha \neq 0.5$. The discernibility matrix of (U, A, F, D, G) is presented in Table 3.

It is easy to compute $\{a_2, a_4\}$ and $\{a_2, a_3\}$ as the lower approximation reductions of (U, A, F, D, G) .

6. Conclusions

The α -plane representation method of type-2 fuzzy sets has been extensively studied and can greatly reduce the computational workload. Based on α -plane theory, the rough type-2 fuzzy approximation operators and the generalized rough type-2 fuzzy approximation operators have been defined, and their properties have been derived. The connections between special binary relations and generalized rough type-2 fuzzy upper and lower approximation operators are examined. The axiomatics are provided to fully characterize the generalized rough type-2 fuzzy upper and lower approximation operators. The attribute reduction of the type-2 fuzzy information system has been investigated, and an attribute reduction algorithm to find all reductions of the type-2 fuzzy information system has also been developed. Further research will focus on the applications of the proposed rough type-2 fuzzy set model.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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