Research Article

The Numerical Analysis of Two-Sided Space-Fractional Wave Equation with Improved Moving Least-Square Ritz Method

Rongjun Cheng, 1 Hongxia Ge, 1 and Yong Wu 2

1 Faculty of Maritime and Transportation, Ningbo University, Ningbo 315211, China
2 Ningbo Institute of Technology, Zhejiang University, Ningbo 315100, China

CorrespondenceshouldbeaddressedtoYongWu;wuyong@nit.zju.edu.cn

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1. Introduction

Due to extensive use in the fields of dynamics [1], fluid mechanics [2], viscoelasticity [3], materials [4], hydrology [5], biology [6, 7], porous media [8], physics [9, 10], engineering [11, 12], and so on, fractional partial differential equations have become a hot research topic. Consequently, scholars pay much attention to the analytical solutions of fractional partial differential equation (PDE) of physical interest. Unfortunately, a fractional PDE has no exact solution in many cases owing to complex series or special functions. So it is extremely important and necessary to resort to numerical solutions.

The fractional diffusion-wave equation has been an interesting topic to invest in during the past decades. There is already some important progress for the fractional diffusion equation or advection-diffusion equations. Deng [13–15] presented the numerical method for fractional diffusion equations. Liu et al. [16] used the difference method for space-time fractional equation and presented the stability and convergence. Meerschaert and Tadjeran [17] applied the finite difference approximation for space-fractional equations, Meng [18] put forward a new approach for solving fractional partial differential equations, Sousa [19] developed numerical approximations for fractional diffusion equations via splines, Zhou and Wu [20] proposed the finite element multigrid method for the boundary value problem of fractional advection dispersion equation. Nigmatullin [21] presented the fractional diffusion equation to describe the diffusion in porous media. Mainardi [22, 23] has shown that the FWE describes the propagation of mechanical diffusive waves in viscoelastic media. Much study has been done for the time fractional PDE in [24–26]. In fact, most of the works focus on the time fractional PDE. In another latest paper, Sweilam et al. considered a 1D fractional wave equation in [27], Deng et al. used the alternating direction implicit algorithm for the space-fractional equation in [28], Jia and Wang used the fast finite difference methods for space-fractional PDE with fractional derivative boundary conditions in [29], and Guan and Gunzburger applied the finite element method (FEM) for the space-time fractional PDE in [30].

In the past few decades, meshless methods have already been a hot research topic in computational mechanics. Meshless methods also become important and powerful tools to research and analyze kinds of PDE. Researchers have presented many kinds of meshless methods, including the diffuse element method (DEM) [31], the smoothed particle
where \(w(x - x_i)\) are compact weight functions and \(x_i\) are the nodes. Equation (2) can be expressed in the matrix form

\[
J = (pa - u)^T W(x) (pa - u),
\]

where

\[
u = (u_1, u_2, \ldots, u_n),
\]

\[
p = \begin{bmatrix}
p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) \\
p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n)
\end{bmatrix},
\]

\[
W(x) = \begin{bmatrix}
w(x - x_1) & 0 & \cdots & 0 \\
0 & w(x - x_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & w(x - x_n)
\end{bmatrix}.
\]

In order to find the coefficients \(a(x)\), we take the extremum of \(J\) by

\[
\frac{\partial J}{\partial a} = A(x) a(x) - B(x) u = 0
\]

which will get the following equation system:

\[
A(x) a(x) = B(x) u.
\]

If the functions \(p_1(x), p_2(x), \ldots, p_m(x)\) meet with the following conditions

\[
\left(p_k, p_j\right) = \sum_{i=1}^{n} w_j p_k(x_i) p_j(x_i) = \begin{cases}
0 & k \neq j \\
A_k & k = j
\end{cases}
\]

\[(k, j = 1, 2, \ldots, m),\]

it will be called a weighted orthogonal function set with a weight function \(\{w_j\}\) with points \(\{x_i\}\). The orthogonal function set \(p = (p_i)\) can be obtained by using the Schmidt method,

\[
p_1 = 1,
\]

\[
p_i = r^{-1} - \sum_{k=1}^{i-1} \left(\frac{r^{i-1}_k}{p_k, p_k}\right) p_k, \quad i = 2, 3, \ldots
\]
Equation (6) can be rewritten as

\[
\begin{bmatrix}
(p_1, p_1) & 0 & \cdots & 0 \\
0 & (p_2, p_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (p_m, p_m)
\end{bmatrix}
\begin{bmatrix}
a_1(x) \\
a_2(x) \\
\vdots \\
a_m(x)
\end{bmatrix} = \begin{bmatrix}
(p_1, u_1) \\
(p_2, u_2) \\
\vdots \\
(p_m, u_m)
\end{bmatrix}
\]  \tag{9}

The coefficients \(a_i(x)\) can be easily founded as follows:

\[
a_i(x) = \frac{(p_i, u_j)}{(p_i, p_j)}; \quad (i = 1, 2, \ldots, m);
\]

that is,

\[
a(x) = \bar{A}(x) B(x) u,
\]

where

\[
\bar{A}(x) = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}
\]

From (1), \(u^h(x)\) is expressed as

\[
u^h(x) = \Phi(x) u = \sum_{l=1}^{n} \Phi_l(x) u_l,
\]

where shape function \(\Phi(x)\) is

\[
\Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x))
\]

\[
= p^T(x) \bar{A}(x) B(x).
\]

Taking derivatives of (14), we will get derivatives of shape function

\[
\Phi_{ij}(x) = \sum_{j=1}^{m} \left[ p_{ij} (\bar{A}B)_{jj} + p_j (\bar{A}B + \bar{A}B_j)_{ij} \right].
\]

The cubic spline weight function is chosen as follows:

\[
\omega_j = \omega(x - x_j) = \omega(r),
\]

where

\[
r = \frac{d_i}{d_m}, \quad d_i = \|x - x_i\|, \quad d_m = d_{\text{max}} c_i,
\]

where \(d_{\text{max}}\) is called a scaling parameter and distance \(c_i\) is chosen to make matrix \(M(x)\) which is no longer singular.

### 3. IMLS-Ritz Formulation for the Two-Sided Space-Fractional Wave Equation

Consider the following two-sided space-fractional wave equation:

\[
\frac{\partial^2 u(x,t)}{\partial t^2} = c_+ (x) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + c_- (x) \frac{\partial^\alpha u(x,t)}{\partial x^{-\alpha}} + d(x,t), \quad a \leq x \leq b, \quad 0 \leq t \leq T
\]

with initial conditions

\[
u(x,0) = u_0 (x),
\]

\[
u_t (x,0) = u_1 (x)
\]

and boundary conditions

\[
u(a,t) = u(b,t) = 0,
\]

where the parameter \(\alpha\) describes the fractional order of spatial derivatives with \(1 < \alpha \leq 2\). Function \(d(x,t)\) refers to a source term, and the coefficient functions \(c_+(x) > 0\) and \(c_-(x) > 0\) to transport related coefficients.

In order to establish the numerical approximation scheme, points \(x_i = (i-1)\Delta x, \quad i = 1, 2, 3, \ldots, N\), are considered, where \(\Delta x = (b-a)/(N-1)\). \(x_1 = a\) and \(x_N = b\) are the boundary points; \(\Delta t = n\Delta t, \quad n = 0, 1, 2, 3, \ldots\), where \(\Delta t\) is the time interval.

The left-handed and right Riemann-Liouville fractional derivatives of order \(\alpha\) are defined as [58]

\[
(D^\alpha_{-a} f)(x) = \frac{\partial^\alpha f(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} f(t) (x-t)^{n-\alpha-1} dt
\]

\[
\forall x \in [a,b], \quad \alpha > 0
\]

\[
(D^\alpha_{+a} f)(x) = \frac{\partial^\alpha f(x,t)}{\partial x^{-\alpha}} = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{b} f(t) (t-x)^{n-\alpha-1} dt
\]

\[
\forall x \in [a,b], \quad \alpha > 0,
\]

where \(n\) is an integer such that \(n-1 < \alpha \leq n\).

The weighted integral form of (18a) is obtained as follows:

\[
\int_{x} w \left[ \frac{\partial^2 u(x,t)}{\partial t^2} - c_+ (x) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - c_- (x) \frac{\partial^\alpha u(x,t)}{\partial x^{-\alpha}} - d(x,t) \right] dt = 0.
\]
Define the energy functional $\Pi(u)$ as

$$
\Pi (u) = \int_\Gamma u \frac{\partial^2 u}{\partial t^2} d\Gamma - \int_\Gamma u \left( c_1 \frac{\partial^2 u}{\partial x^a} + c_2 \frac{\partial^2 u}{\partial x^b} + d \right) d\Gamma.
$$

Due to the boundary condition, the modified energy functional becomes

$$
\Pi^* (u) = \int_\Gamma u \frac{\partial^2 u}{\partial t^2} d\Gamma - \int_\Gamma u \left( c_1 \frac{\partial^2 u}{\partial x^a} + c_2 \frac{\partial^2 u}{\partial x^b} + d \right) d\Gamma
$$

By (13), we can derive the approximate function as follows:

$$
u^n (x,t) = \sum_{i=1}^n \Phi_i (x) \cdot T_i (t) = \Phi (x) \cdot T
$$

where

$$
\Phi (x) = (\Phi_1 (x), \Phi_2 (x), \ldots, \Phi_n (x))
$$

$$
T = (T_1 (t), T_2 (t), \ldots, T_n (t))^T
$$

Substituting (23) into (22), by applying the Ritz minimization procedure to $\Pi^* (u)$, we will derive

$$
\frac{\partial \Pi^* (u)}{\partial \Delta} = 0,
$$

where

$$
\Delta = T_i (t), \frac{\partial^2 T_i (t)}{\partial t^2}, \quad I = 1, 2, \ldots, n.
$$

The results can be expressed as

$$
C T + K T = F,
$$

where

$$
C = \int_\Gamma \Phi^T (x) \Phi (x) d\Gamma
$$

$$
K = \alpha_1 \Phi^T (x) \Phi (x) \bigg|_{x=a,b} - \int_\Gamma \Phi^T (x)
$$

By the shifted Grünwald formula, we can discretize the Riemann-Liouville operator [18]

$$
\frac{\partial^\alpha \Phi (\bar{x}_i)}{\partial x^a} = \frac{1}{h^\alpha} \sum_{j=0}^i w_j \Phi (\bar{x}_i - (j - 1) h) + O (h)
$$

$$
\frac{\partial^\alpha \Phi (\bar{x}_i)}{\partial x^a} = \frac{1}{h^\alpha} \sum_{j=0}^{M+1} w_j \Phi (\bar{x}_i + (j - 1) h) + O (h)
$$

where $\{\bar{x}_k\}$ ($k = 1, 2, \ldots, M$, $M = 2N - 1$) is the set of nodes and Gauss points and $w_j$ are the normalized Grünwald weights. The corresponding coefficients $w_j$ can be easily calculated by iteration formula as follows:

$$
w_0 = 1,
$$

$$
w_j = \left( 1 - \frac{\alpha + 1}{k} \right) w_{j-1}.
$$

Substituting (28a), (28b), and (29) into (26) and discrete time by center difference method, we obtain

$$
C U_{n+1} - 2U_n + U_{n-1} + K \frac{U_{n+1} + U_n}{2} = F_{n+1} + F_n,
$$

where

$$
K = \alpha_1 \Phi^T (x) \Phi (x) \bigg|_{x=a,b} - \int_\Gamma \Phi^T (x) \cdot \left( \frac{1}{h^\alpha} \left( \sum_{j=0}^i g_j \Phi (x_i - (j - 1) h) + \sum_{j=0}^{M+1} g_j \Phi (x_i + (j - 1) h) \right) \right) d\Gamma.
$$

The numerical solution of the space-fractional wave equation is obtained by iterative calculation.

4. Numerical Results

In order to verify the validity and correctness of the proposed IMLS-Ritz method for the space-fractional wave equation, examples are studied and the numerical results are presented. Note that, in all examples considered, the cubic spline function is chosen as weight function and the linear bases are chosen in this paper.

**Example 1** (left-handed space-fractional wave equation). Consider the following left-handed space-fractional wave equation:

$$
\frac{\partial^2 u (x,t)}{\partial t^2} = \Gamma (1.2) x^{1.8} \frac{\partial^{1.8} u (x,t)}{\partial x^{1.8}} + d (x,t)
$$

$$
0 < x < 2
$$
Table 1: Maximum error for Example 1 at $t = 1$.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$\Delta x$</th>
<th>IMLS-Ritz method</th>
<th>Finite difference method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>$2^{-3}$</td>
<td>0.0723</td>
<td>0.1128</td>
</tr>
<tr>
<td>0.005</td>
<td>$2^{-4}$</td>
<td>0.0483</td>
<td>0.0511</td>
</tr>
<tr>
<td>0.002</td>
<td>$2^{-5}$</td>
<td>0.0204</td>
<td>0.0270</td>
</tr>
<tr>
<td>0.002</td>
<td>$2^{-6}$</td>
<td>0.0115</td>
<td>0.0137</td>
</tr>
</tbody>
</table>

Example 2 (two-sided space-fractional). Consider the following left-handed and right-handed space-fractional wave equation:

$$
\frac{\partial^2 u(x,t)}{\partial t^2} = \Gamma (1.2)x^{1.8}\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + \Gamma (1.2)(2-x)^{1.8}\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + d(x,t) \quad 0 < x < 2
$$

with initial conditions

$$u (x, 0) = 4x^2 (2 - x)^2, \quad \left. u_t (x, 0) = -4x^2 (2 - x)^2 \right. \quad (38)$$

with boundary conditions

$$u (0, t) = u (2, t) = 0, \quad (39)$$

Using IMLS-Ritz method to solve the equation with penalty factor $\alpha_1 = 10^7$, time step length $\Delta t = 0.001$, space step length $\Delta x = 0.0125$, and $d_{\text{max}} = 3.8$. Table 1 shows numerical results obtained by the IMLS-Ritz method. The maximum error at time $t = 1$ between the exact solution and the numerical solution and finite difference method [27] at different values of $\Delta x$ and $\Delta t$ is shown in Table 1. In Figure 1, the numerical and analytical solution are plotted at time $t = 0.5, 1$, and $1.5$, respectively. The surface of the numerical and analytical solution is plotted in Figures 2 and 3, respectively. Numerical results show that the IMLS-Ritz method is very effective and accurate.
Table 2: Maximum error for Example 2 at $t = 2$.

<table>
<thead>
<tr>
<th>$\Delta t$</th>
<th>$\Delta x$</th>
<th>IMLS-Ritz method</th>
<th>Finite difference method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.020</td>
<td>$2^{-3}$</td>
<td>0.0382</td>
<td>0.0379</td>
</tr>
<tr>
<td>0.0066</td>
<td>$2^{-4}$</td>
<td>0.0135</td>
<td>0.0164</td>
</tr>
<tr>
<td>0.0050</td>
<td>$2^{-6}$</td>
<td>0.0037</td>
<td>0.0042</td>
</tr>
<tr>
<td>0.0033</td>
<td>$2^{-5}$</td>
<td>0.0079</td>
<td>0.0083</td>
</tr>
</tbody>
</table>

Figure 4: Numerical solution and exact solution of $u(x, t)$ when $t = 0.5, 1$, and $2$ (Example 2).

where the source function is

$$d(x, t) = 4e^{-t}x^2(2-x)^2 - 32e^{-t}\left[x^2 + (2-x)^2 - 2.5\left(x^3 + (2-x)^3\right) + \frac{25}{22}\left(x^4 + (2-x)^4\right)\right].$$

The exact solution is

$$u(x, t) = 4e^{-t}x^2(2-x)^2.$$ (41)

The IMLS-Ritz method is applied to solve the above equation with penalty factor $\alpha_1 = 10^7$ and time step length $\Delta t = 0.002$ and $d_{\text{max}} = 3.8$. Table 2 shows numerical results obtained by the IMLS-Ritz method. The maximum error at time $t = 2$ between the exact solution and the numerical solution and finite difference method [27] at different values of $\Delta x$ and $\Delta t$ is shown in Table 2. In Figure 4, the numerical and analytical solution are plotted at time $t = 0.5, 1$, and 2, respectively. The surface of the numerical and analytical solution is plotted in Figures 5 and 6, respectively. Numerical results show that the IMLS-Ritz method is very effective and accurate.

Example 3 (two-sided space-fractional). Consider the following left-handed and right-handed space-fractional wave equation:

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^{1.8} u(x, t)}{\partial x^{1.8}} + \frac{\partial^{1.8} u(x, t)}{\partial x^{-1.8}}, \quad 0 < x < 5$$ (42)

with initial conditions

$$u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = 0$$ (43)

with boundary conditions

$$u(0, t) = u(5, t) = 0.$$ (44)

Particularly, if the fractional order of spatial derivatives $\alpha = 2$, (42) will be a standard wave equation, and the exact solution to this problem in case $\alpha = 2$ is

$$u(x, t) = \sin(\pi x) \cos(2\pi t).$$ (45)
the energy expressions, the final algebraic equations system is obtained. The system obtained by IMLS technique will be not ill-conditioned any more, and the solution can be easily obtained without matrix inversion. Because of the simplicity of numerical implementation, the proposed IMLS-Ritz method will substitute for the difference method and the finite element method for solving space-fractional wave equation and other fractional partial differential equations.

### Competing Interests

The authors declare that they have no competing interests.

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