Research Article

Intuitionistic Fuzzy Time Series Forecasting Model Based on Intuitionistic Fuzzy Reasoning

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Fuzzy sets theory cannot describe the data comprehensively, which has greatly limited the objectivity of fuzzy time series in uncertain data forecasting. In this regard, an intuitionistic fuzzy time series forecasting model is built. In the new model, a fuzzy clustering algorithm is used to divide the universe of discourse into unequal intervals, and a more objective technique for ascertaining the membership function and nonmembership function of the intuitionistic fuzzy set is proposed. On these bases, forecast rules based on intuitionistic fuzzy approximate reasoning are established. At last, contrast experiments on the enrollments of the University of Alabama and the Taiwan Stock Exchange Capitalization Weighted Stock Index are carried out. The results show that the new model has a clear advantage of improving the forecast accuracy.

1. Introduction

Time series forecasting theory plays an important role in the fields of economy, society, and nature. However, conventional forecasting methods are mainly based on statistical analysis, such as ARMA and ARIMA. These methods have two drawbacks: firstly, they need lots of historical data meeting certain conditions; secondly, they cannot handle linguistic values or imprecise data. Therefore, Song and Chissom [1–3] proposed the fuzzy time series (FTS) forecasting model which could effectively manage fuzzy information with the combination of fuzzy sets and fuzzy logic.

The basic idea of FTS is that historical data are expressed as fuzzy sets and series variation trends are expressed as fuzzy relations. Data are forecasted by fuzzy reasoning while there are not enough historical data or just some imprecise data. The FTS theory has aroused wide concerns since its first appearance, and lots of excellent works have been done in the past twenty years. Song et al. [4] built a fuzzy stochastic fuzzy time series model focusing on a special kind of fuzzy historical data whose probabilities are also fuzzy sets. Hwang et al. [5] used the variations of historical data instead of the data themselves to build a time-variant FTS model. This model is quite different from Song and Chissom's one, but it got a more

accurate result. Cheng et al. [6] used the probabilities of fuzzy relations to construct a weighted 0-1 matrix for forecasting, which is simpler to calculate than previous models. Aladag and coworkers [7, 8] used an optimization algorithm and artificial neural networks to build a few high-order models, which were obviously superior to first-order models. Singh and Borah [9] developed the model of reference [6] by using the importance of fuzzy relations as their weights. According to this change, they also proposed a new defuzzification method. Huarng [10] discussed the effects of different lengths of intervals to forecast accuracy at the first time and put forward distribution-based length and average-based length to approach this issue. Lu et al. [11, 12] integrated the information granules and granular computing with Chen's method to get better approaches for universe partition. S.-M. Chen and S.-W. Chen [13] classified the fuzzy relations into three groups: the “downtrend” group, “equal-trend” group, and “uptrend” group. The probabilities of three groups were used to build a two-factor second-order model. In FTS models, the Zadeh fuzzy set [14] is used to fuzzify historical data; namely, there is only one attribute—membership measuring the subjection degree. This is neither objective nor comprehensive and consequently limits the FTS models to deal with uncertain information and improve their forecast accuracy.

The intuitionistic fuzzy set [15] has three indicators to describe data: the membership, the nonmembership, and the intuitionistic index, which make it more objective and careful in fuzzy information description. Therefore, Castillo et al. [16] combined the intuitionistic fuzzy set with time series analysis and put forward an intuitionistic fuzzy reasoning system for data forecasting. However, the main structure was just a weighted average of two subreasoning systems based on membership and nonmembership functions. Joshi and Kumar [17] built the first intuitionistic fuzzy time series (IFTS) forecasting model based on the FTS model, but there is a drawback in the construction of intuitionistic fuzzy set: the intuitionistic index is 0.2 all the time. Zheng et al. [18, 19] used the intuitionistic fuzzy c-means clustering algorithm to get unequal intervals of the universe of discourse, and they also used the trace-back mechanism and vector quantization to forecast. Their models effectively advanced the forecast results, but how to transform the historical data into a suitable form for the intuitionistic fuzzy c-means clustering algorithm is still an urgent problem. The introduction of intuitionistic fuzzy sets dramatically extends the ability for time series to handle with uncertain and imprecise data. It also sets a new research direction for FTS. However, the study on IFTS theory is just getting started. There are only a few academic achievements, and there is a lack of unification and theoretical depth; the forecast accuracy needs further improvement as well.

In view of the above problems, we propose an IFTS model with modifications in three aspects: universe partition, intuitionistic fuzzy set construction, and forecast rules establishment. The paper is organized as follows: Section 2 briefly reviews some concepts on intuitionistic fuzzy sets and intuitionistic fuzzy time series. Section 3 details how to establish the novel IFTS model in four steps. In Section 4, several existing models as well as the proposed model are used to perform profound experiments and validate the effectiveness of the proposed model. Finally, Section 5 gives some conclusions.

2. Basic Concepts

In this section, some basic definitions of intuitionistic fuzzy set and IFTS are presented.

Definition 1. Let X be a finite universal set. An intuitionistic fuzzy set A in X is an object having the form

$$A = \{ \langle x, \mu_A(x), \gamma_A(x) \rangle \mid x \in X \}, \quad (1)$$

where the function $\mu_A(x) : X \rightarrow [0, 1]$ defines the degree of membership and the function $\gamma_A(x) : X \rightarrow [0, 1]$ defines the degree of nonmembership of the element $x \in X$ to set A . For every $x \in X$, $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$. $\pi_A(x) = 1 - \mu_A(x) - \gamma_A(x)$ is called the intuitionistic index of x in A . It is the hesitancy of x to A .

When $X = \{x_1, x_2, \dots, x_n\}$ is discrete, the intuitionistic fuzzy set A can be noted as

$$A = \sum_{i=1}^n \frac{\langle \mu_A(x_i), \gamma_A(x_i) \rangle}{x_i}, \quad x_i \in X. \quad (2)$$

Definition 2. Let X and Y be two finite universal sets. A binary intuitionistic fuzzy relation R from X to Y is an intuitionistic fuzzy set in the direct product space $X \times Y$:

$$R = \{ \langle \langle x, y \rangle, \mu_R(x, y), \gamma_R(x, y) \rangle \mid (x, y) \in X \times Y \}, \quad (3)$$

where $0 \leq \mu_A(x, y) + \gamma_A(x, y) \leq 1$, $\forall (x, y) \in X \times Y$ for $\mu_R(x, y) : X \times Y \rightarrow [0, 1]$, and $\gamma_R(x, y) : X \times Y \rightarrow [0, 1]$.

Definition 3. Let $X(t)$ ($t = 1, 2, \dots$), a subset of $\mathbf{R}$, be the universe of discourse on which intuitionistic fuzzy sets $f_i(t) = \langle \mu_i(X(t)), \gamma_i(X(t)) \rangle$ ($i = 1, 2, \dots$) are defined. $F(t) = \{f_1(t), f_2(t), \dots\}$ is a collection of $f_i(t)$ and defines an intuitionistic fuzzy time series on $X(t)$.

Definition 4. Let $R(t, t-1)$ be an intuitionistic fuzzy relation from $F(t-1)$ to $F(t)$. Suppose that $F(t)$ is caused only by $F(t-1)$, denoted as

$$F(t) = F(t-1) \circ R(t, t-1), \quad (4)$$

where “ $\circ$ ” is the intuitionistic fuzzy compositional operator. Then $R(t, t-1)$ is called a first-order intuitionistic fuzzy logical relationship of $F(t)$.

Definition 5. If $R(t, t-1)$ is independent of time t ,

$$R(t, t-1) = R(t-1, t-2) \quad \forall t, \quad (5)$$

then $F(t)$ is called a time-invariant intuitionistic fuzzy time series. Otherwise, $F(t)$ is called a time-variant intuitionistic fuzzy time series.

The IFTS model studied in this paper is first order and time-invariant.

3. The Novel Intuitionistic Fuzzy Time Series Forecasting Model

The IFTS model can be summarized in four steps as the FTS model:

- (1) Define and partition the universe of discourse.
- (2) Construct intuitionistic fuzzy set and intuitionistically fuzzify the historical data.
- (3) Establish forecast rules and get the forecasted value.
- (4) Defuzzify and output the forecast result.

The rest of this section will detail the proposed IFTS model following this procedure.

3.1. Unequal Universe Partition Based on Fuzzy Clustering. First of all, the universe of discourse $U = [x_{\min} - \varepsilon_1, x_{\max} + \varepsilon_2]$ should be defined, where $x_{\min}$ and $x_{\max}$ are the minimum and maximum historical data, respectively. ε_1 and ε_2 are two proper positive numbers. Usually, for simplicity, ε_1 and ε_2 are chosen to round down $x_{\min}$ and round up $x_{\max}$ to two proper integers.

Secondly, partition the universe U into several intervals. References [21–23] proved that unequal intervals do not only

have actual meanings for regular understanding but also lead to a better outcome than equal ones. Some researchers [22, 24, 25] have already made achievements in this step by adopting methods such as genetic algorithms, particle swarm optimization, and fuzzy c-means clustering algorithm. But these kinds of methods usually need a huge amount of historical data to get a good performance, which deviates from the small database of historical information of IFTS. What is more, in practice, the IFTS model is generally used for problems which have not too many historical data such as in economic and environmental forecasting. So in this paper, we decide to use a more convenient and real-time method to solve this problem [26].

Let $X = \{x_1, x_2, \dots, x_n\}$ be the universe of objects to be classified, where $x_j = (x_{j1}, x_{j2}, \dots, x_{jm})$ ($j = 1, 2, \dots, n$) has m characteristics. Let $\mathbf{R} = (r_{jk})_{n \times n}$ be the similarity matrix of X , where r_{jk} is the similarity between x_j and x_k ($j, k = 1, 2, \dots, n$). A maximum spanning tree is a tree with all x_j being the vertices and r_{jk} being the weights of every edge. Let $\lambda \in [0, 1]$ be the clustering threshold. Cutting down the edges whose weights are smaller than λ , we can get a few subtrees. Hence, the vertices of different subtrees make up different groups. The main steps are as follows.

Step 1. Standardize historical data. Since the elements of fuzzy matrix should be in $[0, 1]$, data in different dimensions should be transformed into the interval $[0, 1]$ to meet the requirement of similarity matrix $\mathbf{R}$ [26]. Generally, two kinds of transformation are required.

(1) Standard deviation transformation is as follows:

$$x'_{jh} = \frac{x_{jh} - \bar{x}_h}{s_h}, \quad (6)$$

where $\bar{x}_h = (1/n) \sum_{j=1}^n x_{jh}$ and $s_h = \sqrt{(1/n) \sum_{j=1}^n (x_{jh} - \bar{x}_h)^2}$, $j = 1, 2, \dots, n$, $h = 1, 2, \dots, m$. With this transformation, the mean of every variable becomes 0, the standard deviation becomes 1, and the dimensional differences are eliminated. But it cannot ensure that x'_{jh} will locate in $[0, 1]$.

(2) Range transformation is as follows:

$$x''_{jh} = \frac{x'_{jh} - \min_{1 \leq j \leq n} \{x'_{jh}\}}{\max_{1 \leq j \leq n} \{x'_{jh}\} - \min_{1 \leq j \leq n} \{x'_{jh}\}}, \quad (7)$$

where, obviously, $0 \leq x''_{jh} \leq 1$.

Step 2. Establish the fuzzy similarity matrix $\mathbf{R}$.

Let r_{jk} be the similarity between x''_j and x''_k ; then we will have a fuzzy similarity matrix $\mathbf{R} = (r_{jk})_{n \times n}$. There are different ways to get r_{jk} . Since the Euclidean distance is widely used in similarity matrix establishment [26], we also choose it to calculate r_{jk} :

$$r_{jk} = 1 - d(x''_j, x''_k) = 1 - \sqrt{\sum_{h=1}^m (x''_{jh} - x''_{kh})^2}. \quad (8)$$

Step 3. Build a maximum spanning tree and classify historical data.

In this step, the Kruskal algorithm [26] is used to build the maximum spanning tree. Firstly, draw every vertex x_j . Secondly, draw the edges by the value of their weights r_{jk} in descending order, until all of the vertices are connected but with no circles. At last, cut down the edges with smaller weights than the threshold λ . The vertices of each connected branch make up a group.

Step 4. Calculate the best λ .

The value of λ varies from 0 to 1, and the best λ leads to the best classification. So how to get the best λ is an important step. In this paper, we also use a widely used F -statistic to find the best λ :

$$F = \frac{\sum_{i=1}^r n_i \sum_{h=1}^m (\bar{x}_{ih} - \bar{x}_h)^2 / (r-1)}{\sum_{i=1}^r \sum_{j=1}^{n_i} \sum_{h=1}^m (x_{jh} - \bar{x}_{ih})^2 / (n-r)} \quad (9)$$

$$\sim F(r-1, n-r),$$

where r is the number of groups for a given λ , n_i is the number of objects in group i ($i = 1, 2, \dots, r$), $\bar{x}_{ih} = (1/n) \sum_{j=1}^{n_i} x_{jh}$ is the average of the h th ($h = 1, 2, \dots, m$) characteristic of the objects in group i , and $\bar{x}_h$ is the h th characteristic of all objects. In (9), the numerator represents the distances between groups, and the denominator represents the distances within groups. So the bigger F is, the better the classification we get. For a given confidence level α , we can find several values of F which are larger than F_α . The λ which leads to the largest F is the best λ , and the corresponding classification is the best as well.

The best classification can be noted as

$$\{x_1^1, x_1^2, \dots, x_1^{n_1}\}, \dots, \{x_{i-1}^1, x_{i-1}^2, \dots, x_{i-1}^{n_{i-1}}\},$$

$$\{x_i^1, x_i^2, \dots, x_i^{n_i}\}, \{x_{i+1}^1, x_{i+1}^2, \dots, x_{i+1}^{n_{i+1}}\}, \dots,$$

$$\{x_r^1, x_r^2, \dots, x_r^{n_r}\}, \quad (10)$$

where $i = 1, 2, \dots, r$, $x_{i-1}^{n_{i-1}} \leq x_i^1, x_i^1 \leq x_i^2 \leq \dots \leq x_i^{n_i}$, and $x_i^{n_i} \leq x_{i+1}^1$.

Let

$$d_i = \begin{cases} x_{\min} - \varepsilon_1, & i = 0 \\ \frac{x_i^{n_i} + x_{i+1}^1}{2}, & i = 1, 2, \dots, r-1 \\ x_{\max} + \varepsilon_2, & i = r. \end{cases} \quad (11)$$

Therefore, we partition the universe U into r unequal intervals: $u_1 = [d_0, d_1]$, $u_2 = [d_1, d_2]$, ..., and $u_r = [d_{r-1}, d_r]$.

3.2. Construction of Intuitionistic Fuzzy Sets. Corresponding to the above r intervals, we define r intuitionistic fuzzy sets representing r linguistic values:

$$A_i = \{\langle x, \mu_{A_i}(x), \gamma_{A_i}(x) \rangle \mid x \in X\}, \quad i = 1, 2, \dots, r. \quad (12)$$

Constructing their membership functions and nonmembership functions is the key point in this section.

Since the intuitionistic fuzzy set has a special characteristic, intuitionistic index, the design of membership function and nonmembership function has been quite comprehensive. However, existing methods based on fuzzy statistics, trichotomy, or binary comparison sequencing usually set the intuitionistic index to a fixed value, which does not take full advantage of the intuitionistic fuzzy set [27]. Therefore, according to the characteristics of IFTS intervals, a more objective method is proposed in this section.

First of all, two rules based on objective analysis are as follows:

- (1) When x is located in the middle of an interval, namely, $x = (d_{i-1} + d_i)/2$, we define that $\mu_{A_i}((d_{i-1} + d_i)/2) = 1$ and $\gamma_{A_i}((d_{i-1} + d_i)/2) = 0$.
- (2) When x is located on the boundaries of an interval, namely, $x = d_i$, we define that intuitionistic index has the maximum value and $\mu_{A_i}(d_i) = \gamma_{A_i}(d_i)$. Let $\pi_{A_i}(d_i) = \alpha$ ($0 \leq \alpha \leq 1$); then we can get $\mu_{A_i}(d_i) = \gamma_{A_i}(d_i) = (1 - \alpha)/2$.

Then, in view of above rules, the membership function is defined as a Gaussian function:

$$\mu_{A_i} = \exp\left(-\frac{(x - c_{\mu i})^2}{2\sigma_{\mu i}^2}\right). \quad (13)$$

The nonmembership function is a transformation of Gaussian function:

$$\gamma_{A_i}(x) = 1 - \exp\left(-\frac{(x - c_{\gamma i})^2}{2\sigma_{\gamma i}^2}\right). \quad (14)$$

Hence, the intuitionistic index function reads

$$\pi_{A_i}(x) = 1 - \mu_{A_i}(x) - \gamma_{A_i}(x), \quad (15)$$

where $i = 1, 2, \dots, r$ and $c_{\mu i}$, $\sigma_{\mu i}$, $c_{\gamma i}$, and $\sigma_{\gamma i}$ are important function parameters. The calculations of them are based on the above two rules:

$$\begin{aligned} c_{\mu i} &= c_{\gamma i} = \frac{d_{i-1} + d_i}{2}, \\ \sigma_{\mu i}^2 &= -\frac{(d_{i-1} - c_{\mu i})^2}{2 \ln((1 - \alpha)/2)}, \\ \sigma_{\gamma i}^2 &= -\frac{(d_i - c_{\gamma i})^2}{2 \ln(1 + (1 - \alpha)/2)}. \end{aligned} \quad (16)$$

Definition 6. Let A be an intuitionistic fuzzy set in a finite universe X . If

- (1) $0 \leq \mu_A(x) \leq 1, 0 \leq \gamma_A(x) \leq 1$,
- (2) $0 \leq \pi_A(x) \leq 1, 0 \leq \mu_A(x) + \gamma_A(x) \leq 1$,
- (3) $\pi_A(x) + \mu_A(x) + \gamma_A(x) = 1$,

then A is a normal intuitionistic fuzzy set.

Therefore, we obtain the following theorem.

Theorem 7. The membership function and nonmembership function of A_i ($i = 1, 2, \dots, r$) are standard; that is, A_i is a normal intuitionistic fuzzy set.

Proof. (1) $y = \exp(-(x - c)^2/2\sigma^2)$ is a Gaussian function, so we have

$$\begin{aligned} 0 &\leq \exp\left(-\frac{(x - c)^2}{2\sigma^2}\right) \leq 1, \\ 0 &\leq 1 - \exp\left(-\frac{(x - c)^2}{2\sigma^2}\right) \leq 1. \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} 0 &\leq \mu_{A_i}(x) \leq 1, \\ 0 &\leq \gamma_{A_i}(x) \leq 1. \end{aligned} \quad (18)$$

- (2) Given $0 \leq \alpha \leq 1$ and $c_{\mu i} = c_{\gamma i} = (d_{i-1} + d_i)/2$, we have

$$\sigma_{\mu i}^2 \leq \sigma_{\gamma i}^2. \quad (19)$$

Hence,

$$\begin{aligned} \exp\left(-\frac{(x - c_{\mu i})^2}{2\sigma_{\mu i}^2}\right) &\leq \exp\left(-\frac{(x - c_{\gamma i})^2}{2\sigma_{\gamma i}^2}\right), \\ \exp\left(-\frac{(x - c_{\mu i})^2}{2\sigma_{\mu i}^2}\right) - \exp\left(-\frac{(x - c_{\gamma i})^2}{2\sigma_{\gamma i}^2}\right) &\leq 0. \end{aligned} \quad (20)$$

On the other hand, since $0 \leq \exp(-(x - c_{\mu i})^2/2\sigma_{\mu i}^2) \leq 1$ and $0 \leq \exp(-(x - c_{\gamma i})^2/2\sigma_{\gamma i}^2) \leq 1$, we have

$$-1 \leq \exp\left(-\frac{(x - c_{\mu i})^2}{2\sigma_{\mu i}^2}\right) - \exp\left(-\frac{(x - c_{\gamma i})^2}{2\sigma_{\gamma i}^2}\right) \leq 1. \quad (21)$$

Therefore,

$$\begin{aligned} -1 &\leq \exp\left(-\frac{(x - c_{\mu i})^2}{2\sigma_{\mu i}^2}\right) - \exp\left(-\frac{(x - c_{\gamma i})^2}{2\sigma_{\gamma i}^2}\right) \\ &\leq 0, \\ 0 &\leq \exp\left(-\frac{(x - c_{\mu i})^2}{2\sigma_{\mu i}^2}\right) + 1 - \exp\left(-\frac{(x - c_{\gamma i})^2}{2\sigma_{\gamma i}^2}\right) \\ &\leq 1. \end{aligned} \quad (22)$$

That is,

$$\begin{aligned} 0 &\leq \mu_{A_i}(x) + \gamma_{A_i}(x) \leq 1, \\ 0 &\leq \pi_{A_i}(x) = 1 - \mu_{A_i}(x) - \gamma_{A_i}(x) \leq 1. \end{aligned} \quad (23)$$

(3) According to the calculation of $\pi_{A_i}(x)$, it can be easily found that $\pi_{A_i}(x) + \mu_{A_i}(x) + \gamma_{A_i}(x) = 1$. This completes the proof. $\square$

Theorem 7 shows that the calculation of the membership function and nonmembership function of the intuitionistic fuzzy set is correct and appropriate.

3.3. Forecast Rules Based on Intuitionistic Fuzzy Reasoning

3.3.1. Intuitionistic Fuzzy Multiple Modus Ponens. Let A_i ($i = 1, 2, \dots, n$) and A^* be intuitionistic fuzzy sets in universe U and let B_i ($i = 1, 2, \dots, n$) and B^* be intuitionistic fuzzy sets in universe V . The generalized multiple modus ponens based on intuitionistic fuzzy relation [27] is that a new proposition that “ y is B^* ” can be inferred from $n + 1$ propositions: “if x is A_i , then y is B_i ” and “ x is A^* .” The reasoning model is as follows:

Rules:	IF x is A_1	THEN	y is B_1
	IF x is A_2	THEN	y is B_2
	$\vdots$		
	IF x is A_n	THEN	y is B_n
Input:	IF x is A^*		
Output:			y is B^*

Every rule has a corresponding input-output relation R_i . For R_i , different operators result in different μ_R and γ_R , but the reasoning outputs are all the same. Since it has a better performance and is easier to calculate than other operators [27], the Mamdani implication operator R_c is used here:

$$R_i = (A \times B) \quad (24)$$

$$= \{ \langle (x, y), \mu_{R_i}(x, y), \gamma_{R_i}(x, y) \rangle \mid (x, y) \in U \times V \},$$

where

$$\begin{aligned} \mu_{R_i}(x, y) &= \mu_{A_i}(x) \wedge \mu_{B_i}(y), \\ \gamma_{R_i}(x, y) &= \gamma_{A_i}(x) \vee \gamma_{B_i}(y). \end{aligned} \quad (25)$$

Then, according to the compositional operation of intuitionistic fuzzy rules, we get the total relation:

$$R = \bigcup_{i=1}^n R_i \quad (26)$$

$$= \{ \langle (x, y), \mu_R(x, y), \gamma_R(x, y) \rangle \mid (x, y) \in U \times V \},$$

where

$$\begin{aligned} \mu_R(x, y) &= \bigvee_{i=1}^n \mu_{R_i}(x, y) = \bigvee_{i=1}^n (\mu_{A_i}(x) \wedge \mu_{B_i}(y)), \\ \gamma_R(x, y) &= \bigwedge_{i=1}^n \gamma_{R_i}(x, y) = \bigwedge_{i=1}^n (\gamma_{A_i}(x) \vee \gamma_{B_i}(y)). \end{aligned} \quad (27)$$

The reasoning output is

$$B^* = A^* \circ R, \quad (28)$$

where “ $\circ$ ” is defined as the maximum and minimum operators: “ $\vee$ ” and “ $\wedge$ ”:

$$\begin{aligned} \mu_{B^*}(y) &= \bigvee_{x \in U} (\mu_{A^*}(x) \wedge \mu_R(x, y)), \\ \gamma_{B^*}(y) &= \bigwedge_{x \in U} (\gamma_{A^*}(x) \vee \gamma_R(x, y)). \end{aligned} \quad (29)$$

3.3.2. Forecast Rules of IFTS Model. Inspired by the intuitionistic fuzzy multiple modus ponens, we exchange the positions of historical data and intuitionistic fuzzy sets A_i ($i = 1, 2, \dots, r$) in the IFTS model; that is, let the historical data be intuitionistic fuzzy sets, noted as F_j ($j = 1, 2, \dots, t$), and let A_i be the elements in F_j , and let $\mu_{A_i}(x)$ and $\gamma_{A_i}(x)$ be the membership and nonmembership of A_i to F_j . Then F_j is as follows:

$$F_j = \sum_{i=1}^r \frac{\langle \mu_{F_j}(x_i), \gamma_{F_j}(x_i) \rangle}{x_i}, \quad (30)$$

where

$$\begin{aligned} \mu_{F_j}(x_i) &= \mu_{A_i}(f_i(t)), \\ \gamma_{F_j}(x_i) &= \gamma_{A_i}(f_i(t)). \end{aligned} \quad (31)$$

Hence, we apply the intuitionistic fuzzy multiple modus ponens to A_i and F_j . The reasoning model is as follows:

Rules:	IF x is F_1	THEN	y is F_2
	IF x is F_2	THEN	y is F_3
	$\vdots$		
	IF x is F_j	THEN	y is F_{j+1}
	$\vdots$		
	IF x is F_{t-1}	THEN	y is F_t
Input:	IF x is F_t		
Output:			y is F_{t+1}^*

The reasoning output is as follows:

$$F_{t+1}^* = F_t \circ R, \quad (32)$$

where

$$\begin{aligned} \mu_R(x, y) &= \bigvee_{j=1}^{t-1} \mu_{R_j}(x, y) = \bigvee_{j=1}^{t-1} (\mu_{F_j}(x) \wedge \mu_{F_{j+1}}(y)), \\ \gamma_R(x, y) &= \bigwedge_{j=1}^{t-1} \gamma_{R_j}(x, y) = \bigwedge_{j=1}^{t-1} (\gamma_{F_j}(x) \vee \gamma_{F_{j+1}}(y)). \end{aligned} \quad (33)$$

So, the membership and nonmembership of the output intuitionistic fuzzy set are

$$\begin{aligned} \mu_{F_{t+1}^*}(y) &= \bigvee_{x \in U} (\mu_{F_t}(x) \wedge \mu_R(x, y)), \\ \gamma_{F_{t+1}^*}(y) &= \bigwedge_{x \in U} (\gamma_{F_t}(x) \vee \gamma_R(x, y)). \end{aligned} \quad (34)$$

That is to say, the membership and nonmembership of the forecasted result $f_{t+1}(t)$ to every intuitionistic fuzzy set A_i are $\mu_{F_{t+1}^*}(y)$ and $\gamma_{F_{t+1}^*}(y)$.

FIGURE 1: The maximum spanning tree of historical data.

3.4. Defuzzification Algorithm. The widely used defuzzification algorithms include the maximum truth-value algorithm, gravity algorithm, and weighted average algorithm [27]. In this paper, we utilize the gravity algorithm, which has a more obvious and smoother output than others even when the input has tiny changes [27]. The calculation is as follows:

$$C = \frac{\int_U c (\mu_F(c) + (1/2) \pi_F(c)) dc}{\int_U (\mu_F(c) + (1/2) \pi_F(c)) dc} \quad (35)$$

$$= \frac{\int_U c (1 + \mu_F(c) + \gamma_F(c)) dc}{\int_U (1 + \mu_F(c) + \gamma_F(c)) dc},$$

where U is the output domain and F is an intuitionistic fuzzy set in U .

4. Applications

In this section, we focus on two numerical experiments to demonstrate the performance of the proposed IFTS model. In each experiment, several existing FTS and IFTS models are also applied on the same data set to make comparisons. The experimental results and associating analyses are shown, respectively.

4.1. Enrollments of the University of Alabama. The enrollment of the University of Alabama has been firstly used in Song's paper on FTS model [2]. Since then, this data set has been used by most of the scholars to test their FTS or IFTS models. The detailed test process of our model is as follows.

Step 1. Define and partition the universe of discourse.

The enrollments from year 1971 to 1991 are chosen as historical data to forecast the enrollment of year 1992. In the historical data, $x_{\min} = 13055$ and $x_{\max} = 19337$, so the universe of discourse is set to $U = [13000, 20000]$.

Then U is partitioned into unequal intervals based on the fuzzy clustering algorithm designed in Section 3.1. The step-by-step details are as follows:

- (1) Standardize historical data according to (6) and (7).
- (2) Establish the fuzzy similarity matrix $\mathbf{R}$ as shown in Table 1.

(3) Use the Kruskal algorithm mentioned in Section 3.1 to build a maximum spanning tree based on matrix $\mathbf{R}$. The tree is shown in Figure 1.

Let λ be 0.93, 0.94, 0.95, and 0.96, respectively. We can get different classifications of historical data as shown in Table 2.

(4) For different classifications, calculate the values of F according to (9). The results are also shown in Table 2. From Table 2, we can see that when $\lambda = 0.95$, its corresponding F is maximum and bigger than F_α ($\alpha = 0.05$) at the same time. So this classification is the best.

Therefore, the universe of discourse U is partitioned into 9 unequal intervals according to the above classification. The boundaries of each interval are calculated according to (11). The intervals are

$$\begin{aligned} u_1 &= [13000, 13309], \\ u_2 &= [13309, 14282], \\ u_3 &= [14282, 14921], \\ u_4 &= [14921, 16186], \\ u_5 &= [16186, 16598], \\ u_6 &= [16598, 17535], \\ u_7 &= [17535, 18560], \\ u_8 &= [18560, 19149], \\ u_9 &= [19149, 20000]. \end{aligned} \quad (36)$$

Step 2. Construct intuitionistic fuzzy sets and intuitionistically fuzzify the historical data.

Corresponding to the 9 intervals, there should be 9 intuitionistic fuzzy sets $A_1, A_2, \dots, A_9$, and their realistic significance is as follows: "very very very few", "very very few", "very few", "few", "normal", "many", "very many", "very very many", "very very very many". Then calculate the parameters of the membership and nonmembership functions based on Section 3.2. For $\alpha = 0.4$, the parameters are shown in Table 3.

The membership function, nonmembership function, and intuitionistic index function of every intuitionistic fuzzy set are shown in Figures 2, 3, and 4, respectively.

TABLE I: The fuzzy similarity matrix $\mathbf{R}$.

	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
1971	1.000	0.919	0.871	0.739	0.617	0.641	0.594	0.553	0.403	0.385	0.469	0.622	0.611	0.667	0.664	0.534	0.395	0.189	0.058	0.001	0.000
1972	0.919	1.000	0.952	0.820	0.698	0.722	0.675	0.634	0.484	0.466	0.550	0.702	0.692	0.748	0.745	0.615	0.475	0.270	0.139	0.082	0.081
1973	0.871	0.952	1.000	0.868	0.746	0.770	0.724	0.683	0.532	0.514	0.599	0.751	0.741	0.797	0.794	0.663	0.524	0.318	0.188	0.131	0.129
1974	0.739	0.820	0.868	1.000	0.878	0.902	0.856	0.815	0.664	0.646	0.731	0.883	0.873	0.929	0.926	0.795	0.656	0.450	0.320	0.263	0.261
1975	0.617	0.698	0.746	0.878	1.000	0.976	0.977	0.936	0.786	0.768	0.852	0.996	0.994	0.950	0.953	0.917	0.777	0.572	0.441	0.384	0.383
1976	0.641	0.722	0.770	0.902	0.976	1.000	0.954	0.912	0.762	0.744	0.829	0.981	0.970	0.974	0.976	0.893	0.754	0.548	0.418	0.361	0.359
1977	0.594	0.675	0.724	0.856	0.977	0.953	1.000	0.959	0.808	0.791	0.875	0.973	0.983	0.927	0.930	0.939	0.800	0.595	0.464	0.407	0.406
1978	0.553	0.634	0.683	0.815	0.936	0.912	0.959	1.000	0.849	0.832	0.916	0.932	0.942	0.886	0.889	0.980	0.841	0.636	0.505	0.448	0.447
1979	0.403	0.484	0.532	0.664	0.786	0.762	0.808	0.849	1.000	0.982	0.933	0.781	0.791	0.735	0.738	0.869	0.992	0.786	0.656	0.599	0.597
1980	0.385	0.466	0.514	0.646	0.768	0.744	0.791	0.832	0.982	1.000	0.916	0.763	0.774	0.718	0.721	0.851	0.990	0.804	0.673	0.616	0.615
1981	0.469	0.550	0.599	0.731	0.852	0.829	0.875	0.916	0.933	0.915	1.000	0.848	0.858	0.802	0.805	0.936	0.925	0.720	0.589	0.532	0.531
1982	0.621	0.702	0.751	0.883	0.996	0.981	0.973	0.932	0.781	0.763	0.848	1.000	0.990	0.954	0.957	0.912	0.773	0.568	0.437	0.380	0.379
1983	0.611	0.692	0.741	0.872	0.994	0.970	0.983	0.942	0.791	0.774	0.858	0.990	1.000	0.944	0.947	0.922	0.783	0.578	0.447	0.390	0.389
1984	0.667	0.748	0.797	0.929	0.950	0.974	0.927	0.886	0.735	0.718	0.802	0.954	0.944	1.000	0.997	0.866	0.727	0.522	0.391	0.334	0.333
1985	0.664	0.745	0.794	0.926	0.953	0.976	0.930	0.889	0.738	0.720	0.805	0.957	0.947	0.997	1.000	0.869	0.730	0.525	0.394	0.337	0.336
1986	0.534	0.615	0.663	0.795	0.917	0.893	0.939	0.980	0.869	0.851	0.936	0.912	0.922	0.866	0.869	1.000	0.861	0.655	0.525	0.468	0.466
1987	0.394	0.475	0.524	0.656	0.777	0.754	0.800	0.841	0.992	0.990	0.925	0.773	0.783	0.727	0.730	0.861	1.000	0.795	0.664	0.607	0.606
1988	0.189	0.270	0.318	0.450	0.572	0.548	0.595	0.636	0.786	0.804	0.720	0.567	0.578	0.522	0.525	0.655	0.794	1.000	0.869	0.812	0.811
1989	0.058	0.139	0.188	0.320	0.441	0.418	0.464	0.505	0.656	0.674	0.589	0.437	0.447	0.391	0.394	0.525	0.664	0.869	1.000	0.943	0.942
1990	0.001	0.082	0.131	0.263	0.384	0.361	0.407	0.448	0.599	0.617	0.532	0.380	0.390	0.334	0.337	0.468	0.607	0.812	0.943	1.000	0.999
1991	0.000	0.081	0.129	0.261	0.383	0.359	0.406	0.447	0.597	0.615	0.531	0.379	0.389	0.333	0.336	0.466	0.606	0.811	0.942	0.999	1.000

FIGURE 2: Membership functions.

TABLE 2: Classifications of different λ .

λ	Number of classifications	Classification	$F_{0.05}$	F
0.935	7	{1971}, {1972, 1973}, {1974}, {1979, 1980, 1987}, {1988}, {1989, 1990, 1991}, {1975, 1976, 1977, 1978, 1981, 1982, 1983, 1984, 1985, 1986}	2.85	92.85
0.94	8	{1971}, {1972, 1973}, {1974}, {1981}, {1979, 1980, 1987}, {1988}, {1989, 1990, 1991}, {1975, 1976, 1977, 1978, 1982, 1983, 1984, 1985, 1986}	2.83	129.17
0.95	9	{1971}, {1972, 1973}, {1974}, {1981}, {1979, 1980, 1987}, {1988}, {1989}, {1990, 1991}, {1975, 1976, 1977, 1978, 1982, 1983, 1984, 1985, 1986}	2.85	142.13
0.955	10	{1971}, {1972}, {1973}, {1974}, {1981}, {1979, 1980, 1987}, {1988}, {1989}, {1990, 1991}, {1975, 1976, 1977, 1978, 1982, 1983, 1984, 1985, 1986}	2.9	112.67

FIGURE 3: Nonmembership functions.

FIGURE 4: Intuitionistic index functions.

FIGURE 5: Forecast results of TAIEX.

Then we can calculate the membership, nonmembership, and intuitionistic index of every historical value to every intuitionistic fuzzy set.

Step 3. Establish forecast rules and forecast the enrollments.

The enrollments of year 1971 to 1991 can be denoted as $F_1, F_2, \dots, F_{21}$, and the reasoning model based on

$$C = \frac{\int_U c \left(2 + \exp \left(- (c - 18854.5)^2 / (2 \times 189.8^2) \right) - \exp \left(- (c - 18854.5)^2 / (2 \times 347.8^2) \right) \right) dc}{\int_U \left(2 + \exp \left(- (c - 18854.5)^2 / (2 \times 189.8^2) \right) - \exp \left(- (c - 18854.5)^2 / (2 \times 347.8^2) \right) \right) dc} = 18854.5 \approx 18855. \quad (38)$$

That is to say, the enrollment of year 1992 is 18855.

To test the performance of our model, we use the models of inference [2], [12], and [17] as well as ours to forecast every year's enrollment, respectively. The results are shown in Table 4. The models of inference [2, 12] are FTS models, and the model of inference [17] is an IFTS model. In inference [12], there are three kinds of universe partition: 7, 17, and 22 intervals. Since there are only 22 historical data, the 17-interval partition and 22-interval partition are not applicable, so we choose the 7-interval partition.

The root mean square error (RMSE) and average forecast error (AFE) are exploited to evaluate the performance of every model:

$$\begin{aligned} \text{RMSE} &= \sqrt{\frac{1}{n} \sum_{i=1}^n (\text{actual value}_i - \text{forecast value}_i)^2}, \\ \text{AFE} &= \frac{1}{n} \sum_{i=1}^n \frac{|\text{actual value}_i - \text{forecast value}_i|}{\text{actual value}_i} \times 100\%. \end{aligned} \quad (39)$$

The results are shown in Table 5.

The results in Tables 4 and 5 indicate that our model can not only reach the forecast goal but also achieve a better result

Section 3.3.2 is as follows:

Rules: IF x is F_1 THEN y is F_2
 IF x is F_2 THEN y is F_3
 $\vdots$
 IF x is F_{20} THEN y is F_{21}
 Input: IF x is F_{21}
 Output: y is F_{22}^*
 Then we get F_{22}^* :

$$\begin{aligned} F_{22}^* &= \frac{\langle 0, 1 \rangle}{A_1} + \frac{\langle 0, 1 \rangle}{A_2} + \frac{\langle 0, 1 \rangle}{A_3} + \frac{\langle 0, 1 \rangle}{A_4} + \frac{\langle 0, 1 \rangle}{A_5} \\ &+ \frac{\langle 0, 1 \rangle}{A_6} + \frac{\langle 0, 0.895 \rangle}{A_7} + \frac{\langle 0.687, 0.207 \rangle}{A_8} \\ &+ \frac{\langle 0.039, 0.616 \rangle}{A_9}, \end{aligned} \quad (37)$$

where the membership of F_{22}^* to A_8 is the biggest and the nonmembership is the smallest, so the intuitionistic forecasted result is A_8 .

Step 4. Defuzzify and output the forecast result.

The defuzzification result based on Section 3.4 is

than the other tested models. That is to say, the proposed model is feasible and efficient.

4.2. Experiments on TAIEX. The Taiwan Stock Exchange Capitalization Weighted Stock Index (TAIEX) is a typical economic data set widely used in fuzzy time series forecasting [13, 20, 23, 25, 28, 29]. In this experiment, TAIEX values from 11/1/2004 to 12/31/2004 are used as historical data, which are shown in Table 6. The intuitionistic fuzzified value of historical data when forecasted by our model are also shown in Table 6.

For comparison, we also applied the models of reference [2, 13, 17, 20] to forecast TAIEX at the same time. The forecast results of every model are shown in Table 7 and Figure 5.

The performance of all models is compared in Table 8.

Table 8 indicates that the RMSE and MSE of proposed model are both smaller than the other models. Therefore, our two experiments both indicate that the IFTS model proposed in this paper could effectively increase forecast accuracy.

5. Conclusions

In this paper, a novel IFTS model is proposed for improving the performance of FTS model. In order to be succinct, we use

TABLE 3: Function parameters of A_i .

Intuitionistic fuzzy set	$c_{\mu i}$	$\sigma_{\mu i}$	$c_{\gamma i}$	$\sigma_{\gamma i}$
A_1	13154.5	99.5	13154.5	182.9
A_2	13795.5	313.5	13795.5	576
A_3	14601.5	205.9	14601.5	378.3
A_4	15553.5	407.6	15553.5	748.9
A_5	16392	132.8	16392	243.9
A_6	17066.5	301.9	17066.5	554.7
A_7	18047.5	330.3	18047.5	606.8
A_8	18854.5	189.8	18854.5	348.7
A_9	19574.5	274.2	19574.5	503.8

TABLE 4: Forecast results of the enrollments.

Year	Actual enrollment	Forecasted enrollment			
		Model [2]	Model [12]	Model [17]	Our model
1972	13563	14000	14279	14250	13500
1973	13867	14000	14279	14246	14155
1974	14696	14000	14279	14246	14155
1975	15460	15500	15392	15491	15539
1976	15311	16000	15392	15491	15539
1977	15603	16000	15392	15491	15502
1978	15861	16000	16467	16345	15502
1979	16807	16000	16467	16345	16667
1980	16919	16813	17161	15850	16667
1981	16388	16813	17161	15850	15669
1982	15433	16789	14916	15850	15564
1983	15497	16000	15392	15450	15564
1984	15145	16000	15392	15450	15564
1985	15163	16000	15392	15491	15523
1986	15984	16000	15470	15491	15523
1987	16859	16000	16467	16345	16799
1988	18150	16813	17161	17950	18268
1989	18970	19000	19257	18961	18268
1990	19328	19000	19257	18961	18780
1991	19337	19000	19257	18961	19575
1992	18876	19000	19257	18961	18855

TABLE 5: Forecast performance of enrollments.

Criterion	Model [2]	Model [12]	Model [17]	Our model
RMSE	677.1	445.5	418.9	350.9
AFE	3.35%	2.3%	2.07%	1.72%

the maximum spanning tree based fuzzy clustering algorithm to partition the universe of discourse into unequal intervals. According to the characteristics of partitioned data, a more objective method is proposed to ascertain membership function and nonmembership function of the intuitionistic fuzzy set. Besides, intuitionistic fuzzy reasoning is utilized to establish forecast rules, which make the model more sensitive

TABLE 6: Historical data of TAIEX.

Date	TAIEX	Intuitionistic fuzzified value
11/1/2004	5656.17	A_1
11/2/2004	5759.61	A_2
11/3/2004	5862.85	A_5
11/4/2004	5860.73	A_5
11/5/2004	5931.31	A_7
11/8/2004	5937.46	A_7
11/9/2004	5945.2	A_8
11/10/2004	5948.49	A_8
11/11/2004	5874.52	A_5
11/12/2004	5917.16	A_7
11/15/2004	5906.69	A_7
11/16/2004	5910.85	A_7
11/17/2004	6028.68	A_{11}
11/18/2004	6049.49	A_{12}
11/19/2004	6026.55	A_{11}
11/22/2004	5838.42	A_5
11/23/2004	5851.1	A_5
11/24/2004	5911.31	A_7
11/25/2004	5855.24	A_5
11/26/2004	5778.26	A_3
11/29/2004	5785.26	A_3
11/30/2004	5844.76	A_5
12/1/2004	5798.62	A_4
12/2/2004	5867.95	A_5
12/3/2004	5893.27	A_6
12/6/2004	5919.17	A_7
12/7/2004	5925.28	A_7
12/8/2004	5892.51	A_6
12/9/2004	5913.97	A_7
12/10/2004	5911.63	A_7
12/13/2004	5878.89	A_5
12/14/2004	5909.65	A_7
12/15/2004	6002.58	A_{10}
12/16/2004	6019.23	A_{11}
12/17/2004	6009.32	A_{10}
12/20/2004	5985.94	A_9
12/21/2004	5987.85	A_9
12/22/2004	6001.52	A_{10}
12/23/2004	5997.67	A_{10}
12/24/2004	6019.42	A_{11}
12/27/2004	5985.94	A_9
12/28/2004	6000.57	A_{10}
12/29/2004	6088.49	A_{13}
12/30/2004	6100.86	A_{14}
12/31/2004	6139.69	A_{15}

TABLE 7: Forecast results of TAIEX.

Actual TAIEX	Forecasted TAIEX				
	Model [2]	Model [20]	Model [13]	Model [17]	Our model
5656.17	—	—	—	—	—
5759.61	5675	5658.97	—	5680.5	5678.95
5862.85	5825	5754.24	5756.8	5836.8	5738.41
5860.73	5875	5863.09	5865.05	5866.5	5852.11
5931.31	5875	5868.04	5868.44	5890.23	5852.11
5937.46	5900	5927.67	5935.51	5911.52	5886.39
5945.2	5900	5942.9	5932.96	5968	5937.47
5948.49	5925	5942.85	5940.7	5969.6	5954.27
5874.52	5925	5952.69	5945.72	5969.6	5852.11
5917.16	5900	5882.56	5873.1	5923	5909.01
5906.69	5925	5931.63	5919.36	5887	5920.66
5910.85	5925	5908.03	5905.27	5887	5920.66
6028.68	5925	5912.13	5906.35	5995	5954.27
6049.49	5975	6037.08	6030.88	5942	6026.68
6026.55	5975	6042.75	6051.69	5933	6040.36
5838.42	6025	6010.72	6025.63	5877	5909.01
5851.1	5925	5862.23	5837	5896	5852.11
5911.31	5908.33	5855.49	5853.3	5911.52	5886.39
5855.24	5900	5911.03	5913.51	5933	5852.11
5778.26	5900	5858.01	5862.16	5833	5886.39
5785.26	5858.33	5773.75	5785.57	5790	5816.28
5844.76	5775	5783.93	5780.76	5782	5852.11
5798.62	5850	5844.68	5841.95	5790	5780.44
5867.95	5825	5817.26	5797.2	5778	5828.67
5893.27	5850	5861.34	5865.14	5757	5828.67
5919.17	5825	5897.53	5895.47	5824	5888.54
5925.28	5900	5916.83	5916.36	5911	5920.66
5892.51	5925	5910.24	5920.78	5942	5920.66
5913.97	5900	5895.93	5891.09	5900	5920.66
5911.63	5925	5918.24	5946.17	5900	5906.75
5878.89	5925	5915.62	5910.71	5852	5886.39
5909.65	5900	5882.3	5877.47	5852	5886.39
6002.58	5900	5911.86	5911.86	5990	5920.66
6019.23	5941.67	6002.5	6004.78	6000	6003.52
6009.32	6025	6027.15	6026.94	6025	6015.1
5985.94	6025	6012.86	6004.82	6009	6003.39
5987.85	6000	5991.1	5984.52	5946	5991.75
6001.52	5975	6005.92	5995.56	5965	5979.99
5997.67	6000	6005.86	6009.23	6009	5991.75
6019.42	6000	5997.68	6005.38	6030	6026.68
5985.94	6025	6026.35	6016.65	6030	6026.68
6000.57	6025	5987.87	5985.02	6015	6003.39
6088.49	6025	6018.68	6008.28	6033	6015.1
6100.86	6025	6086.83	6085.68	6098	6081.83
6139.69	6050	6099.45	6098.05	6125	6094.66

TABLE 8: Forecast performance of TAIEX.

Criterion	Model [2]	Model [20]	Model [13]	Model [17]	Our model
RMSE	61.17	52.63	53.63	50.27	43.23
AFE	0.83%	0.51%	0.65%	0.65%	0.51%

to the fuzzy variation of uncertain data. Finally, based on experiments with two data sets, the feasibility and advantage of the new model are verified.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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