Research Article

Exact Solutions for a Generalized KdV-MKdV Equation with Variable Coefficients

Bo Tang, 1,2 Xuemin Wang, 3 Yingzhe Fan, 4 and Junfeng Qu 1

1 School of Mathematics and Computer Science, Hubei University of Arts and Science, Xiangyang, Hubei 441053, China
2 School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China
3 Texas University at Dallas, Dallas, TX 75080-3021, USA
4 School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, China

Correspondence should be addressed to Bo Tang; tangbo0809@163.com

Received 24 September 2015; Revised 19 December 2015; Accepted 27 December 2015

Academic Editor: Hassan Askari

Copyright © 2016 Bo Tang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By using solutions of an ordinary differential equation, an auxiliary equation method is described to seek exact solutions of variable-coefficient KdV-MKdV equation. As a result, more new exact nontravelling solutions, which include soliton solutions, combined soliton solutions, triangular periodic solutions, Jacobi elliptic functions solutions, and combined Jacobi elliptic functions solutions, for the KdV-MKdV equation are obtained. It is shown that the considered method provides a very effective, convenient, and powerful mathematical tool for solving many other nonlinear partial differential equations with variable coefficients in mathematical physics.

1. Introduction

In the nonlinear science, many important phenomena in various fields can be described by the nonlinear partial differential equations (NPDEs). Searching for exact soliton solutions of NPDEs plays an important and significant role in the study on the dynamics of those phenomena. Various methods have been used to handle nonlinear partial differential equations, such as the Hirota bilinear method [1], inverse scattering method [2, 3] the Bäcklund transformation method [4], subequation method [5–7], F-expansion method [8–10], sine-cosine method [11, 12], sech-tanh method [13, 14], Exp-function method [15, 16], and Jacobi elliptic function method [17–19].

It is well known that NPDEs with variable coefficients are more realistic in various physical situations than their constant coefficients counterparts. However, most of the above methods are related to the constant-coefficient NPDEs. The present work is motivated by the desire to establish an auxiliary equation method to construct new and more general exact solutions of variable-coefficient NPDEs, such as soliton and soliton-like solutions, triangular periodic solutions, Jacobi elliptic function solutions, and many exact explicit solutions in form of hyperbolic function solutions and trigonometric function solutions. Being concise and straightforward, this method is applied to the generalized variable-coefficient KdV-MKdV equation as the following form [20]:

\[ u_t - 6 f_0(t) uu_x - 6 f_1(t) u^2 u_x + f_2(t) u_{xxx} - f_3(t) u_x + f_4(t) (Au + xu_x) = 0, \]  

(1)

where \( f_0(t), f_1(t), f_2(t), f_3(t), \) and \( f_4(t) \) are arbitrary functions of \( t \).

Equation (1) can be used to describe the propagation of weakly nonlinear long waves in a KdV-typed medium by changing the coefficients of dispersion and nonlinear coefficients. It includes the following three important equations:

(a) Setting \( f_0(t) = 0 \), (1) can be degenerated to the following variable-coefficient and nonspectral KdV equation studied in [21, 22]:

\[ u_t = K_0(t) (u_{xxx} + 6uu_x) + 4K_1(t) u_x - h(t) (2u + xu_x), \]  

(2)

which models many important nonlinear phenomena, including shallow water waves, dust acoustic solitary structures.
in magnetized dusty plasmas, and ion acoustic waves in plasmas.
(b) Setting \( f_0(t) = 0, f_1(t) = f_2(t) = -h_1(t), f_3(t) = 4h_2(t), f_4(t) = h_0(t), A = 1 \), (1) can be degenerated to the following variable-coefficient MKdV equation studied in [23]:

\[
 u_t + h_1(t) \left( u_{xxx} - 6u_x^2 + 4h_2(t) u_x \right) - h_0(t) \left( u + xu_u_x \right),
\]

which models many important nonlinear phenomena, including shallow water waves, dust acoustic solitary structures in magnetized dusty plasmas, and ion acoustic waves in plasmas.
(c) Setting \( f_0(t) = -a(t)/6, f_1(t) = -b(t)/6, f_2(t) = h(t), f_3(t) = -d(t), f_4(t) = f(t)/A \), \( x = 0 \), (1) becomes the following variable-coefficient Gardner equation:

\[
 u_t + a(t) u u_x + b(t) u^2 u_x + d(t) u_x + f(t) u + h(t) u_{xxx} = 0,
\]

which is widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory [24, 25]. It also describes a variety of wave phenomena in plasma and solid state.

The rest of this paper is organized as follows: in Section 2, we will describe the auxiliary equation method for finding out solutions of variable-coefficient NPDEs and give the main steps of the method here. In Section 3, we illustrate the method in detail with the generalized variable-coefficient KdV-MKdV equation. In Section 4, some conclusions are given.

2. Description of the Auxiliary Equation Method

Consider a given variable-coefficient nonlinear partial differential equation with independent variable \( X = (x_0 = t, x_1, x_2, \ldots, x_n) \) and dependent variable \( u \):

\[
P \left( u, u_t, u_x, u_{xx}, \ldots, \right) = 0,
\]

where \( P \) is in general a polynomial function of its argument and the subscripts denote the partial derivatives.

Suppose (5) has the following solution:

\[
u(t, x) = \sum_{i=0}^{l} a_i F^i(\xi), \quad (6)
\]

where \( a_i = a_i(x_1, x_2, \ldots, x_n, t) \) and \( \xi = \xi(t, x_2, \ldots, x_n) \) are functions to be determined later and \( F^i(\xi) = dF/d\xi \) satisfies

\[
 F^{i2}(\xi) = \sum_{j=0}^{m} h_j F^j(\xi),
\]

where the values of \( m \) can be determined by balancing the highest differential term with the nonlinear terms in (5). The main steps by which we get exact solutions to variable-coefficient NPDE are outlined as follows.

<table>
<thead>
<tr>
<th>( h_0 )</th>
<th>( h_2 )</th>
<th>( h_4 )</th>
<th>( F(\xi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>&gt;0</td>
<td>&lt;0</td>
<td>( -\frac{h_2}{h_4} ) sech ( \sqrt{h_2 \xi} )</td>
</tr>
<tr>
<td>0</td>
<td>&gt;0</td>
<td>&gt;0</td>
<td>( \frac{h_2}{h_4} ) \cosh ( \sqrt{h_2 \xi} )</td>
</tr>
<tr>
<td>( h_2^2 )</td>
<td>&lt;0</td>
<td>&gt;0</td>
<td>( \frac{h_2}{2h_4} ) tanh ( \sqrt{\frac{h_2}{2} \xi} )</td>
</tr>
<tr>
<td>0</td>
<td>&lt;0</td>
<td>&gt;0</td>
<td>( \frac{h_2}{2h_4} ) \tanh ( \sqrt{\frac{h_2}{2} \xi} )</td>
</tr>
</tbody>
</table>

Step 1. Substituting (6) along with (7) into (5) and then setting all coefficients of \( x^p F^q(\xi) \) \( p, q = 0, 1; j = 0, 1, 2, \ldots \) of the resulting equation to zero, we get an overdetermined PDEs system for \( a_i (i = 0, 1, 2, \ldots, l) \) and \( \xi \).

Step 2. Solving the set of overdetermined PDEs by use of Mathematica can permit obtaining of explicit expressions of \( a_i (i = 0, 1, 2, \ldots, l) \) and \( \xi \).

Step 3. Substituting \( a_i (i = 0, 1, 2, \ldots, l) \) and \( \xi \) obtained in Step 2 into (6) along with the solutions of (7), we can get the exact solutions of (5).

Remark 1. In this paper, we consider the case of \( m = 4 \). As we know the more solutions of (7) we find, the more exact solutions of (5) may be obtained. However, the general solutions are difficult to be listed because of the complexity of (7). Some special solutions [19, 26–29] are listed as follows.

Case 1. Suppose that \( h_1 = h_3 = 0 \); the solutions of (7) are given in Tables 1 and 2.

Case 2. Suppose that \( h_0 = h_1 = 0 \); the solutions of (7) are given in Table 3.
Table 2: Solutions of Case 2.1 (continued). \(m (0 < m < 1 )\) denotes the modulus of the Jacobi elliptic function, \(k_1 = \sqrt{1 - m^2}\), and \(A, B, C (ABC \neq 0)\), and \(D\) are arbitrary constants.

<table>
<thead>
<tr>
<th>(h_0)</th>
<th>(h_2)</th>
<th>(h_4)</th>
<th>(f(\xi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2m^2 - 1)</td>
<td>(-m^2 (1 - m^2))</td>
<td>(sd \xi = \frac{sn \xi}{dn \xi})</td>
</tr>
<tr>
<td>(1 - m^2)</td>
<td>(2 - m^2)</td>
<td>1</td>
<td>(cs \xi = \frac{cn \xi}{sn \xi})</td>
</tr>
<tr>
<td>(-m^2 (1 - m^2))</td>
<td>(2m^2 - 1)</td>
<td>1</td>
<td>(ds \xi = \frac{dn \xi}{sn \xi})</td>
</tr>
<tr>
<td>(1)</td>
<td>(1 - 2m^2)</td>
<td>(2)</td>
<td>(ns \xi \pm cs \xi)</td>
</tr>
<tr>
<td>(1 - m^2)</td>
<td>(1 + m^2)</td>
<td>(1 - m^2)</td>
<td>(nc \xi \pm sc \xi)</td>
</tr>
<tr>
<td>(m^2)</td>
<td>(m^2 - 2)</td>
<td>(1)</td>
<td>(ns \xi \pm ds \xi)</td>
</tr>
<tr>
<td>(m^2)</td>
<td>(m^2 - 2)</td>
<td>(2)</td>
<td>(sn \xi \pm i cn \xi, \frac{dn(\xi)}{\sqrt{1 - m^2 sn(\xi) + cn(\xi)}})</td>
</tr>
<tr>
<td>(1)</td>
<td>(1 - 2m^2)</td>
<td>(2)</td>
<td>(msn \xi \pm idn \xi, \frac{sn(\xi)}{1 \pm cn(\xi)})</td>
</tr>
<tr>
<td>(1)</td>
<td>(m^2 - 2)</td>
<td>(m^4)</td>
<td>(sn(\xi), \frac{cn(\xi)}{1 \pm dn(\xi)})</td>
</tr>
<tr>
<td>(m^2 - 1)</td>
<td>(m^2 + 1)</td>
<td>(m^2 - 1)</td>
<td>(dn(\xi) \pm msn(\xi) \pm nd(\xi))</td>
</tr>
<tr>
<td>(1 - m^2)</td>
<td>(m^2 + 1)</td>
<td>(1 - m^2)</td>
<td>(cn(\xi), \frac{nc(\xi) + sc(\xi)}{1 \pm sn(\xi)})</td>
</tr>
<tr>
<td>(1)</td>
<td>(m^2 + 1)</td>
<td>((1 - m^2)^2)</td>
<td>(sn(\xi))</td>
</tr>
<tr>
<td>(1)</td>
<td>(m^2 - 2)</td>
<td>(m^4)</td>
<td>(cn(\xi) \sqrt{1 - m^2 \pm dn(\xi)})</td>
</tr>
<tr>
<td>(- (1 - m^2)^2)</td>
<td>(m^2 + 1)</td>
<td>(-1)</td>
<td>(mcn(\xi) \pm dn(\xi))</td>
</tr>
<tr>
<td>(1)</td>
<td>(1)</td>
<td>(4)</td>
<td>(\frac{dn(\xi)}{1 \pm cn(\xi)} \sqrt{1 - m^2 sn(\xi) + dn(\xi)})</td>
</tr>
<tr>
<td>(1 - 2m^2)</td>
<td>(2)</td>
<td>(2)</td>
<td>(\frac{sn(\xi)}{cn(\xi)} \sqrt{1 - m^2 sn(\xi) + dn(\xi)})</td>
</tr>
<tr>
<td>((m - 1)^2)</td>
<td>(1 + m^2 + 6m)</td>
<td>(A^2(m - 1)^2)</td>
<td>(\frac{dn(\xi) cn(\xi)}{(1 + sn(\xi))(1 + msn(\xi))})</td>
</tr>
<tr>
<td>((m + 1)^2)</td>
<td>(1 + m^2 - 6m)</td>
<td>(A^2(m + 1)^2)</td>
<td>(\frac{dn(\xi) cn(\xi)}{(1 + sn(\xi))(1 - msn(\xi))})</td>
</tr>
<tr>
<td>(-2m^3 + m^4 + m^2)</td>
<td>(6m - m^2 - 1)</td>
<td>(-\frac{4}{m})</td>
<td>(\frac{mdn(\xi) cn(\xi)}{1 + msn^2(\xi)})</td>
</tr>
<tr>
<td>(2m^3 + m^4 + m^2)</td>
<td>(-6m - m^2 - 1)</td>
<td>(\frac{4}{m})</td>
<td>(\frac{mdn(\xi) cn(\xi)}{1 + msn^2(\xi)})</td>
</tr>
<tr>
<td>(2 + 2k_1 - m^2)</td>
<td>(6k_1 - m^2 + 2)</td>
<td>(4k_1)</td>
<td>(\frac{m^2 sn(\xi) cn(\xi)}{k_1 + dn^2(\xi)})</td>
</tr>
<tr>
<td>(2 - 2k_1 - m^2)</td>
<td>(-6k_1 - m^2 + 2)</td>
<td>(-4k_1)</td>
<td>(\frac{-m^2 sn(\xi) cn(\xi)}{k_1 + dn^2(\xi)})</td>
</tr>
<tr>
<td>(m^2 - 1)</td>
<td>(m^2 + 1)</td>
<td>((C^2 m^2 - B^2)(m^2 - 1))</td>
<td>(\sqrt{(B^2 - C^2)(B^2 - C^2 m^2) + sn(\xi)})</td>
</tr>
<tr>
<td>((C^2 m^2 - B^2))</td>
<td>(2)</td>
<td>(4)</td>
<td>(Bcn(\xi) + Cdn(\xi))</td>
</tr>
<tr>
<td>((C^2 + B^2))</td>
<td>(2)</td>
<td>(4)</td>
<td>(\sqrt{(B^2 + C^2 - C^2 m^2)(B^2 + C^2) + db(\xi)})</td>
</tr>
<tr>
<td>((C^2 + B^2))</td>
<td>(2)</td>
<td>(4)</td>
<td>(Bsn(\xi) + Ccn(\xi))</td>
</tr>
<tr>
<td>(2m - m^2 - 1)</td>
<td>(2m^2 + 2)</td>
<td>(-B^2 m^2 - B^2 - 2B^2 m)</td>
<td>(\frac{msn^2(\xi) - 1}{B(\msn^2(\xi) + 1)})</td>
</tr>
</tbody>
</table>
Table 2: Continued.

<table>
<thead>
<tr>
<th>$h_0$</th>
<th>$h_2$</th>
<th>$h_4$</th>
<th>$F(\xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2m + m^2 + 1}{B^2}$</td>
<td>$2m^2 + 2$</td>
<td>$-B'm^2 - B^2 - 2B'm$</td>
<td>$\frac{msn^2 (\xi) + 1}{B (msn^2 (\xi) - 1)}$</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Case 3. If $h_0 = k, h_1 = -4k, h_2 = -1 - k^2 + 6k, h_3 = 2(k - 1)^2, h_4 = -(k - 1)^2$, (7) has the following solutions:

$$F(\xi) = \frac{\sqrt{k} \text{sn} (\xi) + 1}{2 \sqrt{k} \text{sn} (\xi) + \text{ksn} (\xi)^2 + 1},$$

(8)

$$F(\xi) = \frac{k^{1/2} \text{sn} (\xi) + 1}{k^{1/2} \text{sn} (\xi) - \text{dn}^2 (\xi) + 1}.$$

Case 4. If $h_0 = -h_1/4, h_1 < 0, h_2 = -h_1(2 - k^2), h_3 = 2h_1(1 - k^2), h_4 = -h_1(1 - k^2)$, (7) has the following solutions:

$$F(\xi) = \frac{\text{sn} \left( \sqrt{h_1} \xi \right)}{\text{sn} \left( \sqrt{-h_1} \xi \right) + \text{cn} \left( \sqrt{-h_1} \xi \right) + 1},$$

(9)

$$F(\xi) = 1 + \text{cn} \left( \sqrt{-h_1} \xi \right) - \text{sn} \left( \sqrt{-h_1} \xi \right) + \text{cn} \left( \sqrt{-h_1} \xi \right) + 1.$$

Case 5. If $h_0 > 0, h_1 = -4h_0, h_2 = 4h_0(2 - k^2), h_3 = -8h_0(1 - k^2), h_4 = 4h_0(1 - k^2)$, (7) has the following solutions:

$$F(\xi) = \frac{\text{cn} \left( 2 \sqrt{h_0} \xi \right)}{\sqrt{1 - k^2} \text{sn} \left( 2 \sqrt{h_0} \xi \right) + \text{cn} \left( 2 \sqrt{h_0} \xi \right) + \text{dn} \left( 2 \sqrt{h_0} \xi \right)},$$

(10)

$$F(\xi) = \frac{\sqrt{1 - k^2} \text{sn} \left( 2 \sqrt{h_0} \xi \right) + \text{dn} \left( 2 \sqrt{h_0} \xi \right)}{\sqrt{1 - k^2} \text{sn} \left( 2 \sqrt{h_0} \xi \right) + \text{cn} \left( 2 \sqrt{h_0} \xi \right) + \text{dn} \left( 2 \sqrt{h_0} \xi \right)}.$$

Case 6. If $h_0 = -h_1/4, h_1 = -(ka)^2, h_2 = -(2h_1 + \alpha^2), h_3 = 2(h_1 + \alpha^2), h_4 = -(h_1 + \alpha^2)$, (7) has the following solutions:

$$F(\xi) = \frac{k \text{sn} (\alpha_0 \xi)}{k \text{sn} (\alpha_0 \xi) + \text{dn} (\alpha_0 \xi) - 1},$$

(11)

$$F(\xi) = \frac{k \text{cn} (\alpha_0 \xi)}{k \text{cn} (\alpha_0 \xi) + \text{dn} (\alpha_0 \xi) + \sqrt{1 - k^2}},$$

$$F(\xi) = \frac{\text{dn} (\alpha_0 \xi) + \sqrt{1 - k^2}}{k \text{sn} (\alpha_0 \xi) + \text{dn} (\alpha_0 \xi) + \sqrt{1 - k^2}}.$$

3. Solutions for the Generalized Variable-Coefficient KdV-MKdV Equation

To solve (1), we first make the transformation

$$u = u(\xi), \quad \xi = p(t) x + q(t).$$

(12)

where $p(t)$ and $q(t)$ are functions to be determined later. We consider that the solutions of (1) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{l} a_i(t, x) F^i(\xi),$$

(13)

where $a_i(t, x) (i = 0, 1, \ldots, l)$ are functions to be determined later.

Substituting (13) into (1), we can easily find that $l = 1$ by balancing $u_{xxx}$ and $u_t$ in (1). Therefore,

$$u(\xi) = a_0(t, x) + a_1(t, x) F(\xi).$$

(14)

With the aid of Mathematica, substituting (14) along with (7) into (1) and then setting each coefficient of $F(\xi)$ to zero, we obtain a set of nonlinear and parameterized partial differential equations with respect to unknowns $a_0$, $a_1$, $p$, and $q$ as follows:

$$a_{0t} + Af_4(t) a_0 = 0,$$

$$a_{tt} + Af_4(t) a_1 = 0,$$

$$-6 f_0(t) a_1^2 p(t) - 12 p(t) a_0 a_1^2,$$

$$+ 3 a_1 h_2 f_2(t) p(t)^3 = 0,$$

$$a_1 q'(t) - 6 a_0 a_1 f_0(t) p(t) - 6 a_0^2 a_1 f_1(t) p(t),$$

$$+ a_1 h_2 f_2(t) p(t)^3 - a_1 f_3(t) p(t) = 0,$$

$$a_1 p'(t) + a_1 f_4(t) p(t) = 0,$$

$$-6 f_1(t) a_1^3 p(t) + 6 h_4 f_2(t) a_1 p(t)^3 = 0,$$

$$a_{0x} f_4(t) = 0,$$

$$a_{1x} f_4(t) = 0,$$

where $a_{it} = \partial a_i / \partial t, a_{tx} = \partial a_i / \partial t, a_{0x} = \partial a_0 / \partial x, a_{1x} = \partial a_1 / \partial x$. 
Table 3: Solutions of Case 2.2.  $\Delta_2 = h_3^2 - 4h_1h_4$, $e = \pm 1$, and $a, b, c,$ and $d$ are the arbitrary constants.

<table>
<thead>
<tr>
<th>$h_2$</th>
<th>$h_3$</th>
<th>$h_4$</th>
<th>$\Delta_2$</th>
<th>$F(\xi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{2c}{a}$</td>
<td>$\frac{c^2 - b^2}{a^2}$</td>
<td>$\frac{a \sech(\xi)}{b + c \sech(\xi)}$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\frac{2c}{a}$</td>
<td>$\frac{c^2 + b^2}{a^2}$</td>
<td>$\frac{a \csch(\xi)}{b + c \csch(\xi)}$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\frac{-4(2b + d)}{a}$</td>
<td>$\frac{c^2 + 4b^2 + 4bd}{a^2}$</td>
<td>$\frac{a \sech^2(\xi)}{(b \sech^2(\xi) + c \tanh(\xi) + d)$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\frac{-4(d - 2b)}{a}$</td>
<td>$\frac{c^2 + 4b^2 - 4bd}{a^2}$</td>
<td>$\frac{a \csch^2(\xi)}{(b \csch^2(\xi) + c \coth(\xi) + d)$</td>
<td></td>
</tr>
<tr>
<td>$a^2$</td>
<td>$2ab$</td>
<td>$b^2$</td>
<td>$\frac{a}{b} (c + \cosh(a \xi) - \sinh(a \xi))$</td>
<td></td>
</tr>
<tr>
<td>$a^2$</td>
<td>$2ab$</td>
<td>$b^2$</td>
<td>$\frac{a}{b} (c + \cosh(a \xi) + \sinh(a \xi))$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$\frac{2c}{a}$</td>
<td>$\frac{-c^2 - b^2}{a^2}$</td>
<td>$\frac{a \text{asec}(\xi)}{b + c \text{asec}(\xi)}$</td>
<td></td>
</tr>
<tr>
<td>$-1$</td>
<td>$\frac{2c}{a}$</td>
<td>$\frac{-c^2 + b^2}{a^2}$</td>
<td>$\frac{a \text{acsc}(\xi)}{b + c \text{acsc}(\xi)}$</td>
<td></td>
</tr>
<tr>
<td>$-4$</td>
<td>$\frac{4(2b + d)}{a}$</td>
<td>$\frac{-c^2 + 4b^2 + 4bd}{a^2}$</td>
<td>$\frac{a \text{asec}(\xi)}{(b \text{asec}(\xi) + c \tan(\xi) + d)$</td>
<td></td>
</tr>
<tr>
<td>$-4$</td>
<td>$\frac{4(2b + d)}{a}$</td>
<td>$\frac{-c^2 + 4b^2 + 4bd}{a^2}$</td>
<td>$\frac{a \text{acsc}(\xi)}{(b \text{acsc}(\xi) + c \cot(\xi) + d)$</td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$h_3 - h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_3 - h_1 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$\frac{4h_2 e^{\sqrt{h_2} \xi}}{(e^{\sqrt{h_2} \xi})^2 - 4h_1h_4}$</td>
<td>$h_3 - h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$h_3 - h_1 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_3 - h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt;0$</td>
<td>$h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_3 - h_1 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&lt;0$</td>
<td>$h_1 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$h_3 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_3 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$h_1 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$=0$</td>
<td>$h_2 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_2 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$h_3 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_3 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$&gt;0$</td>
<td>$h_1 \left(1 + e \tanh \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td>$h_1 \left(1 + e \coth \left(\frac{\sqrt{h_2}}{2} \xi \right) \right)^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
For the sake of simplicity, in the rest of this paper, we introduce four notations:

\[
\varphi(t) = \exp \left[ - \int f_4(s) \, ds \right],
\]
\[
\varphi(s) = \exp \left[ - \int f_4(t) \, dt \right],
\]
\[
\varphi(t) = \exp \left[ - \int f_4(s) \, ds \right],
\]
\[
\varphi(s) = \exp \left[ - \int f_4(t) \, dt \right].
\]

(16)

Solving the system by Mathematica, we get the following solution set:

\[
a_0 = -\frac{a_1 \left[ 2a_1 f_0(t) - p^2(t) h_3 f_2(t) \right]}{4 p^2(t) h_4 f_2(t)},
\]
\[
a_1 = c_2 \varphi(t),
\]
\[
p(t) = c_1 \varphi(t),
\]
\[
f_1(t) = \frac{h_4 p^2(t) f_2(t)}{a_1^2},
\]
\[
q(t) = c_3 + \int \frac{1}{8 h_4 f_2(s) p(s)} \left[ 12 a_1^2 f_0^2(s) \right.
\]
\[
+ \left. \left( 3 h_4^2 - 8 h_4 h_4 \right) p^4(s) f_2^2(s) \right] ds,
\]
\[
+ 8 h_4 p^2(s) f_2(s) f_3(s) \right] ds,
\]

where \(c_1, c_2,\) and \(c_3\) are arbitrary constants.

Therefore, from Cases 1–6, we obtain many kinds of explicit nontravelling solutions of (1). Taking Case 1 as an example, we have the following.

**Case 7.** Soliton and soliton-like solutions are as follows (Figures 1 and 2):

\[
u_1 = -\frac{a_1^2 f_0(t)}{2 p^2(t) h_4 f_2(t)} + c_2 \varphi(t) \sqrt{\frac{-h_2}{h_4}} \cosh \left( \sqrt{h_2 / h_4} \xi \right),
\]

where \( \xi = c_1 \varphi(t) x + \int \left[ \left( 3a_1^2 f_0^2(s) - 2 h_4 h_4 p^4(s) f_2^2(s) + 2 h_4 p^2(s) f_2(s) f_3(s) \right] / 2 h_4 f_2(s) p(s) \right] ds + c_3. \)

Consider

\[
u_2 = -\frac{a_1^2 f_0(t)}{2 p^2(t) h_4 f_2(t)} + c_2 \varphi(t) \sqrt{\frac{h_2}{h_4}} \csc \left( \sqrt{h_2 / h_4} \xi \right),
\]

where \( \xi = c_1 \varphi(t) x + \int \left[ \left( 3a_1^2 f_0^2(s) - 2 h_4 h_4 p^4(s) f_2^2(s) + 2 h_4 p^2(s) f_2(s) f_3(s) \right] / 2 h_4 f_2(s) p(s) \right] ds + c_3. \)

Consider

\[
u_3 = -\frac{a_1^2 f_0(t)}{2 p^2(t) h_4 f_2(t)} + c_2 \varphi(t) \sqrt{\frac{-h_2}{h_4}} \tanh \left( \sqrt{\frac{-h_2}{h_4}} \xi \right),
\]

where \( \xi = c_1 \varphi(t) x + \int \left[ \left( 3a_1^2 f_0^2(s) - 2 h_4 h_4 p^4(s) f_2^2(s) + 2 h_4 p^2(s) f_2(s) f_3(s) \right] / 2 h_4 f_2(s) p(s) \right] ds + c_3.

**Case 8.** Triangular periodic solutions are as follows:

\[
u_5 = -\frac{a_1^2 f_0(t)}{2 p^2(t) h_4 f_2(t)} + c_2 \varphi(t) \sqrt{\frac{-h_2}{h_4}} \sec \left( \sqrt{\frac{-h_2}{h_4}} \xi \right),
\]

where \( \xi = c_1 \varphi(t) x + \int \left[ \left( 3a_1^2 f_0^2(s) - 2 h_4 h_4 p^4(s) f_2^2(s) + 2 h_4 p^2(s) f_2(s) f_3(s) \right] / 2 h_4 f_2(s) p(s) \right] ds + c_3.

Figure 1: The soliton-like solution (18) with \( h_2 = c_1 = c_2 = A = 1, h_4 = -1, c_3 = 0, \) and \( f_0(t) = f_2(t) = f_3(t) = t; f_1(t) = -t. \)

Figure 2: The soliton-like solution (20) with \( h_2 = -2, h_4 = h_0 = c_1 = c_2 = A = 1, c_3 = 0, \) and \( f_0(t) = f_1(t) = f_2(t) = f_3(t) = t. \)
where \( \xi = c_1 \phi(t)x + \int (1/2h_2 f_3(s)p(s))[-3a_1 f_0(s) - 2h_2 h_4 p^4(s) f_2(s) + 2h_4 p^2(s) f_2(s) f_3(s)]ds + c_3. \) Consider

\[
u_6 = -\frac{a_1^2 f_0(t)}{2p^2(t) f_2(t)} + c_2 \phi(t) \csc \left( \frac{h_2}{h_4} \right),
\] (23)

where \( \xi = c_1 \phi(t)x + \int (1/2h_2 f_3(s)p(s))[-3a_1^2 f_0(s) - 2h_2 h_4 p^4(s) f_2(s) + 2h_4 p^2(s) f_2(s) f_3(s)]ds + c_3. \) Consider

\[
u_7 = -\frac{a_1^2 f_0(t)}{2p^2(t) h_4 f_2(t)} + c_2 \phi(t) \tan \left( \frac{h_2}{2h_4} \right),
\] (24)

where \( \xi = c_1 \phi(t)x + \int (1/2h_2 f_3(s)p(s))[-3a_1^2 f_0(s) - 2h_2 h_4 p^4(s) f_2(s) + 2h_4 p^2(s) f_2(s) f_3(s)]ds + c_3. \) Consider

Case 9. Jacobi elliptic function solutions and combined Jacobi elliptic function solutions are as follows:

\[
u_6 = -\frac{a_1^2 f_0(t)}{2p^2(t) m^2 f_2(t)} + c_2 \phi(t) cd(\xi),
\] (25)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2(m^2 + m^2)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2f_2(s) p(s)ds + c_3. \) Consider

\[
u_9 = -\frac{a_1^2 f_0(t)}{2p^2(t) f_2(t)} + c_2 \phi(t) ns(\xi),
\] (26)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2(m^2 - 2)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2f_2(s) p(s)ds + c_3. \) Consider

\[
u_{10} = \frac{a_1^2 f_0(t)}{2p^2(t) f_2(t)} + c_2 \phi(t) dn(\xi),
\] (27)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2(m^2 - 2)m^4)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2f_2(s) p(s)ds + c_3. \) Consider

\[
u_{11} = \frac{a_1^2 f_0(t)}{2m^2 p^2(t) f_2(t)} + c_2 \phi(t) cn(\xi),
\] (28)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2(m^2 - 2)m^4)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{12} = -\frac{a_1^2 f_0(t)}{2p^2(t) (1 - m^2) f_2(t)} + c_2 \phi(t) nc(\xi),
\] (29)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 4m^4 - 6m^2 + 2)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2(1 - m^2) f_2(s) p(s)ds + c_3. \) Consider

\[
u_{13} = -\frac{a_1^2 f_0(t)}{2p^2(t) (m^2 - 1) f_2(t)} + c_2 \phi(t) nd(\xi),
\] (30)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2(m^2 - 3m^2) + 2)p^4(s) f_2^2(s) + (2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{14} = -\frac{a_1^2 f_0(t)}{2p^2(t) f_2(t)} + c_2 \phi(t) cs(\xi),
\] (31)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2m^2 - 4)p^4(s) f_2^2(s) + 2p^2(s) f_2(s) f_3(s)/2f_2(s) p(s)ds + c_3. \) Consider

\[
u_{15} = \frac{a_1^2 f_0(t)}{2p^2(t) (-1 + m^2) f_2(t)} + c_2 \phi(t) sc(\xi),
\] (32)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) - 2m^2 + 3m^2 + 2)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{16} = -\frac{a_1^2 f_0(t)}{2p^2(t) (m^2 + 1) f_2(t)} + c_2 \phi(t) sd(\xi),
\] (33)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) - 2m^2 - 2)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{17} = -\frac{a_1^2 f_0(t)}{2p^2(t) f_2(t)} + c_2 \phi(t) ds(\xi),
\] (34)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2m^2 - 3m^2 + 1)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2(1 - m^2) f_2(s) p(s)ds + c_3. \) Consider

\[
u_{18} = -\frac{2a_1^2 f_0(t)}{p^2(t) f_2(t)} + c_2 \phi(t) \left[ ns(\xi) \pm cs(\xi) \right],
\] (35)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2m^2 - 3m^2 + 1)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{19} = \frac{2a_1^2 f_0(t)}{p^2(t) (m^2 - 1) f_2(t)}
+ c_2 \phi(t) \left[ nc(\xi) \pm sc(\xi) \right],
\] (36)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) + 2m^2 - 3m^2 + 1)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2(1 - m^2) f_2(s) p(s)ds + c_3. \) Consider

\[
u_{20} = -\frac{2a_1^2 f_0(t)}{p^2(t) m^2 f_2(t)} + c_2 \phi(t) \left[ ns(\xi) \pm ds(\xi) \right],
\] (37)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) - 2m^2 - 1)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{21} = -\frac{2a_1^2 f_0(t)}{p^2(t) m^2 f_2(t)} + c_2 \phi(t) \left[ sn(\xi) \pm cn(\xi) \right],
\] (38)

where \( \xi = c_1 \phi(t)x + \int (3a_1^2 f_0(s) - 2m^2 - 1)p^4(s) f_2^2(s) + 2m^2 p^3(s) f_2(s) f_3(s))/2m^2 f_2(s) p(s)ds + c_3. \) Consider

\[
u_{22} = -\frac{2a_1^2 f_0(t)}{p^2(t) m^2 f_2(t)} + c_2 \phi(t) \frac{dn(\xi)}{i \sqrt{1 - m^2} sn(\xi) \pm cn(\xi)},
\] (39)
where \( \xi = c_1 \phi(t)x + \int([(-12a_1^2 f_0^2(s) - (m^2 - 2m + 2)m^2 p^4(s)f_2^2(s) + 2m^2 p^5(s)f_2(s)f_3(s)]m^2 f_2(s)p(s))ds + c_5. \) Consider

\[
u_{23} = -\frac{a_1^2 f_0(t)}{2p^2(t)f_2(t)} + c_2 \phi(t) \frac{sn(\xi)}{cn(\xi)}, \quad (40)
\]

where \( \xi = c_1 \phi(t)x + \int([(-3a_1^2 f_0^2(s) - (2 + 4m^2)^2 p^4(s)f_2^2(s) + 2p^5(s)f_2(s)f_3(s)]2m^2 f_2(s)p(s))ds + c_5. \) Consider

\[
u_{24} = -\frac{2a_1^2 f_0(t)}{p^2(t)A^2(1 + m^2)f_2(t)} + c_2 \phi(t) \frac{dn(\xi)cn(\xi)}{A(1 + sn(\xi))(1 + msn(\xi))}, \quad (41)
\]

where \( \xi = c_1 \phi(t)x + \int([(-12a_1^2 f_0^2(s) - (1 + m^2)m^2 - 6m + 1)p^4(s)f_2^2(s) + 2A^2(1 + m^2)p^3(s)f_2(s)f_3(s)]2A^2(1 + m^2)f_2(s)p(s))ds + c_5. \) Consider

\[
u_{25} = -\frac{2a_1^2 f_0(t)}{p^2(t)A^2(1 + m^2)}f_2(t) + c_2 \phi(t) \frac{dn(\xi)cn(\xi)}{A(1 + sn(\xi))(1 - msn(\xi))}, \quad (42)
\]

where \( \xi = c_1 \phi(t)x + \int([m^2[3ma_1^2 f_0^2(s) - 8(m^2 - 6m + 1)p^4(s)f_2^2(s) + 8p^5(s)f_2(s)f_3(s)f_2(s)]f_2(s)p(s))ds + c_5. \) Consider

\[
u_{26} = \frac{ma_1^2 f_0(t)}{8p^2(t)}f_2(t) + c_2 \phi(t) \frac{m(dn(\xi)cn(\xi)}{1 + msn^2(\xi)}, \quad (43)
\]

where \( \xi = c_1 \phi(t)x + \int([m^2-[3ma_1^2 f_0^2(s) - 8k_1(6k_1 - m^2 + 1)p^4(s)f_2^2(s) + 8k_1p^5(s)f_2(s)f_3(s)]f_2(s)p(s))ds + c_5. \) Consider

\[
u_{27} = \frac{-a_1^2 f_0(t)}{8p^2(t)}f_2(t) + c_2 \phi(t) \frac{m^2sn(\xi)cn(\xi)}{k_1 - 1}, \quad (44)
\]

where \( \xi = c_1 \phi(t)x + \int([(-3a_1^2 f_0^2(s) - 8k_1(6k_1 - m^2 + 2)p^4(s)f_2^2(s) + 8k_1p^5(s)f_2(s)f_3(s)]f_2(s)p(s))ds + c_5. \) Consider

\[
u_{28} = \frac{a_1^2 f_0(t)}{8k_1p^2(t)}f_2(t) - c_2 \phi(t) \frac{m^2sn(\xi)cn(\xi)}{k_1 + dn^2(\xi)}, \quad (45)
\]

where \( \xi = c_1 \phi(t)x + \int([3a_1^2 f_0^2(s)] + 8k_1(6k_1 + 1 + m^2 - 2)p^4(s)f_2^2(s) + 8k_1p^5(s)f_2(s)f_3(s)]f_2(s)p(s))ds + c_5. \) Consider

\[
u_{29} = \frac{a_1^2 f_0(t)}{8k_1p^2(t)}f_2(t) - c_2 \phi(t) \frac{m^2sn(\xi)cn(\xi)}{k_1 + dn^2(\xi)}, \quad (46)
\]

where \( \xi = c_1 \phi(t)x + \int([(-12a_1^2 f_0^2(s) + 2m^2 - 1)p^4(s)f_2^2(s) + 2p^5(s)f_2(s)f_3(s)]2f_2(s)p(s))ds + c_5. \) Consider

\[
u_{30} = -\frac{2a_1^2 f_0(t)}{p^2(t)(m^2 - 1)(C^2m^2 - B^2)f_2(t)} + c_2 \phi(t) \frac{\sqrt{(B^2 - C^2 + B^2 - C^2m^2)} + sn(\xi)}{Bcn(\xi) + Ccn(\xi)}, \quad (47)
\]

where \( \xi = c_1 \phi(t)x + \int([(-12a_1^2 f_0^2(s) - (m^2 - 1)(C^2m^2 - B^2)p^4(s)f_2^2(s) + 2(m^2 - 1)(C^2m^2 - B^2)p^5(s)f_2(s)f_3(s)]/(m^2 - 1)(C^2m^2 - B^2)f_2(s)p(s))ds + c_5. \) Consider

\[
u_{31} = -\frac{2a_1^2 f_0(t)}{p^2(t)(C^2 + B^2)f_2(t)} \quad (48)
\]

\[+ c_2 \phi(t) \frac{\sqrt{(B^2 + C^2 - C^2m^2 + (C^2 + B^2)p^2(s)f_2(s)f_3(s)]/(C^2 + B^2)f_2(s)p(s))ds + c_5. \) Consider

\[
u_{32} = \frac{a_1^2 f_0(t)}{2p^2(t)(m^2 - 1)f_2(t)} + c_2 \phi(t) \frac{m^2sn^2 - 1}{B(m^2sn^2 + 1)}, \quad (49)
\]

where \( \xi = c_1 \phi(t)x + \int([m^2[3ma_1^2 f_0^2(s) - 4B^2(m^2 + 1)^2p^4(s)f_2^2(s) + 2B^2(m^2 + 1)p^5(s)f_2(s)f_3(s)]2B^2(m^2 + 1)f_2(s)p(s))ds + c_5. \) Consider

\[
u_{33} = \frac{-a_1^2 f_0(t)}{2p^2(t)(m^2 - 1)f_2(t)} + c_2 \phi(\xi) \frac{m^2sn^2 + 1}{B(m^2sn^2 - 1)}, \quad (50)
\]

where \( \xi = c_1 \phi(t)x + \int([m^2[3ma_1^2 f_0^2(s) - 4B^2(m^2 - 1)^2p^4(s)f_2^2(s) + 2B^2(m^2 - 1)^2p^5(s)f_2(s)f_3(s)]2B^2(m^2 - 1)f_2(s)p(s))ds + c_5. \) Consider

\[
u_{34} = \frac{-2a_1^2 f_0(t)}{p^2(t)f_2(t)} + c_2 \phi(t) \frac{m^2sn(\xi) + idn(\xi)}{1}, \quad (51)
\]

where \( \xi = c_1 \phi(t)x + \int([((-12a_1^2 f_0^2(s) + 2m^2 - 1)p^4(s)f_2^2(s) + 2p^5(s)f_2(s)f_3(s)]2f_2(s)p(s))ds + c_5. \) Consider

\[
u_{35} = -\frac{2a_1^2 f_0(t)}{p^2(t)f_2(t)} + c_2 \phi(t) \frac{m^2sn(\xi) + idn(\xi)}{1}, \quad (52)
\]

where \( \xi = c_1 \phi(t)x + \int([(-12a_1^2 f_0^2(s) + 2m^2 - 1)p^4(s)f_2^2(s) + 2p^5(s)f_2(s)f_3(s)]2f_2(s)p(s))ds + c_5. \) Consider

\[
u_{36} = -\frac{2a_1^2 f_0(t)}{p^2(t)f_2(t)} + c_2 \phi(t) \frac{sn(\xi)}{1 \pm cn(\xi)}, \quad (53)
\]

where \( \xi = c_1 \phi(t)x + \int([(-12a_1^2 f_0^2(s) + 2m^2 - 1)p^4(s)f_2^2(s) + 2p^5(s)f_2(s)f_3(s)]2f_2(s)p(s))ds + c_5. \) Consider

\[
u_{37} = -\frac{2a_1^2 f_0(t)}{p^2(t)f_2(t)} + c_2 \phi(t) \frac{cn(\xi)}{\sqrt{1 - m^2sn(\xi) \pm dn(\xi)}}, \quad (54)
\]
where $\xi = c_1 \phi(t)x + \int \left[ (-12a^2 f_0^2(s) + (2m^2 - 1)p^4(s)f_2^2(s) + 2p^2(s)f_2(s)f_3(s)/2f_2(s)p(s))ds + c_5 \right]$. Consider

$$u_{38} = -\frac{2a^2 f_0(t)}{p^2(t)(m^2 - 1)f_2(t)} + c_2 \phi(t) \frac{dn(\xi)}{1 \pm m sn(\xi)},$$

where $\xi = c_1 \psi(t)x + \int \left[ (-12a^2 f_0^2(s) - (m^2 - 1)p^4(s)f_2^2(s) + 2(m^2 - 1)p^2(s)f_2(s)f_3(s))/2(f_2(s)p(s))ds + c_5 \right]$. Consider

$$u_{39} = -\frac{2a^2 f_0(t)}{p^2(t)(m^2 - 1)f_2(t)} + c_2 \phi(t) \frac{cn(\xi)}{1 \pm m sn(\xi)},$$

where $\xi = c_1 \psi(t)x + \int \left[ (-12a^2 f_0^2(s) + (m^2 - 1)p^4(s)f_2^2(s) + 2(1 - m^2)p^2(s)f_2(s)f_3(s))/2f_2(s)p(s))ds + c_5 \right]$. Consider

$$u_{40} = -\frac{2a^2 f_0(t)}{p^2(t)(m^2 - 1)f_2(t)} + c_2 \phi(t) \frac{sn(\xi)}{dn(\xi) \pm cn(\xi)},$$

where $\xi = c_1 \psi(t)x + \int \left[ (-12a^2 f_0^2(s) - (m^2 + 1)(m^2 - 1)p^4(s)f_2^2(s) + 2(m^2 - 1)p^2(s)f_2(s)f_3(s))/2m^4 f_2(s)p(s))ds + c_5 \right]$. Consider

$$u_{41} = -\frac{2a^2 f_0(t)}{p^2(t)(m^2 - 1)f_2(t)} + c_2 \phi(t) \frac{sn(\xi)}{dn(\xi) \pm cn(\xi)},$$

where $\xi = c_1 \psi(t)x + \int \left[ (-12a^2 f_0^2(s) - m^2(m^2 - 2)p^4(s)f_2^2(s) + 2m^4 p^2(s)f_2(s)f_3(s))/2m^4 f_2(s)p(s))ds + c_5 \right]$. Consider

$$u_{42} = -\frac{2a^2 f_0(t)}{p^2(t)m^2 f_2(t)} + c_2 \phi(t) \frac{cn(\xi)}{\sqrt{1 - m^2} \pm dn(\xi)},$$

where $\xi = c_1 \psi(t)x + \int \left[ (-12a^2 f_0^2(s) - m^2(m^2 - 2)p^4(s)f_2^2(s) + 2m^4 p^2(s)f_2(s)f_3(s))/2m^4 f_2(s)p(s))ds + c_5 \right]$. Consider

$$u_{43} = -\frac{2a^2 f_0(t)}{p^2(t)m^2 f_2(t)} + c_2 \phi(t) \frac{cn(\xi)}{\sqrt{1 - m^2} \pm dn(\xi)},$$

where $\xi = c_1 \psi(t)x + \int \left[ (-12a^2 f_0^2(s) - m^2(m^2 - 2)p^4(s)f_2^2(s) + 2m^4 p^2(s)f_2(s)f_3(s))/2m^4 f_2(s)p(s))ds + c_5 \right]$. Consider

Remark 2. There are some other hyperbolic function solutions and trigonometric function solutions which can be obtained from solutions (25)–(60) at the limit case when $m \rightarrow 1$ and $m \rightarrow 0$, but we omit them for simplicity.

Remark 3. By using our method, we can obtain many more general exact solutions including all the solutions given in [20] as special cases. More precisely, setting $h_0 = h_1 = h_3 = 0$, $h_2 = 1$, and $h_4 = -1$, solution (18) is equivalent to (22) in [20]. Similarly, solution (20) is equivalent to (38) obtained in [20] with $h_0 = h_4 = 1$, $h_1 = h_3 = 0$, and $h_2 = -1$. Setting $h_0 = 0$, $h_4 = 1/4$, $h_1 = h_3 = 0$, $h_2 = -1/2$, and $m \rightarrow 1$, solution (52) is equivalent to (62) in [20]. The comparison between our method and the well known results in [20] shows that our method is more powerful than the method in [20] in constructing exact solutions.

4. Conclusion

With the help of symbolic computation, we employ the auxiliary equation method to find many new and more general exact solutions of the generalized KdV-MKdV equation with variable coefficients. This paper showed that the auxiliary equation method not only gives a unified formulation to construct a series of exact solutions but also provides a guideline to classify the types of solutions according to the given parameters. Applying the more generalized ansatz and auxiliary equation method to other variable-coefficients NPDEs is our future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is supported by the Hubei Provincial Department of Education (B2015146).

References


