Research Article

Complementary Cycles in Irregular Multipartite Tournaments

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A tournament is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. A digraph D is cycle complementary if there exist two vertex disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. Let D be a locally almost regular c -partite tournament with $c \geq 3$ and $|\gamma(D)| \leq 3$ such that all partite sets have the same cardinality r , and let C_3 be a 3-cycle of D . In this paper, we prove that if $D - V(C_3)$ has no cycle factor, then D contains a pair of disjoint cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to T_7 , $D_{4,2}$, $D_{4,2}^*$, or $D_{3,2}$.

1. Introduction

Denote the vertex set and arc set of a digraph D by $V(D)$ and $E(D)$, respectively. For a subset X of $V(D)$, we define $D[X]$ as the subdigraph induced by X . If xy is an arc of a digraph D , then we write $x \rightarrow y$ and say x dominates y . If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y , then X dominates Y , denoted by $X \rightarrow Y$. We write $X \Rightarrow Y$ if there is no arc from Y to X .

The out-neighborhood $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x , and the in-neighborhood $N_D^-(x) = N^-(x)$ is the set of vertices dominating x . The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the outdegree and indegree of x , respectively. The minimum outdegree and the minimum indegree of D are denoted by $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$ and $\delta(D) = \delta = \min\{\delta^+, \delta^-\}$. The local irregularity is defined by $i_l(D) = \max\{d^+(x) - d^-(x)\}$ over all vertices x of D and the global irregularity is defined by $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} \mid x, y \in V(D)\}$. Clearly, $i_l(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is almost regular. If $i_l(D) \leq 1$, then D is locally almost regular.

By a path or cycle we mean a directed path or cycle. A cycle of length m is an m -cycle. The independence number $\alpha(D) = \alpha$ is the maximum size among all independent sets of

vertices of D . A digraph D is strongly connected or strong if, for each pair of vertices u and v , there is a path from u to v in D . A digraph D with at least $k + 1$ vertices is k -connected if for any set A of at most $k - 1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The connectivity of D , denoted by $\kappa(D)$, is defined to be the largest value of k such that D is k -connected.

A cycle factor is a spanning subdigraph consisting of disjoint cycles. A cycle factor with the minimum number of cycles is called a minimal cycle factor. If we replace every arc xy of D by yx , then we call the resulting digraph, denoted by D^{-1} , the converse digraph of D .

A c -partite graph is a graph whose vertices are or can be partitioned into c different independent sets. Equivalently, it is a graph that can be colored with c colors, so that no two endpoints of an edge have the same color. A complete c -partite graph is a graph that can be partitioned into c independent sets, as above, so that every pair of vertices from two different independent sets have different colors.

A c -partite or multipartite tournament is an orientation of a complete c -partite graph. If D is a c -partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$, then we define $\gamma(D) = |V_1|$. If the vertex x of D belongs to the partite set V_i , then we define $V(x) = V_i$.

A digraph D is cycle complementary if there exist two vertex disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$.

The problem of complementary cycles in tournaments was almost completely solved by Reid [1] in 1985 and by Song [2] in 1993. The authors proved that every 2-connected tournament D on at least 8 vertices has complementary cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \dots, |V(D)| - 3\}$.

There exists only the following conjecture.

Conjecture 1 (see Yeo [3] 1999). *A regular c -partite tournament D with $c \geq 4$ and $|V(D)| \geq 8$ has a pair of vertex disjoint cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \dots, |V(D)| - 3\}$.*

In 2005, Volkmann [4] confirmed the first conjecture for $t = 3$, unless D is a regular 4-partite tournament with two vertices in each partite set. In 2004, Volkmann [5] confirmed conjecture for $t = 4$. In 2009, He et al. [6] confirmed conjecture for $t = 5$. In 2013, Volkmann [7] confirmed every almost regular c -partite tournament D with $|V(D)| \geq 8$ such that all partite sets have the same cardinality r . If $c \geq 5$ or $c = 4$ and $r \geq 6$, then D contains a pair of vertex disjoint directed cycles of lengths 4 and $|V(D)| - 4$. In 2014, Bai et al. [8] confirmed every regular bipartite tournament contains two complementary cycles. In 2014, He et al. [9] characterized all locally almost regular multipartite tournaments D whose partite sets have the same cardinality with $\gamma(D) \geq 4$ that contain two complementary cycles of length 3 and $|V(D)| - 3$. Let D be a locally almost regular c -partite tournament with $c \geq 3$ and $|\gamma(D)| \leq 3$ such that all partite sets have the same cardinality, and let C_3 be a 3-cycle of D . We prove if $D - V(C_3)$ has no cycle factor, then D contains a pair of disjoint cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to T_7 , $D_{4,2}$, $D_{4,2}^*$, or $D_{3,2}$.

2. Preliminary Results

The following results play important roles in the proof of our main theorem (Theorem 12), where the first one is immediate.

Lemma 2 (see [7]). *If D is a regular c -partite tournament with the partite sets $V_1, V_2, \dots, V_c$, then $\alpha(D) = |V_1| = |V_2| = \dots = |V_c|$.*

Lemma 3 (see Yeo [10] 1998). *If D is a multipartite tournament, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3} \right\rceil. \quad (1)$$

Lemma 4 (see Reid [1] 1985 and Song [2] 1993). *If D is a 2-connected tournament with at least 6 vertices, then D contains two complementary cycles of length t and $|V(D)| - t$ for all $3 \leq t \leq |V(D)|/2$, unless it is isomorphic to the tournament T_7 .*

Lemma 5 (see Bondy [11] 1976). *Each strong c -partite tournament with $c \geq 3$ contains an m -cycle for each $m \in \{3, 4, \dots, c\}$.*

Lemma 6 (see Yeo [12] 1999). *Let D be a c -partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $k = |V_1| \leq |V_2| \leq \dots \leq |V_c| = k + i$ for some $i \in \{0, 1\}$. If*

$$i_l(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2}, \quad (2)$$

then D is Hamiltonian.

Lemma 7 (see Volkmann [5] 2004). *Let D be a regular 3-partite tournament with $|V(D)| \geq 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to $D_{3,2}$.*

Lemma 8 (see Volkmann [4] 2005). *Let D be a regular c -partite tournament with $c \geq 4$ and $|V(D)| \geq 6$. Then D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to T_7 , $D_{4,2}$, or $D_{4,2}^*$.*

Lemma 9 (see Tewes [13] 2002). *If D is a multipartite tournament, then*

$$\begin{aligned} \delta(D) &\geq \frac{|V(D)| - i_l(D) - \alpha(D)}{2} \\ &\geq \frac{|V(D)| - i_g(D) - \alpha(D)}{2}. \end{aligned} \quad (3)$$

Lemma 10 (see Yeo [12] 1999 and Gutin and Yeo [14] 2000). *A digraph D has no cycle factor if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1 , and R_2 such that*

$$\begin{aligned} R_1 &\Rightarrow Y, \\ (R_1 \cup Y) &\Rightarrow R_2, \\ |Y| &> |Z|, \end{aligned} \quad (4)$$

where Y is an independent set.

Lemma 11 (see Volkmann and Winzen [15] 2004). *Let $V_1, V_2, \dots, V_c$ be the partite sets of a c -partite tournament D with no cycle factor such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$. According to Lemma 10, the vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1 and R_2 satisfying (4) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with integers $k, t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. If $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$, and $Q_2 = Q \cap R_2$, then*

$$\begin{aligned} i_l(D) &\geq |V(D)| - 3|V_c| + 2t + 2k + 2, \\ i_g(D) &\geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3}{2}, \end{aligned} \quad (5)$$

if $Q_1 = \emptyset$ or $Q_2 = \emptyset$ case part (1) of lemma and

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2}, \quad (6)$$

if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ case part (2) of lemma.

3. Main Result

Theorem 12. *Let D be a locally almost regular c -partite tournament with $c \geq 3$ and $|\gamma(D)| \leq 3$ such that all partite sets have the same cardinality, and let C_3 be a 3-cycle of D . We prove that if $D - V(C_3)$ has no cycle factor, then according to Lemma 10, the vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1 and R_2 satisfying (4) such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$ and $|Y| > |Z|$, where Y is an independent set. If $|Z \cup Y| \leq 4$, then D contains a pair of disjoint cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to T_7 , $D_{4,2}$, $D_{4,2}^*$, or $D_{3,2}$.*

Proof. First of all notice that the condition that all partite sets have the same cardinality shows that $i_l(D) = i_g(D)$. Let $V_1, V_2, \dots, V_c$ be the partite sets of a locally almost regular c -partite tournament D such that $|V_1| = |V_2| = \dots = |V_c| = r$. If $|V(D)| - r$ is even, then it is easy to say that D is regular. Thus, if $c = 3$, then D is a regular 3-partite tournament, and by Lemma 7, D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to $D_{3,2}$.

According to Lemma 3, we have

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3} \right\rceil. \quad (7)$$

If $c \geq 7$ and $r = 1$, which means that D is a tournament, then (7) yields $\kappa(D) \geq 2$. In addition, if $c = 6$ and $r = 1$, then it can be shown that $\kappa(D) \geq 2$ too. Hence Lemma 4 implies that D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to T_7 .

Therefore we only need to consider the case $c \geq 4$ and $2 \leq r \leq 3$. In view of Lemma 3, $\kappa(D) \geq 2$ and thus Lemma 5 ensures the existence of a 3-cycle C_3 in D . If we define in the case $r \geq 2$ the c -partite tournament H by $H = D - V(C_3)$, then $i_l(H) \leq 4$, and $|V(H)| = cr - 3$. If $V'_1, V'_2, \dots, V'_c$ are the partite sets of H such that $|V'_1| \leq |V'_2| \leq \dots \leq |V'_c|$, then $|V'_1| = r - 1$ and $|V'_c| = r$. With exception of the cases $c = 4$ and $r = 2, 3$, $c = 5$ and $r = 2, 3$, $c = 6$ and $r = 2$, and $c = 7$ and $r = 2$, this leads to

$$i_l(H) \leq 4 \leq \frac{|V(H)| - |V'_{c-1}| - 2|V'_c| + 2}{2}. \quad (8)$$

Applying Lemma 6, we conclude that H has a Hamiltonian cycle C , and we obtain the desired result that $V(D) = V(C_3) \cup V(C)$. Since D is a regular c -partite tournament for $c = 4$ and $r = 2$, $c = 5$ and $r = 2, 3$, $c = 6$ and $r = 2$, and $c = 7$ and $r = 2$, by Lemma 8, D contains two complementary cycles of length 3 and $|V(D)| - 3$, unless D is isomorphic to $D_{4,2}$, or $D_{4,2}^*$. Therefore, there only remain the cases $c = 4$ and $r = 3$.

Let $V_1 = \{u_1, u_2, u_3\}$, $V_2 = \{v_1, v_2, v_3\}$, $V_3 = \{w_1, w_2, w_3\}$, $V_4 = \{x_1, x_2, x_3\}$ be the partite sets of D , $C_3 = u_3v_3w_3u_3$, and $H = D - V(C_3)$. Then $\alpha(D) = 3$ and $4 \leq d_D^+(x), d_D^-(x) \leq 5$ for each $x \in V(D)$. Then $i_l(H) \leq 4$, and $|V(H)| = 9$. If V'_1, V'_2, V'_3, V'_4 are the partite sets of H such that $|V'_1| \leq |V'_2| \leq |V'_3| \leq |V'_4|$, then $|V'_1| = |V'_2| = |V'_3| = 2$ and $|V'_4| = 3$. Then by Lemma 10, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y, (R_1 \cup Y) \Rightarrow R_2, |Y| > |Z|$, and Y is an independent set. Let, without loss of generality, $|R_1| \leq |R_2|$ (if $|R_1| \geq |R_2|$, then we consider D^{-1}). By Lemma 9, $\delta(H) \geq 1$.

Case 1. $Z = \emptyset$ and $|Y| = 1, 2, 3$. Then $R_1 \neq \emptyset$. Otherwise $d_D^+(x) \geq 6$ and $d_D^-(x) \leq 3$ for $x \in Y$ which contradicts $i_l(D) \leq 1$. Consider $|R_1| \neq 1, 2$ for $\delta(H) \geq 1$. So $3 \leq |R_1| \leq 4$.

Assume that $|Y| = 1$. We claim that $|R_1| \neq 3$, or else since $d_H^+(x), d_H^-(x) \geq 1$, $x \in V(H)$, R_1 induces a 3-cycle, and thus there is a vertex $y \in R_1$ such that $|V(y) \cap (Y \cup R_2)| \leq 1$. It follows that $d_D^+(y) \geq 6$ and $d_D^-(y) \leq 3$, a contradiction to $i_l(D) \leq 1$. Assume that $|R_1| = 4$. If R_1 is bipartite, then since $\delta(H) \geq 1$, each partite set of R_1 consists of two vertices.

Hence, R_1 induces a 4-cycle and thus there is a vertex $y \in R_1$ such that $|V(y) \cap (Y \cup R_2)| \leq 1$. It follows that $d_D^+(y) \geq 6$ and $d_D^-(y) \leq 3$, a contradiction to $i_l(D) \leq 1$. If R_1 is 3-partite, then there exists a vertex $y \in R_1$ such that $|V(y) \cap (Y \cup R_2)| \leq 2$ and $d_{R_1}^+(y) \geq 2$ or $|V(y) \cap (Y \cup R_2)| \leq 1$ and $d_{R_1}^+(y) \geq 1$. This leads to $d_D^+(y) \geq 6$ and $d_D^-(y) \leq 3$, a contradiction to $i_l(D) \leq 1$. If R_1 is 4-partite, then $y \in R_1$ such that $|V(y) \cap (Y \cup R_2)| \leq 1$ and $d_{R_1}^+(y) \geq 2$; then $d_D^+(y) \geq 6$ and $d_D^-(y) \leq 3$, a contradiction to $i_l(D) \leq 1$.

If $|Y| = 2$, then $|R_1| \leq 3$. Note that $\delta(H) \geq 1$, R_1 induces a 3-cycle. So there is a vertex $y \in R_1$ such that $d_D^+(y) \geq 6$ and $d_D^-(y) \leq 3$, a contradiction to $i_l(D) \leq 1$.

If $|Y| = 3$, then replacing k by 2 and t by 0 in Lemma 11 leads us to $i_l(H) \geq 5$, a contradiction to $i_l(H) \leq 4$.

Case 2. One has $|Z| = 1$ and $|Y| = 2$. We discuss this case by the value of $|R_1|$, where $|R_1| = 0, 1, 2, 3$.

Case 2.1. Assume that $|R_1| = 0$; that is, $R_1 = \emptyset$.

Note that $Z \rightarrow Y$ since $\delta(H) \geq 1$. If $Y \subset V_i$, for some $i \in \{1, 2, 3\}$, then $R_2 \cap V_i = \emptyset$. Then $d_D^+(y) \geq 6$ and $d_D^-(y) \leq 3$ for $y \in Y$, a contradiction. Therefore, $Y \subset V_4$. Assume, without loss of generality, that $Y = \{x_1, x_2\}$, $x_3 \in R_2$, and $Z = \{u_1\}$. Since $i_l(D) \leq 1$, $V(C_3) \rightarrow Y$, and there exists one vertex in $\{v_1, v_2, w_1, w_2\}$, say, v_1 such that $v_1 \rightarrow u_1$.

Case 2.1.1. Assume that $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has a Hamiltonian path.

(1) Assume that there is a Hamiltonian path which does not end at x_3 or begin at x_3 , say, $p_1p_2p_3p_4p_5$ with $x_3 \notin \{p_1, p_5\}$ in $H[\{u_2, v_2, w_1, w_2, x_3\}]$. If p_5 does not dominate p_1 , then p_5 has at least one positive neighbor x in C_3 , and there are the complementary cycles $x_1v_1u_1x_1$ and $x_2p_1p_2p_3p_4p_5xx^+x^{++}x_2$. If $p_5 \rightarrow p_1$, then there exist at least two arcs from $\{p_1, p_2, p_3, p_4, p_5\}$ to C_3 . Let, without loss of generality, $p_i \rightarrow x$ with $p_i^+ \neq x_3$ and $x \in V(C_3)$. This leads to the complementary cycles $x_1v_1u_1x_1$ and $x_2p_{i+1}p_{i+2} \dots p_ixx^+x^{++}x_2$, all indices are taken modulo 5.

(2) If there is a Hamiltonian path ending at x_3 in $H[\{u_2, v_2, w_1, w_2, x_3\}]$, say, $p_1p_2p_3p_4p_5$ with $p_5 = x_3$, then we need only to consider $\{p_1, p_2, p_3\} \rightarrow p_5$. The other cases are easily reduced to (1). Thus p_5 has at least one positive neighbor in C_3 , say, x . This leads to the complementary cycles $x_1v_1u_1x_1$ and $x_2p_1p_2p_3p_4p_5xx^+x^{++}x_2$.

(3) If there is a Hamiltonian path beginning at x_3 in $H[\{u_2, v_2, w_1, w_2, x_3\}]$, say, $p_1p_2p_3p_4p_5$ with $p_1 = x_3$, then we need only to consider $p_1 \rightarrow \{p_2, p_3, p_4, p_5\}$. The other cases are easily reduced to (1) or (2).

Assume that $p_1 \rightarrow v_1$. Then $(V(C_3) \cup \{u_1\}) \rightarrow p_1$, and we obtain a 3-cycle $p_1v_1u_1p_1$. Next we are to find its complementary cycle. Since $i_l(D) \leq 1$, p_5 dominates at least one vertex x of C_3 , and there exists at least one vertex of $\{p_2, p_3, p_4\}$, say, p_i , such that $p_i \rightarrow y$ with $i \in \{2, 3, 4\}$, $y \in V(C_3)$ and $x \neq y$. This leads to the complementary cycles $C_1 = p_1v_1u_1p_1$ and $C_2 = x_1p_2 \dots p_iyy^+x_2p_{i+1} \dots p_5xx_1$ if $y = x^+$ or $C_2 = x_1p_2 \dots p_iyx_2p_{i+1} \dots p_5xx^+x_1$ if $y = x^-$.

Now assume that $v_1 \rightarrow p_1$. If there is a vertex p_i of $\{p_2, p_3, p_4, p_5\}$ that dominates u_1 ($i \in \{2, 3, 4, 5\}$), then let p_i take the place of v_1 , the above case being reduced to (1) of

Case 2.1.1. Therefore, we let $u_1 \Rightarrow \{p_2, p_3, p_4, p_5\}$. Then we have $p_1 \cup V(C_3) \Rightarrow u_1$.

If $u_2 \notin \{p_3, p_5\}$ and $V(p_3) \neq V(p_5)$, then there exists an arc between p_3 and p_5 . Let, without loss of generality, $p_3 \rightarrow p_5$. This leads to the contradiction $d_D^-(p_5) \geq 6$.

If $u_2 \notin \{p_3, p_5\}$ and $V(p_3) = V(p_5)$, then $\{p_3, p_5\} = \{w_1, w_3\}$ and $\{p_3, p_5\} \Rightarrow \{v_1\} \cup V(C_3)$ and $p_5 \rightarrow p_2$. If $p_2 \rightarrow p_4$, then $p_4 = u_2$, $p_2 = v_2$, $\{p_2, p_4\} \Rightarrow \{v_1\} \cup V(C_3)$, and $v_1 \Rightarrow V(C_3)$. This leads to the complementary cycles $p_2 p_4 p_5 p_2$ and $x_1 p_3 u_3 x_2 v_1 p_1 v_3 w_3 u_1 x_1$. If $p_4 \rightarrow p_2$, then $p_2 = u_2$, $p_4 = v_2$, $\{p_2, p_4\} \Rightarrow \{v_1\} \cup V(C_3)$, and $v_1 \Rightarrow V(C_3)$. This leads to the complementary cycles $p_2 p_3 p_4 p_2$ and $x_1 p_5 u_3 x_2 v_1 p_1 v_3 w_3 u_1 x_1$.

If $u_2 \in \{p_3, p_5\}$, let, without loss of generality, $p_5 = u_2$; then we have $V(p_2) = V(p_4) = V(w_1)$, $p_3 = v_2$, $p_3 \rightarrow p_5$, $p_5 \rightarrow p_2$, $\{p_2, p_4, p_5\} \rightarrow v_1$, and $\{v_1, p_2, p_3, p_4, p_5\} \Rightarrow V(C_3)$. This leads to the complementary cycles $p_2 p_3 p_5 p_2$ and $x_1 v_1 p_1 u_1 x_2 p_4 u_3 v_3 w_3 x_1$.

Case 2.1.2. Now we assume that $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path. Then let, without loss of generality, $u_2 \rightarrow v_2$.

(1) Assume that v_2 dominates one vertex $\{w_1, w_2\}$, say, $v_2 \rightarrow w_2$.

(a) If $w_2 \rightarrow x_3$, then since $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path, $w_1 \rightarrow x_3$, $\{u_2, v_2\} \rightarrow \{w_1, x_3\}$, and $u_2 \rightarrow w_2$.

If there is one vertex of $\{w_1, w_2\}$ that dominates u_1 , let, without loss of generality, $w_1 \rightarrow u_1$; then let w_1 take the place of v_1 , the case being reduced to (1) of Case 2.1.1. Therefore, let $u_1 \rightarrow \{w_1, w_2\}$. Then we have $\{w_1, w_2\} \Rightarrow (\{v_1\} \cup V(C_3))$. Since $i_l(D) \leq 1$, $x_3 \rightarrow u_3$ or $x_3 \rightarrow v_3$. In the former case there are complementary cycles $w_1 v_1 u_1 w_1$ and $x_1 u_2 x_3 u_3 x_2 v_2 w_2 v_3 w_3 x_1$; in the latter case complementary cycles are $w_1 v_1 u_1 w_1$ and $x_1 u_2 w_2 u_3 x_2 v_2 x_3 v_3 w_3 x_1$.

(b) Assume that $x_3 \rightarrow w_2$.

If $w_1 \rightarrow u_2$, then since $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path, $\{w_1, u_2, v_2\} \rightarrow x_3$. This leads to the Hamiltonian path $w_1 u_2 v_2 x_3 w_2$, a contradiction.

If $u_2 \rightarrow w_1$ and $w_1 \rightarrow v_2$, then since $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path, $x_3 \rightarrow \{w_1, u_2, v_2\}$. This leads to the Hamiltonian path $x_3 u_2 w_1 v_2 w_2$, a contradiction.

If $u_2 \rightarrow w_1$ and $v_2 \rightarrow w_1$, then since $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path, $x_3 \rightarrow w_1$, $u_2 \rightarrow w_2$, and $\{w_1, w_2\} \Rightarrow (\{v_1, u_1\} \cup V(C_3))$. If $v_2 \rightarrow x_3$ or $x_3 \rightarrow v_2$ and $u_2 \rightarrow x_3$, then let w_1 take the place of v_1 , the case being reduced to (1) of Case 2.1.1. If $x_3 \rightarrow v_2$ and $x_3 \rightarrow u_2$, then let w_1 take the place of v_1 , the case being reduced to (3) of Case 2.1.1.

(2) Assume that $\{w_1, w_2\} \rightarrow v_2$. Then we have $v_2 \Rightarrow (Z \cup V(C_3) \cup \{x_3\})$.

(a) If u_2 dominates one vertex of $\{w_1, w_2\}$, say, w_1 , then since $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path, $\{w_1, w_2\} \rightarrow x_3$ and $u_2 \rightarrow \{x_3, w_2\}$.

(i) If $x_3 \rightarrow v_1$ and v_1 dominates one vertex of $\{w_1, w_2\}$, then let v_2 take the place of v_1 , the case being reduced to (3) of Case 2.1.1.

(ii) If $x_3 \rightarrow v_1$ and $\{w_1, w_2\} \rightarrow v_1$, then let v_2 take the place of v_1 , the case being reduced to (2) of Case 2.1.1.

(iii) Assume that $v_1 \rightarrow x_3$; then we have $x_3 \rightarrow (Z \cup V(C_3))$.

If v_3 dominates one vertex of $\{u_2, w_1, w_2\}$, say, $v_3 \rightarrow w_1$, then we obtain the complementary cycles $w_1 x_3 v_3 w_1$ and $x_1 v_1 u_1 x_2 u_2 w_2 v_2 w_3 u_3 x_1$.

If $\{u_2, w_1, w_2\} \rightarrow v_3$, then we have $v_1 \Rightarrow (\{u_2\} \cup V(C_3))$. Then let v_2 take the place of v_1 , the case being reduced to (2) of Case 2.1.1.

(b) Assume that $\{w_1, w_2\} \rightarrow u_2$. Since $H[\{u_2, v_2, w_1, w_2, x_3\}]$ has no Hamiltonian path, $\{w_1, w_2\} \rightarrow x_3$.

(i) Assume that $x_3 \rightarrow u_2$. Then $u_2 \Rightarrow (V(C_3) \cup \{v_1\})$.

If $x_3 \rightarrow v_1$, then v_1 dominates at least one vertex of $\{w_1, w_2\}$; then let v_2 take the place of v_1 , the case being reduced to (1) of Case 2.1.1.

If $v_1 \rightarrow x_3$ and there exists one vertex of $\{w_1, w_2\}$ that dominates at least one vertex of C_3 , say, $w_1 \rightarrow v_3$, then we obtain the complementary cycles $C_1 = x_3 u_2 v_1 x_3$ and $C_2 = x_1 w_2 v_2 u_1 x_2 u_2 v_3 w_3 u_3 x_1$.

If $v_1 \rightarrow x_3$ and $V(C_3) \Rightarrow \{w_1, w_2\}$, then $R_2 \setminus \{w_1, w_2\} \Rightarrow \{u_3, v_3\}$. This leads to the complementary cycles $C_1 = w_1 v_2 u_3 w_1$ and $C_2 = x_1 v_1 u_1 x_2 u_2 x_3 u_2 v_3 w_3 x_1$.

(ii) Assume that $u_2 \rightarrow x_3$.

If $x_3 \rightarrow v_1$ and v_1 dominates one vertex of $\{w_1, w_2\}$, then let v_2 take the place of v_1 , the case being reduced to (1) of Case 2.1.1.

Assume that $\{w_1, w_2, x_3\} \rightarrow v_1$. Then we have $v_1 \Rightarrow (V(C_3) \cup \{u_2\})$ and $u_2 \Rightarrow V(C_3)$. If there exists one vertex of $\{w_1, w_2\}$ that dominates u_1 , say, $w_1 \rightarrow u_1$, then let w_1 take the place of v_1 , the case being reduced to (2) of Case 2.1.1. Therefore, let $u_1 \rightarrow \{w_1, w_2\}$. This leads to the complementary cycles $C_1 = w_1 v_1 u_1 w_1$ and $C_2 = x_1 v_2 u_3 x_2 u_2 x_3 v_3 w_3 x_1$.

Now assume that $v_1 \rightarrow x_3$. Then we have $x_3 \rightarrow (\{u_1\} \cup V(C_3))$.

Let v_1 dominate one vertex of $\{w_1, w_2\}$, say, $v_1 \rightarrow w_1$.

If $w_2 \rightarrow u_1$, then we have the complementary cycles $C_1 = x_1 w_2 u_1 x_1$ and $C_2 = x_2 v_1 u_1 u_2 v_2 x_3 u_3 v_3 w_3 x_2$.

If $u_1 \rightarrow w_2$, then since $i_l(D) \leq 1$, u_2 dominates at least one vertex of C_3 . This leads to the complementary cycles $C_1 = w_2 v_2 u_1 w_2$ and $C_2 = x_1 w_1 u_2 v_3 w_3 x_2 v_1 x_3 u_3 x_1$ if $u_2 \rightarrow v_3$ or $C_2 = x_1 w_1 u_2 w_3 x_2 v_1 x_3 u_3 v_3 x_1$ if $u_2 \rightarrow w_3$.

Now let $\{w_1, w_2\} \rightarrow v_1$.

If u_1 dominates one vertex of $\{w_1, w_2\}$, say, $u_1 \rightarrow w_1$, then we obtain the complementary cycles $C_1 = v_1 u_1 w_1 v_1$ and $C_2 = x_1 w_2 v_2 u_3 x_2 u_2 x_3 v_3 w_3 x_1$.

If $\{w_1, w_2\} \rightarrow u_1$, then we have $u_1 \Rightarrow V(C_3)$ and $V(C_3) \Rightarrow \{w_1, w_2\}$. This leads to the complementary cycles $C_1 = w_1 u_1 v_3 w_1$ and $C_2 = x_1 w_2 u_2 v_2 w_3 x_2 v_1 x_3 u_3 x_1$.

Case 2.2. Assume that $|R_1| = 1$. If $R_1 \cap V_4 = \emptyset$, then $|V(x) \cap (Y \cup R_2)| = 1$ for $x \in R_1$. Therefore, we have $d_D^+(x) \geq 6$ and $d_D^-(x) \leq 3$ for $x \in R_1$, a contradiction. Hence, let, without loss of generality, $R_1 = \{x_3\}$. If $Y = \{x_1, x_2\}$, then the case is equivalent to the case of " $|Y| = 2$, $x_3 \in R_2$, and $x_3 \rightarrow R_2 \setminus \{x_3\}$," which has been proved in (3) of Case 2.1.1 and Case 2.1.2. If $Z \cap V_4 \neq \emptyset$, then $d_D^+(x_3) \geq 6$, a contradiction. So, let, without loss of generality, $Y = \{u_1, u_2\}$ and $Z = \{v_1\}$. Then $(V(C_3) \cup Z) \Rightarrow R_1$ and $(V(C_3) \cup Z) \Rightarrow Y$.

(1) Assume that there is an arc between $\{x_1, x_2\}$ to v_1 , say, $x_1 \rightarrow v_1$. Then we have $C_1 = x_1 v_1 u_1 x_1$.

(a) Assume that $H[\{w_1, w_2, v_2, x_2\}]$ has a Hamiltonian path.

If there is a Hamiltonian path which does not end at x_2 or begin at x_2 , let, without loss of generality, $p_1 p_2 p_3 p_4$ with $x_2 \notin \{p_1, p_4\}$ be a Hamiltonian path in $H[\{w_1, w_2, v_2, x_2\}]$; then p_4 has at least one positive neighbor in C_3 , say, $p_4 \rightarrow u_3$, and we have the complementary cycle $C_2 = w_3 x_3 u_2 p_1 p_2 p_3 p_4 u_3 v_3 w_3$.

If there is a Hamiltonian path ending at x_2 , let, without loss of generality, $p_1 p_2 p_3 p_4$ with $p_4 = x_2$ be a Hamiltonian path in $H[\{w_1, w_2, v_2, x_2\}]$; then we need only to consider $\{p_1, p_2, p_3\} \rightarrow p_4$. The other cases are easily reduced to the above case which does not end at x_2 . Thus $p_4 \rightarrow (\{v_1\} \cup V(C_3))$ and there is the complementary cycle $C_2 = u_3 x_3 u_2 p_1 p_2 p_3 p_4 v_3 w_3 u_3$.

If there is a Hamiltonian path beginning at x_2 , let, without loss of generality, $p_1 p_2 p_3 p_4$ with $p_1 = x_2$ be a Hamiltonian path in $H[\{w_1, w_2, v_2, x_2\}]$; then we need only to consider $p_1 \rightarrow \{p_2, p_3, p_4\}$. The other cases are easily reduced to the above case which does not begin at x_2 . Thus $p_4 \Rightarrow (\{v_1, x_1\} \cup V(C_3))$ and we have the complementary cycle $C_2 = x_3 u_2 p_1 p_2 p_3 p_4 u_3 v_3 w_3 x_3$.

(b) Now we assume that $H[\{w_1, w_2, v_2, x_2\}]$ has no Hamiltonian path. Let, without loss of generality, $v_2 \rightarrow w_2$.

If $w_2 \rightarrow x_2$, then since $H[\{w_1, w_2, v_2, x_2\}]$ has no Hamiltonian path, $w_1 \rightarrow x_2$, $v_2 \rightarrow x_2$, and $v_2 \rightarrow w_1$. Then we have $x_2 \rightarrow V(C_3)$ and w_1 dominates at least one vertex of C_3 , say, $w_1 \rightarrow u_3$. This leads to the complementary cycle $C_2 = u_3 x_3 v_2 w_2 x_2 v_3 w_3 u_2 w_1 u_3$.

If $x_2 \rightarrow w_2$ and $v_2 \rightarrow x_2$, then $v_2 \rightarrow w_1$ and $x_2 \rightarrow w_1$. This leads to $\{w_1, w_2\} \Rightarrow V(C_3)$ and the complementary cycle $C_2 = u_3 x_3 v_2 x_2 w_2 v_3 w_3 u_2 w_1 u_3$.

If $x_2 \rightarrow w_2$ and $x_2 \rightarrow v_2$, then $\{x_2, v_2\} \rightarrow w_1$ and $\{w_1, w_2\} \Rightarrow V(C_3)$. This leads to the complementary cycle $C_2 = u_3 x_3 v_2 w_1 v_3 w_3 u_2 x_2 w_2 u_3$.

(2) Assume that $v_1 \rightarrow \{x_1, x_2\}$. Then we have $(\{w_1, w_2\} \cup V(C_3)) \Rightarrow v_1, R_2 \Rightarrow w_3$, and $C_1 = w_1 v_1 u_1 w_1$.

(a) Assume that $H[\{v_2, w_2, x_1, x_2\}]$ has a Hamiltonian path. Then, since $d_D^+(x) \leq 5$, there exists a Hamiltonian ending at x_1 or x_2 ; let, without loss of generality, $p_1 p_2 p_3 p_4$ with $p_4 = x_1$ be a Hamiltonian path $H[\{v_2, w_2, x_1, x_2\}]$. This leads to the complementary cycle $C_2 = w_3 u_3 v_3 x_3 u_2 p_1 p_2 p_3 p_4 w_3$.

(b) Assume that $H[\{v_2, w_2, x_1, x_2\}]$ has no Hamiltonian path.

If $v_2 \rightarrow w_2$, then $w_2 \rightarrow x_1$ or $w_2 \rightarrow x_2$, say, $w_2 \rightarrow x_1$. Since $H[\{v_2, w_2, x_1, x_2\}]$ has no Hamiltonian path, $\{v_2, w_2\} \rightarrow x_2$ and $v_2 \rightarrow x_1$. This leads to $\{x_1, x_2\} \rightarrow V(C_3)$ and we obtain the complementary cycle $C_2 = u_3 x_3 v_2 w_2 x_1 v_3 w_3 u_2 x_2 u_3$.

If $w_2 \rightarrow v_2$, then $v_2 \rightarrow x_1$ or $v_2 \rightarrow x_2$, say, $v_2 \rightarrow x_1$. Since $H[\{v_2, w_2, x_1, x_2\}]$ has no Hamiltonian path, $\{v_2, w_2\} \rightarrow x_2$ and $w_2 \rightarrow x_1$. This leads to $\{x_1, x_2\} \rightarrow V(C_3)$ and we obtain the complementary cycle $C_2 = u_3 x_3 w_2 v_2 x_1 v_3 w_3 u_2 x_2 u_3$.

Case 2.3. Assume that $|R_1| = 2$. If $R_1 \cap V_4 = \emptyset$, then there exists a vertex $x \in R_1$ such that $|V(x) \cap R_1| = 2$ or $d_R^+(x) = 1$. Each case leads to $d_D^+(x) \geq 6$ and $d_D^-(x) \leq 3$, a contradiction to $i_l(D) \leq 1$. Therefore, $R_1 \cap V_4 \neq \emptyset$. Let, without loss of generality, $x_1 \in R_1$.

(1) Assume that $R_1 \subset V_4$. Let, without loss of generality, $R_1 = \{x_1, x_2\}$ and $Y = \{u_1, u_2\}$. Then $x_3 \in R_2$. Let, without loss of generality, $Z = \{v_1\}$. Since $i_l(D) \leq 1$, $(Z \cup V(C_3)) \Rightarrow R_1$, $v_2 \rightarrow w_1$, or $v_2 \rightarrow w_2$, say, $v_2 \rightarrow w_2$. Then we have $w_2 \Rightarrow (\{v_1, x_3\} \cup V(C_3))$.

Assume that there is an arc from Z to Y , say, $v_1 \rightarrow u_1$.

If $w_1 \rightarrow v_2$, then we have $v_2 \Rightarrow (V(C_3) \cup \{x_3\})$ and x_3 dominates at least two vertices of C_3 . This leads to the complementary cycles $C_1 = u_1 w_2 v_1 u_1$ and $C_2 = u_2 x_3 u_3 v_3 x_1 w_1 v_2 w_3 x_2 u_2$ if $x_3 \rightarrow u_3$, or $C_2 = u_2 x_3 v_3 x_1 w_1 v_2 w_3 x_2 u_2$ if $x_3 \rightarrow v_3$.

If $v_2 \rightarrow w_1$, then $w_1 \Rightarrow (V(C_3) \cup \{x_3\})$, and x_3 dominates at least two vertices of C_3 . This leads to the complementary cycles $C_1 = u_1 w_2 v_1 u_1$ and $C_2 = u_2 x_3 u_3 x_1 v_2 w_1 v_3 w_3 x_2 u_2$ if $x_3 \rightarrow u_3$, or $C_2 = u_2 x_3 v_3 x_1 v_2 w_1 w_3 u_3 x_2 u_2$ if $x_3 \rightarrow v_3$.

Now assume that $Y \rightarrow Z$. Then we have $V(C_3) \rightarrow Y$, $R_2 \Rightarrow w_3$, and $C_1 = u_1 v_1 x_1 u_1$.

If $w_1 \rightarrow v_2$, then we have $v_2 \Rightarrow (V(C_3) \cup \{w_2, x_3\})$ and $w_2 \rightarrow x_3$. This leads to the complementary cycle $C_2 = u_2 w_1 v_2 w_2 x_3 w_3 u_3 v_3 u_2$.

If $v_2 \rightarrow w_1$, then $w_1 \Rightarrow (V(C_3) \cup \{x_3\})$ and we obtain the complementary cycle $C_2 = u_2 v_2 w_1 x_3 w_3 u_3 x_2 w_2 v_3 u_2$.

(2) Assume that $|R_1 \cap V_4| = 1$. Let, without loss of generality, $R_1 = \{x_1, u_1\}$. Then $x_1 \rightarrow u_1$, $u_2 \in R_2$, and $(Z \cup V(C_3)) \Rightarrow R_1$. Since $i_l(D) \leq 1$, $Z \cap V_4 = \emptyset$. Let, without loss of generality, $Z = \{v_1\}$.

(a) Assume that $Y \cap V_4 = \emptyset$. Let, without loss of generality, $Y = \{w_1, w_2\}$. Then $R_2 = \{x_2, x_3, u_2, v_2\}$.

(i) Assume that there is an arc from Y to Z , say, $w_1 \rightarrow v_1$. Then we have $C_1 = w_1 v_1 x_1 w_1$.

If $H[R_2]$ has a Hamiltonian path, then it ends at x_2 (or x_3). Let, without loss of generality, $p_1 p_2 p_3 p_4$ with $p_4 \in \{x_2, x_3\}$ be a Hamiltonian path in $H[R_2]$. Then p_4 dominates at least two vertices of C_3 . This leads to the complementary cycle $C_2 = w_2 p_1 p_2 p_3 p_4 u_3 v_3 w_3 u_1 w_2$ if $p_4 \rightarrow u_3$ or $C_2 = w_2 p_1 p_2 p_3 p_4 u_3 v_3 u_1 w_2$ if $p_4 \rightarrow w_3$.

Now assume that $H[R_2]$ has no Hamiltonian path.

If $u_2 \rightarrow v_2$, then $v_2 \Rightarrow (\{x_2, x_3\} \cup V(C_3))$. Since $H[R_2]$ has no Hamiltonian path, $u_2 \rightarrow \{x_2, x_3\}$. Then we have $\{x_2, x_3\} \rightarrow (Z \cup V(C_3))$. If $u_3 \rightarrow w_2$, then we have the complementary cycle $C_2 = u_2 v_2 x_2 v_3 u_1 x_3 w_3 u_3 w_2 u_2$. If $w_2 \rightarrow u_3$, then $v_3 \rightarrow w_2$ and we have the complementary cycle $C_2 = u_2 v_2 x_2 w_3 u_1 x_3 u_3 v_3 w_2 u_2$.

If $v_2 \rightarrow u_2$, then u_2 dominates one vertex of $\{x_2, x_3\}$, say, $u_2 \rightarrow x_2$. Since $H[R_2]$ has no Hamiltonian path, $v_2 \rightarrow \{x_2, x_3\}$ and $u_2 \rightarrow x_3$. Then we have $\{x_2, x_3\} \rightarrow (Z \cup V(C_3))$. If $u_3 \rightarrow w_2$, then we have the complementary cycle $C_2 = v_2 u_2 x_2 v_3 u_1 x_3 w_3 u_3 w_2 v_2$. If $w_2 \rightarrow u_3$, then $v_3 \rightarrow w_2$ and we have the complementary cycle $C_2 = v_2 u_2 x_2 w_3 u_1 x_3 u_3 v_3 w_2 v_2$.

(ii) Assume that $Z \rightarrow Y$. Then there is an arc from $\{x_2, x_3\}$ to Z , say, $x_2 \rightarrow v_1$. This leads to the 3-cycle $C_1 = w_1 x_2 v_1 w_1$.

If $u_2 \rightarrow v_2$, then $v_2 \Rightarrow x_3$ and x_3 dominates at least two vertices of C_3 . This leads to the complementary cycle $C_2 = u_2 v_2 x_3 u_3 v_3 x_1 u_1 w_2 u_2$ if $x_3 \rightarrow u_3$ or $C_2 = u_2 v_2 x_3 v_3 w_3 u_3 x_1 u_1 w_2 u_2$ if $x_3 \rightarrow v_3$.

If $v_2 \rightarrow u_2$ and $u_2 \rightarrow x_3$, then x_3 dominates at least two vertices of C_3 . This leads to the complementary cycle $C_2 = v_2 u_2 x_3 u_3 v_3 w_3 x_1 u_1 w_2 v_2$ if $x_3 \rightarrow u_3$ or $C_2 = v_2 u_2 x_3 v_3 w_3 u_3 x_1 u_1 w_2 v_2$ if $x_3 \rightarrow v_3$.

If $v_2 \rightarrow u_2$ and $x_3 \rightarrow u_2$, then $u_2 \Rightarrow V(C_3)$. This leads to the complementary cycle $C_2 = v_2x_3u_2v_3w_3u_3x_1u_1w_2v_2$ if $v_2 \rightarrow x_3$ or $C_2 = x_3v_2u_2v_3w_3u_3x_1u_1w_2v_2$ if $x_3 \rightarrow v_2$.

(b) Assume that $Y \subset V_4$. Then let, without loss of generality, $Y = \{x_2, x_3\}$. Then we have $R_2 = \{u_2, v_2, w_1, w_2\}$.

(i) Assume that there is an arc from Z to Y , say, $v_1 \rightarrow x_2$. Then there is an arc from R_2 to Z .

If there is an arc from $\{w_1, w_2\}$ to Z , say, $w_1 \rightarrow v_1$, then we have $C_1 = x_2w_1v_1x_2$. Let, without loss of generality, $v_2 \rightarrow w_2$. Then $w_2 \Rightarrow (\{u_2\} \cup V(C_3))$ and u_2 dominates at least one vertex of C_3 . This leads to the complementary cycle $C_2 = v_2w_2u_2v_3w_3u_3x_1u_1x_3v_2$ if $u_2 \rightarrow v_3$ or $C_2 = v_2w_2u_2w_3u_3v_3x_1u_1x_3v_2$ if $u_2 \rightarrow w_3$.

Otherwise, $u_2 \rightarrow v_1$. Then we have 3-cycle $C_1 = x_2u_2v_1x_2$. Let, without loss of generality, $v_2 \rightarrow w_2$. Then $w_2 \Rightarrow (Z \cup V(C_3))$.

If $w_1 \rightarrow v_2$, then we have the complementary cycle $C_2 = x_3w_1v_2w_2u_3v_3w_3x_1u_1x_3$.

If $v_2 \rightarrow w_1$, then $w_1 \Rightarrow Z \cup V(C_3)$ and we have the complementary cycle $C_2 = v_2w_2u_3x_1w_1v_3w_3u_1x_3v_2$.

(ii) Assume that $Y \rightarrow Z$. Then $V(C_3) \rightarrow Y$ and we have 3-cycle $C_1 = x_2v_1u_1x_2$.

If $w_2 \rightarrow v_2$, then $v_2 \Rightarrow \{w_1, u_2\} \cup V(C_3)$, $w_1 \Rightarrow \{u_2\} \cup V(C_3)$, and $u_2 \Rightarrow V(C_3)$. This leads to the complementary cycle $C_2 = w_2v_2u_3x_1w_1u_2v_3w_3x_3w_2$.

Now assume that $v_2 \rightarrow w_2$. Then $w_2 \Rightarrow \{u_2\} \cup V(C_3)$.

If $u_2 \rightarrow w_1$, then $w_1 \Rightarrow V(C_3)$ and we have the complementary cycle $C_2 = u_2w_1v_3w_3x_1v_2w_2u_3x_3u_2$.

If $w_1 \rightarrow u_2$, then $u_2 \Rightarrow V(C_3)$ and we have the complementary cycle $C_2 = w_1u_2v_3w_3x_1v_2w_2u_3x_3w_1$.

Case 2.4. Assume that $|R_1| = 3$.

Let $R_1 \cap V_4 = \emptyset$. If $R_2 \cap V_4 \neq \emptyset$, then there exists one vertex $x \in R_1$ such that $d_D^+(x) \geq 6$, a contradiction to $i_l(D) \leq 1$. Therefore, $Z \cup Y = \{x_1, x_2, x_3\}$. Let, without loss of generality, $Z = \{x_1\}$, $Y = \{x_2, x_3\}$. Since $i_l(D) \leq 1$, let, without loss of generality, the set R_1 induce the 3-cycle $u_1v_1w_1u_1$, and the set R_2 induces the 3-cycle $u_2v_2w_2u_2$ such that $R_2 \Rightarrow Z \Rightarrow R_1$. In addition, $i_l(D) \leq 1$ implies $R_2 \Rightarrow V(C_3) \Rightarrow R_1$. It is evident that $u_2x_1v_1u_2$ and $x_2v_2u_3w_1x_3w_2v_3w_3u_1x_2$ are complementary cycles in D .

If $R_1 = V_4$, then there exist two vertices $x, y \in R_2$ such that $(V(x) \cup V(y)) \cap (Y \cup R_1) = \emptyset$. Since there is an arc between x and y , $d_D^+(x) \geq 6$ or $d_D^+(y) \geq 6$, a contradiction.

Therefore, $R_1 \cap V_4 \neq \emptyset$ and $R_1 \neq V_4$. Thus $|R_1 \cap V_4| = 1$ or 2 .

(1) Assume that $|R_1 \cap V_4| = 1$ first. Let, without loss of generality, $\{x_1, u_1\} \subset R_1$.

(a) If $u_2 \in R_1$, then $x_1 \rightarrow \{u_1, u_2\}$ and $\{x_2, x_3\} \subset (Y \cup R_2)$.

(i) If $Y = \{x_2, x_3\}$, then there exists one vertex $x \in R_2$ such that $d_D^+(x) \geq 6$, a contradiction.

(ii) If, without loss of generality, $Y = \{w_1, w_2\}$ and $Z = \{v_1\}$, then $R_2 = \{x_2, x_3, v_2\}$. Then we have $v_2 \rightarrow \{x_2, x_3\}$, $R_2 \Rightarrow (V(C_3) \cup Z)$, and $(V(C_3) \cup Z) \Rightarrow R_1$. This leads to the complementary cycles $C_1 = x_2v_2u_1x_2$ and $C_2 = w_1x_3u_3v_3u_2w_2v_2w_3x_1w_1$.

(b) If, without loss of generality, $v_1 \in R_1$ and we let $u_1 \rightarrow v_1$, then $x_1 \rightarrow u_1$, $C_3 \Rightarrow u_1$, and $u_2 \in R_2$.

(i) Let $x_1 \rightarrow v_1$.

Assume that $\{x_2, x_3\} \subset R_2$. Then $Y = \{w_1, w_2\}$ and $Z = \{v_2\}$. In addition, $i_l(D) \leq 1$ implies $(V(C_3) \cup Z) \Rightarrow R_1$.

If $x_2 \rightarrow u_2$ or $x_3 \rightarrow u_2$, say, $x_2 \rightarrow u_2$, then $u_2 \Rightarrow (\{x_3, v_2\} \cup V(C_3))$ and $x_3 \rightarrow (\{v_2\} \cup V(C_3))$. Since $i_l(D) \leq 1$, $x_2 \rightarrow u_3$ or $x_2 \rightarrow v_3$. In the former case there exist two complementary cycles $x_3v_2u_1x_3$ and $w_1x_2u_3x_1w_2u_2v_3w_3v_1w_1$, and in the latter case two complementary cycles are $x_3v_2u_1x_3$ and $w_1x_2v_3x_1w_2u_2w_3u_3v_1w_1$.

If $u_2 \rightarrow \{x_2, x_3\}$, then $\{x_2, x_3\} \rightarrow (\{v_2\} \cup V(C_3))$. In addition, if $u_2 \rightarrow v_2$, we have the complementary cycles $u_2v_2x_1u_2$ and $w_1x_2u_3v_1w_2x_3v_3w_3u_1w_1$; if $v_2 \rightarrow u_2$ and there exists an arc from Y to Z , say, $w_1 \rightarrow v_2$, then we have the complementary cycles $C_1 = w_1v_2x_1w_1$ and $C_2 = x_2u_3v_1w_2u_2x_3v_3w_3u_1x_2$; if $v_2 \rightarrow (\{u_2\} \cup Y)$, then $V(C_3) \Rightarrow v_2$, and we have the complementary cycles $C_1 = v_2w_1x_2v_2$ and $C_2 = w_2u_2x_3u_3v_3w_3x_1u_1v_1w_2$.

Now assume that $x_2 \notin R_2$ or $x_3 \notin R_2$. Then $Y = \{x_2, x_3\}$ and $v_2 \in R_2$. Otherwise, there exists one vertex $x \in R_2$ such that $d_D^-(x) \geq 6$, a contradiction. Therefore, let, without loss of generality, $Z = \{w_1\}$. Then we have $R_2 = \{u_2, v_2, w_2\}$, $(Z \cup V(C_3)) \Rightarrow \{x_1, u_1\}$, and $w_2 \rightarrow \{u_2, v_2\}$. Let, without loss of generality, $u_2 \rightarrow v_2$. This leads to the contradiction $d_D^-(v_2) \geq 6$.

(ii) Let $v_1 \rightarrow x_1$. Then $v_2 \in R_2$ and $(Z \cup V(C_3)) \Rightarrow v_1$. Let, without loss of generality, $u_2 \rightarrow v_2$. If $Y = \{w_1, w_2\}$, let, without loss of generality, $Z = \{x_2\}$, $x_3 \in R_2$; then $v_2 \rightarrow x_3 \rightarrow u_2$, $R_2 \Rightarrow (V(C_3) \cup Z)$, and $(Z \cup V(C_3)) \Rightarrow R_1$. It is evident that $u_2x_2v_1u_2$ and $w_1x_3w_3x_1w_2v_2u_3v_3u_1w_1$ are complementary cycles in D . If $Y = \{x_2, x_3\}$, let, without loss of generality, $Z = \{w_1\}$; then $v_2 \rightarrow w_2 \rightarrow u_2$. This leads to the contradiction $d_D^+(w_2) \geq 6$.

(2) Assume that $|R_1 \cap V_4| = 2$. Let, without loss of generality, $R_1 = \{x_1, x_2, u_1\}$ and $Y = \{v_1, v_2\}$.

(a) If $Z = \{x_3\}$, then $R_2 = \{u_2, w_1, w_2\}$ and $\{w_1, w_2\} \rightarrow u_2$. This leads to the contradiction $d_D^-(u_2) \geq 6$.

(b) If $Z = \{u_2\}$, then $R_2 = \{x_3, w_1, w_2\}$, $\{w_1, w_2\} \rightarrow x_3$, $R_2 \Rightarrow (V(C_3) \cup Z)$, and $(V(C_3) \cup Z) \Rightarrow R_1$. This leads to the complementary cycles $C_1 = w_1u_2x_1w_1$ and $C_2 = v_1w_2u_3x_2v_2x_3v_3w_3u_1v_1$.

(c) If, without loss of generality, $Z = \{w_1\}$, then $R_2 = \{x_3, u_2, w_2\}$, $w_2 \rightarrow \{u_2, x_3\}$, and $u_2 \rightarrow x_3$. This leads to $R_2 \Rightarrow (V(C_3) \cup Z)$ and $(V(C_3) \cup Z) \Rightarrow R_1$.

If $u_1 \rightarrow x_1$ or $u_1 \rightarrow x_2$, say, $u_1 \rightarrow x_1$, then $x_2 \Rightarrow u_1$ and $(Z \cup V(C_3)) \Rightarrow \{u_1, x_2\}$. This leads to the complementary cycles $C_1 = u_2w_1x_1u_2$ and $C_2 = v_1w_2u_3x_2v_2x_3v_3w_3u_1v_1$.

If $\{x_1, x_2\} \rightarrow u_1$, then $Z \cup V(C_3) \Rightarrow \{x_1, x_2\}$. This leads to the complementary cycles $C_1 = x_3w_1u_1x_3$ and $C_2 = v_1u_2v_3w_3x_1v_2w_2u_3x_2v_1$.

Thus we finish proof of case $(|Z|, |Y|) = (1, 2)$.

Case 3. One has $|Z| = 1$ and $|Y| = 3$. Then $Y = V_4 = \{x_1, x_2, x_3\}$.

The case is equivalent to the case of " $|Y| = 2$, $x_3 \in R_2$, and $x_3 \rightarrow R_2 \setminus \{x_3\}$," which has been proved in Case 2. $\square$

4. Concluding Remarks

In this paper, we proved the existence of complementary cycles in a kind of irregular multipartite tournaments.

Problems on complementary cycles in irregular multidigraphs have important significance.

In 2002, Volkmann gave a new type of connectivity condition for a multipartite tournament to be cycle complementary. They proposed the following conjecture.

Conjecture 13 (see Volkmann [16] 2002). *Let D be a multipartite tournament. If $k(D) \geq \alpha(D) + 1$, then D is cycle complementary, unless D is a member of a finite family of multipartite tournaments.*

In 2010, Li et al. [17] proved every $(\alpha(D) + 1)$ -strong n -partite tournament D with $n \geq 6$ contains two disjoint cycles which contain vertices from exactly 3 and $n - 3$ partite sets, respectively, unless it is isomorphic to T_7^1 .

The vertex set of a digraph D is denoted by $V(D)$. A c -partite tournament is an orientation of a complete c -partite graph. A multipartite tournament D is called weakly cycle complementary if there exist two vertex disjoint cycles C_1 and C_2 such that $V(C_1) \cup V(C_2)$ contains vertices of all partite sets of D . In 2008, Volkmann and Winzen [18] proved all 3-connected c -partite tournaments with $c \geq 3$ that are weakly cycle complementary. In 2011, He and Wang [19] proved all 3-partite tournaments that are weakly cycle complementary.

In 2008, Volkmann and Winzen [18] also proposed the following problem.

Problem 14 (see Volkmann and Winzen [18] 2008). Characterize all 2-strong c -partite tournaments D with $c \geq 3$ and $|V(D)| \geq 6$ that are weakly cycle complementary.

All these problems may stimulate our further study for complementary cycles of digraphs.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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