Research Article

Construction of Fractional Power Series Solutions to Fractional Boussinesq Equations Using Residual Power Series Method

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This paper is aimed at constructing fractional power series (FPS) solutions of time-space fractional Boussinesq equations using residual power series method (RPSM). Firstly we generalize the idea of RPSM to solve any-order time-space fractional differential equations in high-dimensional space with initial value problems in $\mathbb{R}^n$. Using RPSM, we can obtain FPS solutions of fourth-, sixth-, and $2n$ th-order time-space fractional Boussinesq equations in $\mathbb{R}$ and fourth-order time-space fractional Boussinesq equations in $\mathbb{R}^2$ and $\mathbb{R}^n$. Finally, by numerical experiments, it is shown that RPSM is a simple, effective, and powerful method for seeking approximate analytic solutions of fractional differential equations.

1. Introduction

The study of nonlinear problems is of crucial importance not only in mathematics but also in physics, engineering, economic, and other disciplines, since most phenomena in our world are essentially nonlinear. But it is very difficult to establish mathematical models for nonlinear problem. In the field of engineering and science, we come across physical and natural phenomena which are represented by mathematical models, happening to differential equations. For example, simple harmonic motion, equation of motion, and deflection of a beam are represented by differential equations. Hence, the behaviour of differential equations is a necessity in such studies. A natural idea is to seek the exact solution, but it is often too difficult to succeed. So, as in [1], we attempt to construct approximate solution. Many assumptions have to be made artificially or unnecessarily in the integer-order differential system to make the practical problems solvable, leading to loss of most significant information. So it is necessary to construct new systems to solve these problems and fractional-order system is one of these. That is to say, fractional differential system is much well suited to physical problems compared to its differential partner since it makes

less unnecessary or overrestricted assumption which may change the problem being solved, sometimes seriously [2].

The subject of fractional-order system dates back from the idea of derivatives of noninteger order which initially appeared in a letter from Leibnitz to L'Hospital in 1695 about the notation $d^n y/dx^n$. L'Hospital posed a question to Leibnitz: "what would the result be if $n = 1/2$?" Leibnitz replied [3] "it follows that $d^{1/2}x$ will be equal to $x\sqrt{dx}:x$. This is an apparent paradox, from which, one day useful consequences will be drawn." From these words, fractional calculus was born. And so most authors on this topic will cite a particular date (September 30, 1695) as the birthday of the so-called "fractional calculus" [4]. However, at that time, there are little specific models based on this kind of derivative, so the study of fractional order system attracts little attention.

Nowadays, more and more researchers pay massive attention to fractional calculus and they find that fractional systems can seize the missed information in the integral order systems. It is well known that the integer-order differential operators and the integer-order integral operators are local, but on the aspect as well, the fractional-order differential operators and the fractional-order integral operators are nonlocal operators. This means that the next state of a system

depends not only upon its current state but also upon all of its historical states. In fact, this is the main reason why differential operators of fractional order provide an excellent instrument for description of memory and hereditary properties of various mathematical, physical, and engineering processes [5, 6]. More details about the fractional differential theory can be found in [7–13].

Particularly, power series has played an important role in the study of elementary functions in the theory of calculus and it has been widely used in computational science for easily obtaining an approximation of functions. In physics, chemistry, and many other sciences, this power series expansion has allowed scientists to make an approximate study of many systems, neglecting higher-order terms around the equilibrium point. This is a fundamental tool to linearize a problem, which guarantees easy analysis [1, 6, 14–16].

The motivation of this paper is to study the time-space fractional Boussinesq equations. The motion of water waves plays an important role in oceanographic engineering and for most geographical areas, which are produced by environmental actions on beaches or on man-made fixed or floating structures. Natural wave trains are irregular in shape and they interact because the propagation process is nonlinear. Classical wave theory failed to describe the combined effect of these processes and they can be divided into two categories:

- (1) a nonlinear description of monochromatic waves of a specific frequency or wave length;
- (2) a linear description of irregular waves based on superposition of individual frequency components with random phases [17, 18].

So many mathematicians and physicists tried to contract new model to describe the process of wave propagation on the surface of water. In 1872, the French scientist Joseph Boussinesq (1842–1929) derived a model equation [18]

$$u_{tt} - u_{xx} + \mu u_{xxxx} = \eta (u^2)_{xx}, \quad (1)$$

$$-\infty < x < +\infty, \quad t \geq 0,$$

for the propagation of water waves from Euler's equations to describe the propagation process of long waves on the surface of water with small amplitude. This equation was formulated for analysis of long waves in shallow water. Later the general form of Boussinesq equation [17]

$$u_{tt} - (f(u))_{xx} - u_{xxxx} = h(x, t), \quad (2)$$

$$-\infty < x < +\infty, \quad t \geq 0,$$

was applied to problems not only in the study of the dynamics of thin inviscid layers with free surface but also in the study of nonlinear string, the shape-memory alloys, the propagation of waves in elastic rods, shallow wave waters (waves with long wavelength compared to the depth), and the continuum limit of lattice dynamics or coupled electrical circuits. More and more researchers pay attention to this equation and get a number of interesting results which were applied to many areas [17–20]. In recent years, with

the development of fractional system rapidly, the classical Boussinesq equation evolves into the fractional Boussinesq equation, which is suitable for studying the water propagation through heterogeneous porous media [21–24].

This paper is concerned with the following initial value problem for time-space fractional Boussinesq equation in high-dimensional space:

$$D_t^{2\alpha} u(x, t) = \sum_{p=1}^{n_1} \beta_p D_{x_p}^{4\sigma_p} u(x, t) + \sum_{q=1}^{n_2} \gamma_q D_{x_q}^{2\tau_q} u(x, t) + \sum_{r=1}^n \theta_r D_{x_r}^{2\lambda_r} u^2(x, t) - \sum_{r=1}^n 4\theta_r u^2(x, t), \quad (3)$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t)|_{t=0} = a_1(x),$$

where $u(x, t)$ is a field function, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, $3/4 < \sigma_p \leq 1$, $p = 1, 2, \dots$, $n_1 \leq n \in \mathbb{N}$, $1/2 < \tau_q \leq 1$, $q = 1, 2, \dots$, $n_2 \leq n \in \mathbb{N}$, $1/2 < \lambda_r \leq 1$, $r = 1, 2, \dots, n$, $1/2 < \alpha \leq 1$, describing the order of the time-fractional derivative. The constants β_p ($p = 1, 2, \dots, n_1$) and γ_q ($q = 1, 2, \dots, n_2$) are the fourth-order and second-order coefficients of the space derivative, respectively, and the constants θ_r ($r = 1, 2, \dots, n$) are the nonlinear coefficients. The symbol $D_t^{2\alpha}$ means the Caputo fractional derivative of order 2α , $a_0(x)$, and $a_1(x) \in C^\infty(\mathbb{R}^n)$. In [25], Jafari et al. considered the one-dimensional time-fractional Boussinesq equation; by finding the Lie point symmetries of the equation, they obtain the invariant solutions of the one-dimensional time-fractional Boussinesq equation. By the fractional subequation method, in [23], they obtained the solutions in terms of fractional hypergeometric functions, fractional triangle functions, and a rational function. Different from the solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations in $\mathbb{R}$ obtained by Arqub et al. [15], in this paper we attempt to construct the more general approximate fractional power series solutions in $\mathbb{R}^n$ using RPSM. Furthermore, this method is valid for high-dimensional space and any-order time-space partial differential equations.

The organization of the rest of this paper is as follows: in Section 2, an introduction of some concepts on fractional calculus theory is presented. In Section 3, we introduce algorithm of RPSM for any-order time-space fractional differential equations with initial value problems. In Section 4, error analysis of RPSM is discussed simply. In Section 5, we apply RPSM to obtain fractional power series solutions of time-space fractional Boussinesq equations with initial value problems. The paper ends with some simple conclusions.

2. Concepts on Fractional Calculus Theory

There are several definitions of the fractional integration of order $\alpha \geq 0$, not necessarily equivalent to each other;

see [9]. Riemann-Liouville and Caputo fractional definitions are the two most used from all the other definitions of fractional calculus which have been introduced recently [5, 6, 26–29].

Definition 1. The Mittag-Leffler function is defined as follows:

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0. \quad (4)$$

Definition 2. A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, +\infty)$, and it is said to be in the space C_μ^n if $f^{(n)}(x) \in C_\mu$, $n \in \mathbb{N}$.

Definition 3. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as follows:

$$J_s^\alpha f(x) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_s^x (x-t)^{\alpha-1} f(t) dt, & x > t > s \geq 0, \alpha > 0, \\ f(x), & \alpha = 0. \end{cases} \quad (5)$$

Property 1. Properties of the operator J_s^α : for $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta, C \in \mathbb{R}$, and $\gamma \geq -1$.

Property 1.1. Consider $J_s^\alpha J_s^\beta f(x) = J_s^{\alpha+\beta} f(x) = J_s^\beta J_s^\alpha f(x)$.

Property 1.2. Consider $J_s^\alpha C = (C/\Gamma(1+\alpha))(x-s)^\alpha$.

Property 1.3. Consider $J_s^\alpha (x-s)^\gamma = (\Gamma(\gamma+1)/\Gamma(\gamma+1+\alpha))(x-s)^{\gamma+\alpha}$.

Definition 4. The Caputo fractional derivative of order $\alpha > 0$ of $f \in C_{-1}^n$, $n \in \mathbb{N}$, is defined as follows:

$$D_s^\alpha f(x) := \begin{cases} J_s^{n-\alpha} f^{(n)}(x), & x > s \geq 0, n-1 < \alpha < n, \\ \frac{d^n f(x)}{dx^n}, & \alpha = n \in \mathbb{N}. \end{cases} \quad (6)$$

Property 2. Properties of the operator D_s^α : for $\gamma > -1$, $x > s \geq 0$ and $C \in \mathbb{R}$.

Property 2.1. Consider $D_s^\alpha D_s^\beta f(x) = D_s^{\alpha+\beta} f(x) = D_s^\beta D_s^\alpha f(x)$.

Property 2.2. Consider $D_s^\alpha C = 0$.

Property 2.3. Consider $D_s^\alpha (x-s)^\gamma = (\Gamma(\gamma+1)/\Gamma(\gamma+1-\alpha))(x-s)^{\gamma-\alpha}$.

Definition 5. For n to be the smallest integer that exceeds α , the Caputo time-fractional derivative operator of order α of $u(x, t)$ is defined as follows:

$$D_t^\alpha u(x, t) := \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n, \\ \frac{\partial^n u(x, t)}{\partial t^n}, & \alpha = n \in \mathbb{N}. \end{cases} \quad (7)$$

And the space fractional derivative of order β is defined as follows:

$$D_x^\beta u(x, t) := \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x (x-\tau)^{n-\beta-1} \frac{\partial^n u(\tau, t)}{\partial \tau^n} d\tau, & n-1 < \beta < n, \\ \frac{\partial^n u(x, t)}{\partial x^n}, & \beta = n \in \mathbb{N}. \end{cases} \quad (8)$$

Definition 6. A power series representation of the form

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} := c_0 + c_1 (t-t_0)^\alpha + c_2 (t-t_0)^{2\alpha} + \dots, \quad (9)$$

where $m-1 < \alpha < m$ and $t \geq t_0$, is called a fractional power series about t_0 , where t is a variable and c_n are the coefficients of the series.

Theorem 7 (see [5, 6]). Suppose that f has a FPS representation at t_0 of the form

$$f(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha}, \quad (10)$$

$$0 \leq m-1 < \alpha \leq m, \quad t_0 \leq t < t_0 + R,$$

and R is the radius of convergence of the FPS. If $f(t) \in C[t_0, t_0+R)$ and $D_t^{n\alpha} f(t) \in C(t_0, t_0+R)$ for $n = 0, 1, 2, \dots$, then the coefficients c_n will take the form of

$$c_n = \frac{D_t^{n\alpha} f(t)|_{t=t_0}}{\Gamma(n\alpha+1)}, \quad (11)$$

where $D_t^{n\alpha} = D_t^\alpha \cdot D_t^\alpha \cdot \dots \cdot D_t^\alpha$ (n -times).

Remark 8. If $f(t) = u(x, t)$, then we have

$$u(x, t) = \sum_{n=0}^{\infty} f_n(x) (t-t_0)^{n\alpha}, \quad (12)$$

$$0 \leq m-1 < \alpha \leq m, \quad x \in I \subset \mathbb{R}^n, \quad t_0 \leq t < t_0 + R,$$

and R is the radius of convergence of the FPS. If $D_t^{n\alpha} \in C(I \times (t_0, t_0+R))$, $n = 0, 1, 2, \dots$, then the coefficients are $f_n(x) = D_t^{n\alpha} u(x, t)|_{t=t_0} / \Gamma(n\alpha+1)$.

3. Algorithm of RPSM

In [14], Abu Arqub et al. give RPSM to obtain power series solution of higher-order ordinary differential equation with initial conditions in $\mathbb{R}$ in the integer system. In [5], El-Ajou et al. give the iterative progress of RPSM for time-fractional KdV-Burgers equation in $\mathbb{R}$. In this section, we give the more general RPSM to find out fractional power series solution for any-order time-space fractional differential equations with initial conditions in $\mathbb{R}^n$.

Let us consider the higher-order time-space fractional differential equation as the following form:

$$\begin{aligned} D_t^{m\alpha} u(x, t) + G(x, t) &= 0, \\ u(x, 0) &= a_0(x), \\ D_t^\alpha u(x, t)|_{t=0} &= a_1(x), \dots, D_t^{(m-1)\alpha} u(x, t)|_{t=0} \\ &= a_{m-1}(x), \end{aligned} \quad (13)$$

where $x = (x_1, x_2, \dots, x_n) \in I \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, and

$$\begin{aligned} G(x, t) &= F(u, D_t^\alpha u, D_t^{2\alpha} u, \dots, D_t^{(m-1)\alpha} u, D_{x_1}^{\beta_{11}} u, D_{x_2}^{\beta_{12}} u, \\ &\dots, D_{x_n}^{\beta_{1n}} u, D_{x_1}^{\beta_{21}} u, D_{x_2}^{\beta_{22}} u, \dots, D_{x_n}^{\beta_{2n}} u, \dots, D_{x_1}^{\beta_{m1}} u, D_{x_2}^{\beta_{m2}} u, \dots, \\ &D_{x_n}^{\beta_{mn}} u). \end{aligned} \quad (14)$$

$u(x, \cdot)$ and $G(x, \cdot)$ are analytic functions about t , $(m-1)/m < \alpha \leq 1$, $m \in \mathbb{N}$, $p-1 < \beta_{pj} \leq p$ ($p = 1, 2, \dots, l$; $j = 1, 2, \dots, n$), and $a_q(x) \in C^\infty(\mathbb{R}^n)$, $q = 0, 1, 2, \dots, m-1$.

Since $u(x, t)$ is analytic about t , the solution of the system can be written in the form of

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad (15)$$

where $u_n(x, t)$ are terms of approximations and given as

$$\begin{aligned} u_n(x, t) &= C_n(x) t^{n\alpha}, \\ x \in I \subset \mathbb{R}^n, |t| < R, n &= 0, 1, 2, \dots, \end{aligned} \quad (16)$$

where R is the radius of convergence of above series. Obviously, when $i = 0, 1, 2, \dots, m-1$, since the terms $D_t^{i\alpha} u(x, t)$ satisfy the initial condition, we can get

$$\begin{aligned} a_i(x) &= D_t^{i\alpha} u(x, t)|_{t=0} = C_i(x) \Gamma(i\alpha + 1) \implies \\ u_i(x, t) &= C_i(x) t^{i\alpha} = \frac{D_t^{i\alpha} u(x, t)|_{t=0}}{\Gamma(i\alpha + 1)} t^{i\alpha} \\ &= \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \\ i &= 0, 1, \dots, m-1. \end{aligned} \quad (17)$$

So we have the initial guess approximation of $u(x, t)$ in the following form:

$$\begin{aligned} u^{\text{initial}}(x, t) &:= u_0(x, t) + u_1(x, t) + \dots + u_{m-1}(x, t) \\ &= a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^\alpha + \dots \\ &\quad + \frac{a_{m-1}(x)}{\Gamma((m-1)\alpha + 1)} t^{(m-1)\alpha}. \end{aligned} \quad (18)$$

On the other aspect as well, if we choose $u^{\text{initial}}(x, t)$ as initial guess approximation $u_i(x, t)$ for $i = m, m+1, m+2, \dots$, the approximate of the solution of $u(x, t)$ of (13) by the k th truncated series is

$$\begin{aligned} u^k(x, t) &:= u^{\text{initial}}(x, t) + \sum_{i=m}^k C_i(x) t^{i\alpha}, \\ k &= m, m+1, m+2, \dots \end{aligned} \quad (19)$$

Before applying RPS method for solving (13), we give some notes:

$$\text{Res}(x, t) := D_t^{m\alpha} u(x, t) + G(x, t). \quad (20)$$

Substituting the k th truncated series $u^k(x, t)$ into (13), we can obtain the following definition for k th residual function:

$$\text{Res}^k(x, t) := D_t^{m\alpha} u^k(x, t) + G^k(x, t), \quad (21)$$

where

$$\begin{aligned} G^k(x, t) &= F(u^k, D_t^\alpha u^k, D_t^{2\alpha} u^k, \dots, D_t^{(m-1)\alpha} u^k, D_{x_1}^{\beta_{11}} u^k, \\ &D_{x_2}^{\beta_{12}} u^k, \dots, D_{x_n}^{\beta_{1n}} u^k, D_{x_1}^{\beta_{21}} u^k, D_{x_2}^{\beta_{22}} u^k, \dots, D_{x_n}^{\beta_{2n}} u^k, \dots, \\ &D_{x_1}^{\beta_{m1}} u^k, D_{x_2}^{\beta_{m2}} u^k, \dots, D_{x_n}^{\beta_{mn}} u^k). \end{aligned} \quad (22)$$

Then we have the following facts:

- (1) $\lim_{k \rightarrow \infty} u^k(x, t) = u(x, t)$;
- (2) $\text{Res}(x, t) = 0$;
- (3) $\text{Res}^\infty(x, t) = \lim_{k \rightarrow +\infty} \text{Res}^k(x, t) = \text{Res}(x, t) = 0$, $x \in I \subseteq \mathbb{R}^n$, $|t| < R$.

These show that the residual function $\text{Res}^\infty(x, t)$ is infinitely many times differentiable at $t = 0$. On the other hand, we can show that

$$\begin{aligned} 0 &= D_t^{(k-m)\alpha} \text{Res}^\infty(x, t)|_{t=0} \\ &= C_k(x) \Gamma(k\alpha + 1) + D_t^{(k-m)\alpha} G^k(x, t)|_{t=0} \implies \\ C_k(x) &= -\frac{D_t^{(k-m)\alpha} G^k(x, t)|_{t=0}}{\Gamma(k\alpha + 1)}, \\ k &= m, m+1, m+2, \dots \end{aligned} \quad (23)$$

Since $-D_t^{(k-m)\alpha} G^k(x, t)|_{t=0}$ is not dependent on t , denoting it by $f_k(x)$, then (23) can be written as

$$C_k(x) = \frac{f_k(x)}{\Gamma(k\alpha + 1)},$$

$$f_k(x) = -D_t^{(k-m)\alpha} G^k(x, t)|_{t=0}, \quad (24)$$

$$k = m, m+1, m+2, \dots$$

In fact, the relation of (24) is a fundamental rule in RPSM and its applications. So the FPS solution of (13) is

$$u(x, t) = u^{\text{initial}}(x, t) + \sum_{i=m}^{\infty} C_i(x) t^{i\alpha}$$

$$= \sum_{i=0}^{m-1} \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} + \sum_{i=m}^{\infty} \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (25)$$

where $a_i(x)$ ($i = 0, 1, 2, \dots, m-1$) are given and $f_i(x)$ ($i = m, m+1, m+2, \dots$) have been constructed by RPSM in the form (24).

The algorithm of RPSM is considered a direct and simple approach because it depends on the recursive differentiation of time-fractional derivative and the use of given initial conditions to calculate coefficients of the multiple FPS solution using minimal calculations. The RPS does not require linearization, perturbation, or discretization of the variables; is not affected by computational round-off errors; and does not require large computer memory and extensive time.

4. Error Analysis of RPSM

It will be convenient to have a notation for the error in the approximation of (13):

$$u(x, t) \approx u^k(x, t), \quad (k = 0, 1, 2, \dots),$$

$$\text{Rem}^k(x, t) = u(x, t) - u^k(x, t), \quad (26)$$

where $u(x, t)$ denotes the exact solution of (13) and $u^k(x, t)$ denotes k th approximate solution of (13) obtained by RPSM and $\text{Rem}^k(x, t)$ denotes the difference between $u(x, t)$ and $u^k(x, t)$. The functions $\text{Rem}^k(x, t)$ are called the k th remainder for the approximation of $u(x, t)$. In fact, it often happens that the remainder $\text{Rem}^k(x, t)$ become smaller and smaller, approaching zero, as k tends to infinity.

To show the accuracy and efficiency of RPSM, we give two types of error:

$$\text{Exa}^k(x, t) := |\text{Rem}^k(x, t)| = |u(x, t) - u^k(x, t)|, \quad (27)$$

where $\text{Exa}^k(x, t)$ denotes the absolute error;

$$\text{Rel}^k(x, t) := \frac{|\text{Rem}^k(x, t)|}{|u(x, t)|} = \frac{|u(x, t) - u^k(x, t)|}{|u(x, t)|}, \quad (28)$$

where $\text{Rel}^k(x, t)$ denotes the relative error.

RPSM provides an approximate analytical solution in terms of an infinite FPS and it is necessary to give the errors of the approximate solution. In the next section, we apply RPSM to time-space fractional Boussinesq equations and take fourth-order time-space fractional Boussinesq equations as an example to illustrate the absolute error and relative error of the approximate solution.

5. Application of RPSM to Time-Fractional Boussinesq Equations

In this section, the procedure of constructing approximate analytical solutions to time-space fractional Boussinesq equations using RPSM is presented.

5.1. Fourth-Order Time-Space Fractional Boussinesq Equations in $\mathbb{R}$. Consider the fourth-order time-space fractional Boussinesq equation with one-dimensional space variable:

$$D_t^{2\alpha} u(x, t) = \beta D_x^{4\sigma} u(x, t) + \gamma D_x^{2\tau} u(x, t)$$

$$+ \theta D_x^{2\lambda} u^2(x, t) - 4\theta u^2(x, t), \quad (29)$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t)|_{t=0} = a_1(x),$$

where $u(x, t)$ is a field function, $(x, t) \in \mathbb{R} \times \mathbb{R}$, $1/2 < \alpha \leq 1$, $3/4 < \sigma \leq 1$, $1/2 < \tau \leq 1$, $1/2 < \lambda \leq 1$, β, γ , and θ are constant coefficients, and $a_0(x)$ and $a_1(x) \in C^\infty(\mathbb{R})$.

According to (25), the solution of (29) can be written in the following form:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t),$$

$$u_n(x, t) = C_n(x) t^{n\alpha} = \frac{f_n(x)}{\Gamma(n\alpha + 1)} t^{n\alpha}, \quad (30)$$

$$n = 0, 1, 2, \dots$$

Apparently, according to (17), it yields

$$f_0(x) = a_0(x),$$

$$f_1(x) = a_1(x). \quad (31)$$

By (18)–(22), we can show that

$$u^{\text{initial}}(x, t) = a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^\alpha,$$

$$u^k(x, t) = u^{\text{initial}}(x, t) + \sum_{i=2}^k \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (32)$$

$$k = 2, 3, \dots,$$

$$\text{Res}(x, t) = D_t^{2\alpha} u(x, t) + G(x, t),$$

where $G(x, t) = -\beta D_x^{4\sigma} u(x, t) - \gamma D_x^{2\tau} u(x, t) - \theta D_x^{2\lambda} u^2(x, t) + 4\theta u^2(x, t)$,

$$\begin{aligned} \text{Res}^k(x, t) &= D_t^{2\alpha} u^k(x, t) + G^k(x, t), \\ G^k(x, t) &= -\beta D_x^{4\sigma} u^k(x, t) - \gamma D_x^{2\tau} u^k(x, t) \\ &\quad - \theta D_x^{2\lambda} (u^k(x, t))^2 + 4\theta (u^k(x, t))^2, \end{aligned} \quad (33)$$

$k = 2, 3, \dots$

According to (23) and (24), we can obtain

$$\begin{aligned} f_i(x) &= -D_t^{(i-2)\alpha} G^i(x, t) \Big|_{t=0} \\ &= \beta D_t^{(i-2)\alpha} D_x^{4\sigma} u^i(x, t) \Big|_{t=0} \\ &\quad + \gamma D_t^{(i-2)\alpha} D_x^{2\tau} u^i(x, t) \Big|_{t=0} \\ &\quad + \theta D_t^{(i-2)\alpha} D_x^{2\lambda} (u^i(x, t))^2 \Big|_{t=0} \\ &\quad - 4\theta D_t^{(i-2)\alpha} (u^i(x, t))^2 \Big|_{t=0} = \beta D_x^{4\sigma} a_{i-2}(x) \\ &\quad + \gamma D_x^{2\tau} a_{i-2}(x) + \theta D_x^{2\lambda} \Gamma((i-2)\alpha + 1) \\ &\quad \cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \\ &\quad - 4\theta \Gamma((i-2)\alpha + 1) \\ &\quad \cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \triangleq a_i(x), \end{aligned} \quad (34)$$

$i = 2, 3, 4, \dots$

So the k th approximate solution of (29) is

$$u^k(x, t) = \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad k = 0, 1, 2, 3, \dots, \quad (35)$$

where $a_i(x)$ ($i = 0, 1, 2, 3, \dots$) can be found from (29) and (34).

Particularly, when $m = 2$, $\beta = 1$, $\gamma = 1$, $\theta = 1$, $\sigma = 1$, $\tau = 1$, $\lambda = 1$, $a_0(x) = e^x$, and $a_1(x) = \sqrt{2}e^x$, the exact solution of (29) is

$$u(x, t) = e^x E_\alpha(\sqrt{2}t^\alpha). \quad (36)$$

Under these special conditions, according to RPSM, that is (31)–(34), we have

$$a_i(x) = (\sqrt{2})^i e^x, \quad i = 0, 1, 2, 3, \dots \quad (37)$$

And the k th approximate solution of (29) is

$$\begin{aligned} u^k(x, t) &= e^x \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} = e^x \sum_{i=0}^k \frac{(\sqrt{2})^i}{\Gamma(i\alpha + 1)} t^{i\alpha}, \\ k &= 0, 1, 2, 3, \dots \end{aligned} \quad (38)$$

It is easy to see that

$$\begin{aligned} u(x, t) &= \lim_{k \rightarrow \infty} u^k(x, t) = \lim_{k \rightarrow \infty} e^x \sum_{i=0}^k \frac{(\sqrt{2})^i}{\Gamma(i\alpha + 1)} t^{i\alpha} \\ &= e^x \sum_{n=0}^{\infty} \frac{(\sqrt{2}t^\alpha)^n}{\Gamma(n\alpha + 1)} = e^x E_\alpha(\sqrt{2}t^\alpha). \end{aligned} \quad (39)$$

So the FPS solution of (29) is

$$u(x, t) = e^x E_\alpha(\sqrt{2}t^\alpha), \quad (40)$$

which is equal to the exact solutions (36).

Since we choose the special constant coefficients and give the initial value $a_0(x) = e^x$ and $a_1(x) = \sqrt{2}e^x$, so the exact solution in (36) is easily obtained. However, the initial value in application may be “complex” which leads to difficulty to obtain the exact solution and even we cannot obtain it, so it is necessary to construct the approximate solution by RPSM. Thus it is necessary to illustrate the validity and reliability of RPSM in order to construct numerical solution for generalized initial value.

In the numerical experiment, we take $\alpha = 1$; then $u(x, t) = e^{x+\sqrt{2}t}$. We plot the image of $u(x, t)$ and $u^9(x, t)$ with the Mathematica software (see Figures 1 and 2). It is obvious that $u^k(x, t)$ is closely approximate to $u(x, t)$ when $k = 9$. In order to give precise comparison, we plot the intersection of $u(x, t)$ and $u^k(x, t)$ at $x = t$ when $k = 3, 5, 7, 9$ with MATLAB (see Figures 3, 4, 5, and 6). It is clear that $u^k(x, t)$ (see the red curve) is very close to $u(x, t)$ (see the blue curve) when $k = 9$, especially around the point (0, 0) which corresponds to the form of the fractional power series solution assumed in (15).

In the Appendix, from the comparison of Tables 1–4, if the variable x is given, the absolute error $\text{Exa}^k(x, t)$ and the relative error $\text{Rel}^k(x, t)$ tend to 0 as t tends to 0. So the behaviour of the exact solution can be described by the approximate fractional power series solution around the point (0, 0).

Furthermore, if we want to know the behaviour of $u(x, t)$ around the point $(x_0, t_0) \in \mathbb{R} \times (-R, R)$, we can get the information by the transformation $\Gamma : (0, 0) \mapsto (x_0, t_0)$, namely, the expansion of the exact solution at point (x_0, t_0) . And this analysis indicates that RPSM is effective and powerful in approximating the exact solution.

5.2. Sixth-Order Time-Space Fractional Boussinesq Equations in $\mathbb{R}$. Let us consider the sixth-order time-space fractional Boussinesq equation with one-dimensional space variable:

$$\begin{aligned} D_t^{2\alpha} u(x, t) &= \delta D_x^{6\rho} u(x, t) + \beta D_x^{4\sigma} u(x, t) \\ &\quad + \gamma D_x^{2\tau} u(x, t) + \theta D_x^{2\lambda} u^2(x, t) \\ &\quad - 4\theta u^2(x, t), \\ u(x, 0) &= a_0(x), \\ D_t^\alpha u(x, t) \Big|_{t=0} &= a_1(x), \end{aligned} \quad (41)$$

FIGURE 1: The image of $u(x, t)$.

FIGURE 2: The image of $u^9(x, t)$.

— Exact solution $u(x, t)$
 - - - Series approximation $u^3(x, t)$

FIGURE 3: The image of $u(x, t)$ and $u^3(x, t)$.

FIGURE 4: The image of $u(x, t)$ and $u^5(x, t)$.

FIGURE 5: The image of $u(x, t)$ and $u^7(x, t)$.

where $u(x, t)$ is a field function, $(x, t) \in \mathbb{R} \times \mathbb{R}$, $1/2 < \alpha \leq 1$, $5/6 < \rho \leq 1$, $3/4 < \sigma \leq 1$, $1/2 < \tau \leq 1$, $1/2 < \lambda \leq 1$, δ, β, γ , and θ are constants, and $a_0(x)$ and $a_1(x) \in C^\infty(\mathbb{R})$.

TABLE 1: Absolute error and relative error of approximating the solution of (29) when $\alpha = \beta = \gamma = \theta = 1$ and $a_0(x) = e^x$, $a_1(x) = \sqrt{2}e^x$ using RPSM at $x = 5$.

k	t	$u(x, t)$	$u^k(x, t)$	$\text{Exa}^k(x, t)$	$\text{Rel}^k(x, t)$
3	1	610.4605	576.6769	33.7836	0.0553
	0.5	300.9990	299.2057	1.7932	0.0060
	0.25	211.3580	211.2541	0.1039	0.0005
	0.125	177.1110	177.1047	0.0063	0
5	1	610.4605	608.4086	2.0518	0.0034
	0.5	300.9990	300.9703	0.0286	0.0001
	0.25	211.3580	211.3576	0.0004	0
	0.125	177.1110	177.1110	0	0
7	1	610.4605	610.3908	0.0696	0.0001
	0.5	300.9990	300.9987	0.0002	0
	0.25	211.3580	211.3580	0	0
	0.125	177.1110	177.1110	0	0
9	1	610.4605	610.4590	0.0015	0
	0.5	300.9990	300.9990	0	0
	0.25	211.3580	211.3580	0	0
	0.125	177.1110	177.1110	0	0

FIGURE 6: The image of $u(x, t)$ and $u^9(x, t)$.

According to (25), the solution of (41) can be written in the following form:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t),$$

$$u_n(x, t) = C_n(x) t^{n\alpha} = \frac{f_n(x)}{\Gamma(n\alpha + 1)} t^{n\alpha}, \quad (42)$$

$$n = 0, 1, 2, \dots$$

TABLE 2: Absolute error and relative error of approximating the solution of (29) when $\alpha = \beta = \gamma = \theta = 1$ and $a_0(x) = e^x$, $a_1(x) = \sqrt{2}e^x$ using RPSM at $x = 2.5$.

k	t	$u(x, t)$	$u^k(x, t)$	$\text{Exa}^k(x, t)$	$\text{Rel}^k(x, t)$
3	1	50.1096	47.3365	2.7731	0.0553
	0.5	24.7075	24.5603	0.1472	0.0060
	0.25	17.3493	17.3408	0.0085	0.0005
	0.125	14.5382	14.5376	0.0005	0
5	1	50.1096	49.9412	0.1684	0.0034
	0.5	24.7075	24.7051	0.0023	0.0001
	0.25	17.3493	17.3493	0	0
	0.125	14.5382	14.5382	0	0
7	1	50.1096	50.1039	0.0057	0.0001
	0.5	24.7075	24.7075	0	0
	0.25	17.3493	17.3493	0	0
	0.125	14.5382	14.5382	0	0
9	1	50.1096	50.1095	0.0001	0
	0.5	24.7075	24.7075	0	0
	0.25	17.3493	17.3493	0	0
	0.125	14.5382	14.5382	0	0

TABLE 3: Absolute error and relative error of approximating the solution of (29) when $\alpha = \beta = \gamma = \theta = 1$ and $a_0(x) = e^x$, $a_1(x) = \sqrt{2}e^x$ using RPSM at $x = 1.25$.

k	t	$u(x, t)$	$u^k(x, t)$	$\text{Exa}^k(x, t)$	$\text{Rel}^k(x, t)$
3	1	14.3567	13.5621	0.7945	0.0553
	0.5	7.0788	7.0366	0.0422	0.0060
	0.25	4.9707	4.9682	0.0024	0.0005
	0.125	4.1653	4.1651	0.0001	0
5	1	14.3567	14.3084	0.0483	0.0034
	0.5	7.0788	7.0781	0.0007	0.0001
	0.25	4.9707	4.9707	0	0
	0.125	4.1653	4.1653	0	0
7	1	14.3567	14.3550	0.0016	0.0001
	0.5	7.0788	7.0788	0	0
	0.25	4.9707	4.9707	0	0
	0.125	4.1653	4.1653	0	0
9	1	14.3567	14.3566	0	0
	0.5	7.0788	7.0788	0	0
	0.25	4.9707	4.9707	0	0
	0.125	4.1653	4.1653	0	0

Apparently, according to (17), we have

$$f_0(x) = a_0(x),$$

$$f_1(x) = a_1(x). \quad (43)$$

TABLE 4: Absolute error and relative error of approximating the solution of (29) when $\alpha = \beta = \gamma = \theta = 1$ and $a_0(x) = e^x$, $a_1(x) = \sqrt{2}e^x$ using RPSM at $x = 0.625$.

k	t	$u(x, t)$	$u^k(x, t)$	Exa ^k (x, t)	Rel ^k (x, t)
3	1	7.6846	7.2593	0.4253	0.0553
	0.5	3.7890	3.7664	0.0226	0.0060
	0.25	2.6606	2.6593	0.0013	0.0005
	0.125	2.2295	2.2294	0.0001	0
5	1	7.6846	7.6587	0.0258	0.0034
	0.5	3.7890	3.7887	0.0004	0.0001
	0.25	2.6606	2.6606	0	0
	0.125	2.2295	2.2295	0	0
7	1	7.6846	7.6837	0.0009	0.0001
	0.5	3.7890	3.7890	0	0
	0.25	2.6606	2.6606	0	0
	0.125	2.2295	2.2295	0	0
9	1	7.6846	7.6845	0	0
	0.5	3.7890	3.7890	0	0
	0.25	2.6606	2.6606	0	0
	0.125	2.2295	2.2295	0	0

Using (18)–(22), we can get

$$u^{\text{initial}}(x, t) = a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^\alpha,$$

$$u^k(x, t) = u^{\text{initial}}(x, t) + \sum_{i=2}^k \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (44)$$

$$k = 2, 3, \dots,$$

$$\text{Res}(x, t) = D_t^{2\alpha} u(x, t) + G(x, t),$$

where $G(x, t) = -\delta D_x^{6\rho} u(x, t) - \beta D_x^{4\sigma} u(x, t) - \gamma D_x^{2\tau} u(x, t) - \theta D_x^{2\lambda} u^2(x, t) + 4\theta u^2(x, t)$,

$$\text{Res}^k(x, t) = D_t^{2\alpha} u^k(x, t) + G^k(x, t),$$

$$G^k(x, t) = -\delta D_x^{6\rho} u^k(x, t) - \beta D_x^{4\sigma} u^k(x, t) - \gamma D_x^{2\tau} u^k(x, t) - \theta D_x^{2\lambda} (u^k(x, t))^2 + 4\theta (u^k(x, t))^2, \quad k = 2, 3, \dots \quad (45)$$

By (23) and (24), it is shown that

$$f_i(x) = -D_t^{(i-2)\alpha} G^i(x, t) \Big|_{t=0}$$

$$= \delta D_t^{(i-2)\alpha} D_x^{6\rho} u^i(x, t) \Big|_{t=0}$$

$$+ \beta D_t^{(i-2)\alpha} D_x^{4\sigma} u^i(x, t) \Big|_{t=0}$$

$$+ \gamma D_t^{(i-2)\alpha} D_x^{2\tau} u^i(x, t) \Big|_{t=0}$$

$$+ \theta D_t^{(i-2)\alpha} D_x^{2\lambda} (u^i(x, t))^2 \Big|_{t=0}$$

$$- 4\theta D_t^{(i-2)\alpha} (u^i(x, t))^2 \Big|_{t=0} = \delta D_x^{6\rho} a_{i-2}(x)$$

$$+ \beta D_x^{4\sigma} a_{i-2}(x) + \gamma D_x^{2\tau} a_{i-2}(x)$$

$$+ \theta D_x^{2\lambda} \Gamma((i-2)\alpha + 1)$$

$$\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)}$$

$$- 4\theta \Gamma((i-2)\alpha + 1)$$

$$\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \triangleq a_i(x),$$

$$i = 2, 3, 4, \dots \quad (46)$$

So the k th approximate solution of (41) is

$$u^k(x, t) = \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad k = 0, 1, 2, 3, \dots, \quad (47)$$

where $a_i(x)$ ($i = 0, 1, 2, 3, \dots$) can be found from (41) and (46).

Particularly, when $m = 2$, $\beta = 1$, $\gamma = 1$, $\theta = 1$, $\rho = 1$, $\sigma = 1$, $\tau = 1$, $\lambda = 1$, $a_0(x) = e^x$, and $a_1(x) = \sqrt{3}e^x$, the exact solution of (41) is

$$u(x, t) = e^x E_\alpha(\sqrt{3}t^\alpha). \quad (48)$$

According to RPSM, we can obtain

$$a_i(x) = (\sqrt{3})^i e^x, \quad i = 0, 1, 2, 3, \dots \quad (49)$$

And the k th approximate solution of (41) is

$$u^k(x, t) = e^x \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} = e^x \sum_{i=0}^k \frac{(\sqrt{3})^i}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (50)$$

$$k = 0, 1, 2, 3, \dots$$

It is easy to see that

$$u(x, t) = \lim_{k \rightarrow \infty} u^k(x, t) = \lim_{k \rightarrow \infty} e^x \sum_{i=0}^k \frac{(\sqrt{3})^i}{\Gamma(i\alpha + 1)} t^{i\alpha}$$

$$= e^x \sum_{n=0}^{\infty} \frac{(\sqrt{3}t^\alpha)^n}{\Gamma(n + 1)} = e^x E_\alpha(\sqrt{3}t^\alpha). \quad (51)$$

So the FPS solution of (41) is

$$u(x, t) = e^x E_\alpha(\sqrt{3}t^\alpha), \quad (52)$$

which is equal to (48).

When $\alpha = 1$, the exact solution is $u(x, t) = e^{x+\sqrt{3}t}$, and we also plot the images (see Figures 7, 8, 9, 10, 11, and 12). And it is clear that when $k = 9$, $u^k(x, t)$ is closely approximate to $u(x, t)$.

FIGURE 7: The image of $u(x, t)$. FIGURE 8: The image of $u^9(x, t)$.

5.3. 2nth-Order Time-Space Fractional Boussinesq Equations in $\mathbb{R}$. Let us consider the sixth-order time-space fractional Boussinesq equation:

$$D_t^{2\alpha} u(x, t) = \sum_{p=1}^n \beta_p D_x^{2p\tau_p} u(x, t) + \theta D_x^2 u^2(x, t) - 4\theta u^2(x, t), \quad (53)$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t)|_{t=0} = a_1(x),$$

where $u(x, t)$ is a field function, $(x, t) \in \mathbb{R} \times \mathbb{R}$, $1/2 < \alpha \leq 1$, $(2p-1)/2p < \tau_p \leq 1$, β_p are constant coefficients, $p = 1, 2, \dots, n$, and $a_0(x)$ and $a_1(x) \in C^\infty(\mathbb{R})$.

According to (25), the solution of (53) can be written in the following form:

$$u(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \quad (54)$$

$$u_i(x, t) = C_i(x) t^{i\alpha} = \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} \quad (i = 0, 1, 2, \dots).$$

FIGURE 9: The image of $u(x, t)$ and $u^3(x, t)$. FIGURE 10: The image of $u(x, t)$ and $u^5(x, t)$.

Apparently, according to (17), we have

$$\begin{aligned} f_0(x) &= a_0(x), \\ f_1(x) &= a_1(x). \end{aligned} \quad (55)$$

— Exact solution $u(x, t)$
 - - Series approximation $u^7(x, t)$

FIGURE 11: The image of $u(x, t)$ and $u^7(x, t)$.

— Exact solution $u(x, t)$
 - - Series approximation $u^9(x, t)$

FIGURE 12: The image of $u(x, t)$ and $u^9(x, t)$.

By (18)–(22), we can obtain

$$u^{\text{initial}}(x, t) = a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^\alpha,$$

$$u^k(x, t) = u^{\text{initial}}(x, t) + \sum_{i=2}^k \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (56)$$

$$(k = 2, 3, \dots),$$

$$\text{Res}(x, t) = D_t^{2\alpha} u(x, t) + G(x, t),$$

$$\text{where } G(x, t) = -\sum_{p=1}^n \beta_p D_x^{2p\tau_p} u(x, t) - \theta D_x^2 u^2(x, t) + 4\theta u^2(x, t),$$

$$\text{Res}^k(x, t) = D_t^{2\alpha} u^k(x, t) + G^k(x, t),$$

$$G^k(x, t) = -\sum_{p=1}^n \beta_p D_x^{2p\tau_p} u^k(x, t) - \theta D_x^2 (u^k(x, t))^2 + 4\theta (u^k(x, t))^2, \quad k = 2, 3, \dots \quad (57)$$

Using (23) and (24), it is shown that

$$f_i(x) = -D_t^{(i-2)\alpha} G^i(x, t) \Big|_{t=0}$$

$$= D_t^{(i-2)\alpha} \left(\sum_{p=1}^n \beta_p D_x^{2p\tau_p} u^i(x, t) \right) \Big|_{t=0}$$

$$+ \theta D_t^{(i-2)\alpha} (u^i(x, t))^2 \Big|_{t=0}$$

$$- 4\theta D_t^{(i-2)\alpha} (u^i(x, t))^2 \Big|_{t=0} = \sum_{p=1}^n \beta_p D_x^{2p\tau_p} a_{i-2}(x)$$

$$+ \theta D_x^2 \Gamma((i-2)\alpha + 1)$$

$$\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)}$$

$$- 4\theta \Gamma((i-2)\alpha + 1)$$

$$\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \triangleq a_i(x),$$

$$i = 2, 3, 4, \dots \quad (58)$$

So the k th approximate solution of (53) is

$$u^k(x, t) = \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad k = 0, 1, 2, 3, \dots, \quad (59)$$

where $a_i(x)$ ($i = 0, 1, 2, 3, \dots$) can be found from (53) and (58).

When $m = 2$, $\beta = 1$, $\gamma = 1$, $\theta = 1$, $\tau_p = 1$, $p = 1, 2, \dots, n$, $\lambda = 1$, $a_0(x) = e^x$, and $a_1(x) = \sqrt{n}e^x$, the exact solution of (53) is

$$u(x, t) = e^x E_\alpha(\sqrt{n}t^\alpha). \quad (60)$$

According to RPSM, we have

$$a_i(x) = (\sqrt{n})^i e^x, \quad i = 0, 1, 2, 3, \dots \quad (61)$$

And the k th approximate solution of (53) is

$$u^k(x, t) = e^x \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} = e^x \sum_{i=0}^k \frac{(\sqrt{n})^i}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (62)$$

$$k = 0, 1, 2, 3, \dots$$

It is easy to see that

$$u(x, t) = \lim_{k \rightarrow \infty} u^k(x, t) = \lim_{k \rightarrow \infty} e^x \sum_{i=0}^k \frac{(\sqrt{n})^i}{\Gamma(i\alpha + 1)} t^{i\alpha} \quad (63)$$

$$= e^x \sum_{n=0}^{\infty} \frac{(\sqrt{nt}^\alpha)^n}{\Gamma(n\alpha + 1)} = e^x E_\alpha(\sqrt{nt}^\alpha).$$

So the FPS solution of (53) is

$$u(x, t) = e^x E_\alpha(\sqrt{nt}^\alpha), \quad (64)$$

which is equal to (60).

5.4. Fourth-Order Time-Space Fractional Boussinesq Equations in $\mathbb{R}^2$. Consider the fourth-order time-space fractional Boussinesq equation with two-dimensional space variables:

$$D_t^{2\alpha} u(x, t) = \beta_1 D_{x_1}^{4\sigma_1} u(x, t) + \beta_2 D_{x_2}^{4\sigma_2} u(x, t) \quad (65)$$

$$+ \gamma_1 D_{x_1}^{2\tau_1} u(x, t) + \gamma_2 D_{x_2}^{2\tau_2} u(x, t)$$

$$+ \theta_1 D_{x_1}^{2\lambda_1} u^2(x, t)$$

$$+ \theta_2 D_{x_2}^{2\lambda_2} u^2(x, t) - 4\theta_1 u^2(x, t)$$

$$- 4\theta_2 u^2(x, t),$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t)|_{t=0} = a_1(x),$$

where $u(x, t)$ is a field function, $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$, $1/2 < \alpha \leq 1$, $3/4 < \sigma_i \leq 1$, $1/2 < \tau_i \leq 1$, $1/2 < \lambda_i \leq 1$, $\beta_i, \gamma_i, \theta_i$ ($i = 1, 2$) are constants, and $a_0(x)$ and $a_1(x) \in C^\infty(\mathbb{R}^2)$.

Before solving this problem, we give several notations for convenience:

$$x = (x_1, x_2),$$

$$\bar{x} := x_1 + x_2,$$

$$\beta = (\beta_1, \beta_2),$$

$$\gamma = (\gamma_1, \gamma_2),$$

$$\theta = (\theta_1, \theta_2),$$

$$\lambda = (\lambda_1, \lambda_2),$$

$$\bar{\theta} := \theta_1 + \theta_2,$$

$$D_x^{j\omega}(\cdot) := (D_{x_1}^{j\omega_1}, D_{x_2}^{j\omega_2})^T(\cdot),$$

$$\omega = (\omega_1, \omega_2), \quad j = 2, 4,$$

$$\lambda D_x^j(\cdot) := (\lambda_1, \lambda_2) (D_{x_1}^j, D_{x_2}^j)^T(\cdot)$$

$$= \lambda_1 D_{x_1}^{j\omega_1}(\cdot) + \lambda_2 D_{x_2}^{j\omega_2}(\cdot).$$

(66)

So system (65) is equal to

$$D_t^\alpha u(x, t) = \beta D_x^{4\sigma} u(x, t) + \gamma D_x^{2\tau} u(x, t) \quad (67)$$

$$+ \theta D_x^{2\lambda} u^2(x, t) - 4\bar{\theta} u^2(x, t),$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t)|_{t=0} = a_1(x).$$

According to (25), the solution of (67) can be written in the following form:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad (68)$$

$$u_n(x, t) = C_n(x) t^{n\alpha} = \frac{f_n(x)}{\Gamma(n\alpha + 1)} t^{n\alpha},$$

$$n = 0, 1, 2, \dots$$

Apparently, according to (17), we have

$$f_0(x) = a_0(x), \quad (69)$$

$$f_1(x) = a_1(x).$$

Using (18)–(22), we can obtain

$$u^{\text{initial}}(x, t) = a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^\alpha,$$

$$u^k(x, t) = u^{\text{initial}}(x, t) + \sum_{i=2}^k \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (70)$$

$$k = 2, 3, \dots,$$

$$\text{Res}(x, t) = D_t^{2\alpha} u(x, t) + G(x, t),$$

$$\text{where } G(x, t) = -\beta D_x^{4\sigma} u(x, t) - \gamma D_x^{2\tau} u(x, t) - \theta D_x^{2\lambda} u^2(x, t) + 4\bar{\theta} u^2(x, t),$$

$$\text{Res}^k(x, t) = D_t^{2\alpha} u^k(x, t) + G^k(x, t),$$

$$G^k(x, t) = -\beta D_x^{4\sigma} u^k(x, t) - \gamma D_x^{2\tau} u^k(x, t) \quad (71)$$

$$- \theta D_x^{2\lambda} (u^k(x, t))^2 + 4\bar{\theta} (u^k(x, t))^2,$$

$$k = 2, 3, \dots$$

Using (23) and (24), it is shown that

$$\begin{aligned}
 f_i(x) &= -D_t^{(i-2)\alpha} G^i(x, t) \Big|_{t=0} \\
 &= \beta D_t^{(i-2)\alpha} D_x^{4\sigma} u^i(x, t) \Big|_{t=0} \\
 &+ \gamma D_t^{(i-2)\alpha} D_x^{2\tau} u^i(x, t) \Big|_{t=0} \\
 &+ \theta D_t^{(i-2)\alpha} D_x^{2\lambda} (u^i(x, t))^2 \Big|_{t=0} \\
 &- 4\bar{\theta} D_t^{(i-2)\alpha} (u^i(x, t))^2 \Big|_{t=0} = \beta D_x^{4\sigma} a_{i-2}(x) \\
 &+ \gamma D_x^{2\tau} a_{i-2}(x) + \theta D_x^{2\lambda} \Gamma((i-2)\alpha + 1) \\
 &\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \\
 &- 4\bar{\theta} \Gamma((i-2)\alpha + 1) \\
 &\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \triangleq a_i(x), \\
 &i = 2, 3, 4, \dots
 \end{aligned} \quad (72)$$

So the k th approximate solution of (67) is

$$u^k(x, t) = \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad k = 0, 1, 2, 3, \dots, \quad (73)$$

where $a_i(x)$ ($i = 0, 1, 2, 3, \dots$) can be found from (67) and (72).

Particularly when $m = 2$, $\beta = (1, 1)$, $\gamma = (1, 1)$, $\theta = (1, 1)$, $\sigma = (1, 1)$, $\tau = (1, 1)$, $\lambda = (1, 1)$, $a_0(x) = e^{\bar{x}}$, and $a_1(x) = \sqrt{2}e^{\bar{x}}$, the exact solution of (67) is

$$u(x, t) = e^{\bar{x}} E_\alpha(\sqrt{2}t^\alpha). \quad (74)$$

According to RPSM, we have

$$a_i(x) = (\sqrt{2})^i e^{\bar{x}}, \quad i = 0, 1, 2, 3, \dots \quad (75)$$

And the k th approximate solution of (67) is

$$\begin{aligned}
 u^k(x, t) &= e^{\bar{x}} \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} = e^{\bar{x}} \sum_{i=0}^k \frac{(\sqrt{2})^i}{\Gamma(i\alpha + 1)} t^{i\alpha}, \\
 &k = 0, 1, 2, 3, \dots
 \end{aligned} \quad (76)$$

It is easy to see that

$$u(x, t) = \lim_{k \rightarrow \infty} u^k(x, t) = \lim_{k \rightarrow \infty} e^{\bar{x}} \sum_{i=0}^k \frac{(\sqrt{2})^i}{\Gamma(i\alpha + 1)} t^{i\alpha} \quad (77)$$

$$= e^{\bar{x}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}t^\alpha)^n}{\Gamma(n\alpha + 1)} = e^{\bar{x}} E_\alpha(\sqrt{2}t^\alpha).$$

So the FPS solution of (67) is

$$u(x, t) = e^{\bar{x}} E_\alpha(\sqrt{2}t^\alpha), \quad (78)$$

which is equal to (74).

5.5. Fourth-Order Time-Space Fractional Boussinesq Equations in $\mathbb{R}^n$. Let us consider the fourth-order time-space fractional Boussinesq equation with n -dimensional space variables:

$$\begin{aligned}
 D_t^{2\alpha} u(x, t) &= \sum_{p=1}^{n_1} \beta_p D_{x_p}^{4\sigma_p} u(x, t) + \sum_{q=1}^{n_2} \gamma_q D_{x_q}^{2\tau_q} u(x, t) \\
 &+ \sum_{r=1}^n \theta_r D_{x_r}^{2\lambda_r} u^2(x, t) \\
 &- \sum_{r=1}^n 4\bar{\theta}_r u^2(x, t),
 \end{aligned} \quad (79)$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t) \Big|_{t=0} = a_1(x),$$

where $u(x, t)$ is a field function, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, $1/2 < \alpha \leq 1$, $3/4 < \sigma_p \leq 1$, $p = 1, 2, \dots, n_1 \leq n \in \mathbb{N}$, $1/2 < \tau_q \leq 1$, $q = 1, 2, \dots, n_2 \leq n \in \mathbb{N}$, $1/2 < \lambda_r \leq 1$, $r = 1, 2, \dots, n$, β_p ($p = 1, 2, \dots, n_1$) and γ_q ($q = 1, 2, \dots, n_2$) are constants, and $a_0(x)$ and $a_1(x) \in C^\infty(\mathbb{R}^n)$.

For convenience, we give several notations:

$$\begin{aligned}
 x &= (x_1, x_2, \dots, x_n), \\
 \bar{x} &:= \sum_{i=1}^n x_i, \\
 \beta &= (\beta_1, \beta_2, \dots, \beta_{n_1}), \\
 \gamma &= (\gamma_1, \gamma_2, \dots, \gamma_{n_2}), \\
 \theta &= (\theta_1, \theta_2, \dots, \theta_n), \\
 \lambda &= (\lambda_1, \lambda_2, \dots, \lambda_n), \\
 \bar{\theta} &= \sum_{i=1}^n \theta_i,
 \end{aligned} \quad (80)$$

$$\begin{aligned}
 D_x^{j\omega}(\cdot) &:= (D_{x_1}^{j\omega_1}, D_{x_2}^{j\omega_2}, \dots, D_{x_n}^{j\omega_n})^T(\cdot), \\
 \omega &= (\omega_1, \omega_2, \dots, \omega_n), \quad j = 2, 4,
 \end{aligned}$$

$$\lambda D_x^j(\cdot) := (\lambda_1, \lambda_2, \dots, \lambda_n) (D_{x_1}^j, D_{x_2}^j, \dots, D_{x_n}^j)^T(\cdot)$$

$$= \sum_{i=1}^n (\lambda_i D_{x_i}^{j\omega_i}(\cdot)).$$

So system (79) is equal to

$$\begin{aligned}
 D_t^{2\alpha} u(x, t) &= \beta D_x^{4\sigma} u(x, t) + \gamma D_x^{2\tau} u(x, t) \\
 &+ \theta D_x^{2\lambda} u^2(x, t) - 4\bar{\theta} u^2(x, t),
 \end{aligned} \quad (81)$$

$$u(x, 0) = a_0(x),$$

$$D_t^\alpha u(x, t) \Big|_{t=0} = a_1(x).$$

According to (25), the solution of (81) can be written in the following form:

$$u(x, t) = \sum_{i=0}^{\infty} u_i(x, t), \quad (82)$$

$$u_i(x, t) = C_i(x) t^{i\alpha} = \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} \quad (i = 0, 1, 2, \dots).$$

Apparently, according to (17), we have

$$\begin{aligned} f_0(x) &= a_0(x), \\ f_1(x) &= a_1(x). \end{aligned} \quad (83)$$

By (18)–(22), we can show that

$$u^{\text{initial}}(x, t) = a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^\alpha,$$

$$u^k(x, t) = u^{\text{initial}}(x, t) + \sum_{i=2}^k \frac{f_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (84)$$

$$(k = 2, 3, \dots),$$

$$\text{Res}(x, t) = D_t^{2\alpha} u(x, t) + G(x, t),$$

where $G(x, t) = -\beta D_x^{4\sigma} u(x, t) - \gamma D_x^{2\tau} u(x, t) - \theta D_x^{2\lambda} u^2(x, t) + 4\bar{\theta} u^2(x, t)$,

$$\begin{aligned} \text{Res}^k(x, t) &= D_t^{2\alpha} u^k(x, t) + G^k(x, t), \\ G^k(x, t) &= -\beta D_x^{4\sigma} u^k(x, t) - \gamma D_x^{2\tau} u^k(x, t) \\ &\quad - \theta D_x^{2\lambda} (u^k)^2(x, t) + 4\bar{\theta} (u^k)^2(x, t), \end{aligned} \quad (85)$$

$$k = 2, 3, \dots$$

According to (23) and (24), we can obtain

$$\begin{aligned} f_i(x) &= -D_t^{(i-2)\alpha} G^i(x, t) \Big|_{t=0} \\ &= \beta D_t^{(i-2)\alpha} D_x^{4\sigma} u^i(x, t) \Big|_{t=0} \\ &\quad + \gamma D_t^{(i-2)\alpha} D_x^{2\tau} u^i(x, t) \Big|_{t=0} \\ &\quad + \theta D_t^{(i-2)\alpha} D_x^{2\lambda} (u^i(x, t))^2 \Big|_{t=0} \\ &\quad - 4\bar{\theta} D_t^{(i-2)\alpha} (u^i(x, t))^2 \Big|_{t=0} = \beta D_x^{4\sigma} a_{i-2}(x) \\ &\quad + \gamma D_x^{2\tau} a_{i-2}(x) + \theta D_x^{2\lambda} \Gamma((i-2)\alpha + 1) \\ &\quad \cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \\ &\quad - 4\bar{\theta} \Gamma((i-2)\alpha + 1) \end{aligned}$$

$$\cdot \sum_{j=0}^{i-2} \frac{a_j(x) a_{i-2-j}(x)}{\Gamma(j\alpha + 1) \Gamma((i-2-j)\alpha + 1)} \triangleq a_i(x), \quad i = 2, 3, 4, \dots \quad (86)$$

So the k th approximate solution of (81) is

$$u^k(x, t) = \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad k = 0, 1, 2, 3, \dots, \quad (87)$$

where $a_i(x)$ ($i = 0, 1, 2, 3, \dots$) can be found from (81) and (86).

When $m = 2$, $\beta = \underbrace{(1, 1, \dots, 1)}_n$, $\gamma = \underbrace{(1, 1, \dots, 1)}_n$, $\theta = \underbrace{(1, 1, \dots, 1)}_n$, $\sigma = \underbrace{(1, 1, \dots, 1)}_{n_1}$, $\tau = \underbrace{(1, 1, \dots, 1)}_{n_2}$, $\lambda = \underbrace{(1, 1, \dots, 1)}_n$, $a_0(x) = e^{\bar{x}}$, and $a_1(x) = \sqrt{2}e^{\bar{x}}$. The exact solution of (81) is

$$u(x, t) = e^{\bar{x}} E_\alpha(\sqrt{2}t^\alpha). \quad (88)$$

According to RPSM, we have

$$a_i(x) = (\sqrt{2})^i e^{\bar{x}}, \quad i = 0, 1, 2, 3, \dots \quad (89)$$

And the k th approximate solution of (81) is

$$u^k(x, t) = e^{\bar{x}} \sum_{i=0}^k \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha} = e^{\bar{x}} \sum_{i=0}^k \frac{(\sqrt{2})^i}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad (90)$$

$$k = 0, 1, 2, 3, \dots$$

It is easy to see that

$$\begin{aligned} u(x, t) &= \lim_{k \rightarrow \infty} u^k(x, t) = \lim_{k \rightarrow \infty} e^{\bar{x}} \sum_{i=0}^k \frac{(\sqrt{2})^i}{\Gamma(i\alpha + 1)} t^{i\alpha} \\ &= e^{\bar{x}} \sum_{n=0}^{\infty} \frac{(\sqrt{2}t^\alpha)^n}{\Gamma(n\alpha + 1)} = e^{\bar{x}} E_\alpha(\sqrt{2}t^\alpha). \end{aligned} \quad (91)$$

So the FPS solution of (81) is

$$u(x, t) = e^{\bar{x}} E_\alpha(\sqrt{2}t^\alpha), \quad (92)$$

which is equal to (88).

6. Conclusion

The RPSM method for constructing the fractional power series solutions for any-order time-space fractional Boussinesq equation with the initial values in $\mathbb{R}^n$ is presented in this paper. The present technique is performed, based mainly on generating a residual error function and then applying the generalized power series formula. The numerical simulation results reveal the validity and reliability of RPSM. That is to say, RPSM is extremely powerful in terms of constructing fractional power series solutions or approximate fractional partial differential equations in scientific applications.

Appendix

See Tables 1, 2, 3, and 4.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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