Adaptive Finite-Time Stabilization of High-Order Nonlinear Systems with Dynamic and Parametric Uncertainties

Meng-Meng Jiang and Xue-Jun Xie

Institute of Automation, Qufu Normal University, Shandong 273165, China

Correspondence should be addressed to Meng-Meng Jiang; jmm725@163.com

Received 1 May 2016; Accepted 31 July 2016

Academic Editor: Ricardo Aguilar-López

Copyright © 2016 M.-M. Jiang and X.-J. Xie. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Under the weaker assumption on nonlinear functions, the adaptive finite-time stabilization of more general high-order nonlinear systems with dynamic and parametric uncertainties is solved in this paper. To solve this problem, finite-time input-to-state stability (FTISS) is used to characterize the unmeasured dynamic uncertainty. By skillfully combining Lyapunov function, sign function, backstepping, and finite-time input-to-state stability approaches, an adaptive state feedback controller is designed to guarantee high-order nonlinear systems are globally finite-time stable.

1. Introduction

Since the concept of finite-time stability was introduced in [1], many efforts have been made to study the problem of finite-time stabilization because of faster convergence rates, higher accuracies, and better disturbance rejection properties. Based on the finite-time stability theorem in [2–4], some finite-time stabilization results have been achieved by combining finite-time stability with backstepping design method, for example, [5–9] and the references therein.

Recently, more attention of finite-time stability has been focused on a family of high-order nonlinear systems of the form

\[
\dot{x}_i (t) = x_i^{p_i} (t) + \phi_i (x_1 (t), \ldots, x_i (t), d), \quad i = 1, \ldots, n - 1, \tag{1}
\]

\[
\dot{x}_n (t) = u^{p_n} (t) + \phi_n (x_1 (t), \ldots, x_n (t), d),
\]

where \(u \in R\) is the control input, \(x = (x_1, \ldots, x_n)^T \in R^n\) is the measured state, and \(d \in R^r\) denotes the unknown parameter vector. For \(i = 1, \ldots, n, \phi_i : R^r \times R^{p_i} \times R \rightarrow R\) is an unknown and Lipschitz continuous function. \(p_i \in R^r_{od} = \{p/q \in R^r : p\) and \(q\) are odd integers, \(p \geq q\). System (1) is called high-order system if there exists at least \(p_i > 1, i \in \{1, \ldots, n\}\).

For system (1), when \(d\) is known, [10, 11] studied finite-time stability, where the order of state \(x_j (j = 1, \ldots, i)\) in (10) and (11) can be taken value in \((0, 1/(p_j \cdots p_{i-1}))\) with \(p_j p_{i-1} = 1\). The restrictive condition was relaxed by [12], in which all the states in the bounding condition were allowed to be of both low order and high order. When \(d\) is unknown, it is well known that adaptive technique is an effective way to deal with control problem of nonlinear systems with parametric uncertainty. Reference [13] developed a continuous adaptive finite-time controller with the bounding condition of \(\phi\) being an order equal to 1. The latest paper [14] weakened the growth condition by allowing the order greater than 0. However, there is no dynamic uncertainty considered by these papers.

The analysis and control problem of nonlinear systems with dynamic uncertainty have been an active research topic because dynamic uncertainty often arises from many different control engineering applications; see [15–19] and the references therein. In view of the benefits of finite-time convergence, finite-time stabilization of nonlinear systems with dynamic uncertainty has been regarded as one of the important issues. By characterizing dynamic uncertainty with finite-time input-to-state stability (FTISS), [20] constructed a finite-time adaptive state feedback controller for one-order nonlinear systems with dynamic and parametric uncertainties. Reference [21] gave the explicit definition of FTTSS and
developed a framework for the finite-time control analysis and synthesis based on FTISS. However, for more general high-order nonlinear systems, to the best of the authors’ knowledge, no result on finite-time stabilization has been achieved until now.

Based on the above discussion, an interesting problem is put forward spontaneously: for more general high-order systems with dynamic and parametric uncertainties,

\[ \dot{z}(t) = \psi(z(t), x_i(t)), \]
\[ \dot{x}_i(t) = x_i^{\alpha}(t) + \phi_i(x_1(t), \ldots, x_n(t), z(t), d), \]

\[ i = 1, \ldots, n-1, \]
\[ \dot{x}_n(t) = u^\alpha(t) + \phi_n(x_1(t), \ldots, x_n(t), z(t), d), \]

where \( f : U_0 \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous with respect to \( x \) on an open neighborhood \( U_0 \) of the origin \( x = 0 \). The equilibrium \( x = 0 \) of the system is (local) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood \( U \subseteq U_0 \) of the origin. By “finite-time convergence” one means the following: if, for any initial condition \( x(t_0) = x_0 \in U \) at any given initial time \( t_0 \), there is a settling time \( T > 0 \), such that every solution \( x(t; t_0, x_0) \) of system (3) is defined with \( x(t; t_0, x_0) \in U/0 \) for \( t \in [t_0, T] \) and \( \lim_{t \to T} x(t; t_0, x_0) = 0, x(t; t_0, x_0) = 0 \) for any \( t > T \). When \( U = \mathbb{R}^n \), the origin is a globally finite-time stable equilibrium.

Definition 2 (see [21]). Consider a system

\[ x = f(x, v), \quad f(0, 0) = 0, \quad x \in \mathbb{R}^n, \quad v \in \mathbb{R}^m, \]

where \( v \) is the input and \( f \) is continuous with respect to \( (x, v) \). A continuous function \( V(x) \) is called FTISS-Lyapunov function for system (4) if there exist \( \mathcal{X}_\infty \)-functions \( \phi_1, \phi_2 \), and \( \phi_3 \) and a positive constant \( c \) such that

\[ \phi_1(||x||) \leq V(x) \leq \phi_2(||x||), \quad \forall x \in \mathbb{R}^n, \]

\[ \dot{V}(x(t)) \leq -cV^n(x(t)) + \phi_3(||v(t)||), \quad 0 < \alpha < 1. \]

In the remainder of this section, we list several lemmas that serve as the basis for the design of state feedback controller for system (2). Lemma 3 is finite-time stability theorem. Lemmas 4–8 are used to enlarge inequalities. Lemmas 9 and 10 are used to deal with sign function.

Lemma 3 (see [13]). Suppose that, for system (3), there is a \( \mathcal{C}^1 \) positive-definite function \( V(x, t) \) (defined on \( \bar{U} \times \mathbb{R}^n \), where \( \bar{U} \subseteq U_0 \in \mathbb{R}^n \) is a neighborhood of the origin), and a real number \( c > 0 \) and \( 0 < \alpha < \beta < 1 \), such that \( \dot{V}(x(t), t) \leq V^n(x(t)) + \alpha \dot{V}(x(t), t) \leq (c/(1-\alpha)) \leq (c/(1-\alpha)) \) for any initial condition \( x(t_0) \) in a neighborhood of the origin in \( \bar{U} \).

Lemma 4 (Young’s inequality). Let real numbers \( p \geq 1 \) and \( q \geq 1 \) satisfy \( 1/p + 1/q = 1 \); then for any \( x, y \in \mathbb{R} \) and any given positive number \( y > 0 \),

\[ xy \leq |x|^p + (1/q)(p/y)^{-q/p} |y|^q. \]

Lemma 5 (Jensen’s inequality). If \( 0 < a_1 < a_2 \), then

\[ (\sum_{i=1}^n x_i^{a_1})^{1/a_1} \leq (\sum_{i=1}^n x_i^{a_2})^{1/a_2} \]

for any \( x_i \geq 0, i = 1, \ldots, n \).

Lemma 6 (see [22]). If \( p \geq 1 \), then

\[ |x + y|^p \leq |x|^p + |y|^p, \]

\[ |x + y|^p \leq 2^{p-1} |x|^p + |y|^p \]

for any \( x, y \in \mathbb{R} \).

Lemma 7 (see [22]). If \( p \in (0, 1) \), then

\[ |x - y|^p \leq 2^{p-1} |x|^p + |y|^p \]

for any \( x, y \in \mathbb{R} \).

Lemma 8 (see [23]). For a continuous function \( f(x, y) \) with \( x \in \mathbb{R}^n, y \in \mathbb{R}^m \), there exist smooth functions \( a(x) \geq 0, b(y) \geq 0, c(x) \geq 1, \) and \( d(y) \geq 1 \), such that

\[ |f(x, y)| \leq a(x) + b(y), \]

\[ |f(x, y)| \leq c(x)d(y). \]

Lemma 9 (see [24]). If \( p = a/b \in \mathbb{R}^d_+ \), then

\[ |x|^p - |y|^p \leq 2^{1-\beta} |x|^a - |y|^a \leq 2^{1-\beta} |x|^a - |y|^a \]

for any \( x, y \in \mathbb{R} \).
Lemma 10 (see [24]). \( f(x) = \text{sgn}(x)|x|^a \) is continuously differentiable, and \( \dot{f}(x) = a|x|^a-1, \) where \( a \geq 1, x \in R. \) Moreover, if \( x = x(t), t \geq 0, \) then \( df(x(t))/dt = a|x(t)|^{a-1}\dot{x}(t). \)

3. Finite-Time Convergence Analysis

3.1. Problem Formulation and Assumptions. The purpose of this paper is to achieve a global finite-time control design for high-order nonlinear system (2) with dynamic and parametric uncertainties.

To achieve this purpose, we need the following assumptions.

Assumption 1. The \( z \)-subsystem has an FTISS-Lyapunov function \( V_0(z) \) that satisfies

\[
\dot{V}_0 \leq -c_0 V_0 + \gamma_0 |x_1|^p, \tag{6}
\]

where \( c_0 > 0, 0 < a_0 < 1 \) are constants, and \( \gamma_0 \) is a \( \mathcal{K} \) function. Moreover, \( \pi_1 \) and \( \pi_2 \) are \( \mathcal{K} \) functions such that

\[
\pi_i(\|z\|) \leq \pi_2(\|z\|), \tag{7}
\]

Assumption 2. For each \( i = 1, \ldots, n \), there is a constant \( \nu \in (0,1/(\sum_{j=1}^{n} p_j + 1)) \) satisfying \( (2-\nu)/(2-\nu+r_i) > 2a_0 \) and known \( \mathcal{E}^{\frac{1}{2}} \) nonnegative function \( \kappa_1 \) and \( \mathcal{E} \) nonnegative function \( \kappa_2 \) with \( \kappa_1(0) = 0 \) and \( \kappa_2(0) = 0 \) such that

\[
|\phi_i(x_1, \ldots, x_i, z, d)| \
\leq \kappa_1(\|z\|) + \gamma_1 |x_{i+1}|^p \
+ \theta \kappa_2(x_1, \ldots, x_i) \sum_{j=1}^{i} |x_j|^{(r_j+\gamma)/r_j+\mu_j}, \tag{8}
\]

where \( \theta > 0 \) is an unknown constant, \( \gamma_1 \in [0,1), \mu_j \in [0, +\infty), \) \( P_0 = 1 \), \( x_{i+1}(t) = u(t) \), and

\[
r_1 = 1, \
r_i = \frac{r_i-\nu}{P_{i-1}}, \quad i = 2, \ldots, n. \tag{9}
\]

Assumption 3. We assume that \( \lim \sup_{s \to 0^+}(\gamma_0(s)/s^2) < +\infty \) and \( \lim \sup_{s \to 0^+}(\kappa_1(s)/\pi_1(s)) < +\infty. \)

Remark 4. Assumption 1 implies that \( z \)-subsystem is characterized by finite-time input-to-state stability (FTISS). The inequalities in Assumption 3 are the small-gain conditions.

Remark 5. The following discussions in order demonstrate that Assumption 2 encompasses and generalizes the existing results.

(i) When \( z = 0 \) and \( \theta = 0 \), Assumption 2 includes the growth condition in [11]

\[
|\phi_i| \leq \gamma_1(x_1, \ldots, x_i) \sum_{j=1}^{i} |x_j|^{(r_j+\gamma)/r_j}, \tag{10}
\]

as its special case (i.e., \( \mu_j = 0 \) in (8)), as well as the growth condition in [10]

\[
|\phi_i| \leq \gamma_1(x_1, \ldots, x_i) \sum_{j=1}^{i} |x_j|^{(r_j+\gamma)/r_j}, \quad \tau \in \left(-\frac{1}{\sum_{j=1}^{n} P_0 \cdots P_{j-1}}, 0\right) \tag{11}
\]

as a special case (i.e., \( \mu_j = 0 \) and \( \kappa_2 = c \) is a constant).

From \( v \in (0,1/(\sum_{j=1}^{n} P_0 \cdots P_{j-1})) \), \( r_1 = 1, r_{i+1} = (r_i-\nu)/P_i \), it is easy to see that \( 0 < (r_i-\nu)/r_i < 1/(\sum_{j=1}^{n} P_0 \cdots P_{j-1}) \), which implies that the power in condition (8) defined by \( (r_i-\nu)/r_i + \mu_j \) can take any value in an interval \((0, +\infty)\), while, for \([10,11]\), the powers only take values in \((0,1/(\sum_{j=1}^{n} P_0 \cdots P_{j-1})\).

(ii) When \( z = 0 \) and \( \theta = 0 \), Assumption 1 in [14]:

\[
|\phi_i| \leq \beta_i |x_{i+1}|^p + \theta \kappa_2(x_1, \ldots, x_i) \sum_{j=1}^{i} |x_j|^{(r_j+\gamma)/r_j+\mu_j}, \tag{12}
\]

\( \omega \in \left(-\frac{1}{\sum_{j=1}^{n} P_0 \cdots P_{j-1}}, 0\right), \mu_j \in [0, +\infty), \beta_i \in [0,1). \)

(iii) When \( p_i = 1 \) for all \( i = 1, \ldots, n \), system (2) becomes

\[
\dot{x}_i = x_{i+1} + \phi_i(x_1, \ldots, x_i, z, d), \quad i = 1, \ldots, n-1, \tag{13}
\]

\( \dot{x}_n = u + \phi_n(x_1, \ldots, x_n, z, d), \)

which is studied by [20, 21].

By the discussions, it is highlighted that this paper substantially extends the results of these papers; namely, for more general high-order nonlinear systems (2) with dynamic and parametric uncertainties, the finite-time control problem is to be solved under weaker condition (8).

3.2. Design of Adaptive Finite-Time Controller. In what follows, we denote \( \sigma = \max(\sigma(t), \sigma(t) - (\sigma(t)-\sigma(t-\tau)))/\tau) \), which is unknown because \( \theta \) is unknown, \( \sigma(t) \) is the estimate of \( \sigma(t) \) and \( \sigma(t-\tau) \) is the estimation error. Denote \( x = (x_1, \ldots, x_n)^T, \quad \Sigma_i = (x_1, \ldots, x_i)^T \in R^n, i = 1, \ldots, n. \) For simplicity, denote \( [x]^p = |x|^p \text{sgn}(x), \quad x \in R^n, \text{sgn}(\cdot) \) is sign function whose definition is in the notations explained in Section 2, \([x]^p \cdot x = |x|^p \text{sgn}(x), \quad [x]^p \) is obviously \( \mathcal{E}^{\frac{1}{2}} \) if \( p \geq 1 \).

To give the design of controller, we first define the parameters \( \beta_0, \beta_1, \ldots, \beta_n \) recursively as

\[
\beta_0 = 1, \tag{14}
\]

\[
(\beta_i P_i + 1) r_{i+1} = (\beta_{i-1} P_{i-1} + 1) r_i > 0, \quad i = 1, \ldots, n. \]

From (14), it follows that

\[
1 < r_i \beta_{i-1} P_{i-1} < r_{i+1} \beta_i P_i, \quad 1 < \beta_i P_i < \ldots < \beta_n P_n. \tag{15}
\]

\( \beta_i > 1, \quad i = 1, \ldots, n. \)
Besides, from (15), it leads to
\[ r_i p_{i-1} + r_{i-1} \beta_{i-2} \beta_{i-11} = r_{i+1} p_i + r_i \beta_{i+1} \beta_{i-11}; \]
then
\[ r_i p_{i-1} + r_{i-1} \beta_{i-2} \beta_{i-11} = r_2 p_1 + r_1 \beta_0 p_0 = 2 - \nu, \]
(16)
\[ i = 1, \ldots, n. \]

Secondly, we introduce the following coordinate transformation:
\[ w_k (t) = \left( x_k (t) \right)^{\beta_k/p_k-1}, \]
\[ \alpha_k (t) = \left( x_k (t) \right)^{\beta_k/p_k-1}, \]
\[ k = 1, \ldots, n, \]
\[ \alpha_k (t) = \left( x_k (t) \right)^{\beta_k/p_k-1}, \]
\[ k = 1, \ldots, n, \]
\[ u (t) = \alpha_k (x (t), \sigma (t)), \]
(17)
\[ \beta_k (x_k (t), \sigma (t)) \]
\[ = -\left[ \omega_k (t)^{\left(n-\nu\right)/n} \right]^p_k \Phi_k (x_k (t), \sigma (t)), \]
\[ k = 1, \ldots, n, \]
where \( \Phi_k, k = 1, \ldots, n, \) are \( \mathcal{C}^1 \) positive functions to be specified later. For the sake of consistency, we let \( \Phi_0 = 0 \) and \( \nu_0 = 0. \)

Finally, to obtain the detailed expression of \( u, \) we determine \( \Phi_1, \ldots, \Phi_n \) by induction.

**Initial Step.** Consider
\[ \dot{x}_1 = x_1^p_1 + \phi_1 (x_1, z, d). \]
(18)

Take \( V_1 (x_1) = x_1^2/2; \) then \( V_1 \) is \( \mathcal{C}^1 \) positive-definite and \( \dot{V}_1 (x_1) = x_1 (x_1^{p_1} - v_1^{p_1}) + x_1 v_1^{p_1} + x_1 \phi_1. \) By (15), (17), Lemmas 4 and 6, Assumption 2, and \( \sigma \geq \theta, \)
\[ x_1 \phi_1 \leq \left| x_1 \right| \kappa_1 \left( \| z \| \right) + x_1 \cdot y_1 \left| \left| x_2 \right| \right|^{p_1} \]
\[ + \left| x_1 \right| \kappa_{12} \left( x_1 \right) \left| x_1 \right|^{-\nu_1/p_1} \theta \]
\[ \leq \left| w_1 \right|^{2-\nu} + \kappa_{1}^{(2-\nu)/(1-\nu)} + \theta \kappa_{12} \left( x_1 \right) \left| x_1 \right|^{2-\nu} \]
\[ + \gamma_1 \left| w_1 \right| \left| w_2 \right|^{1/p_1} + \gamma_1 \left| w_1 \right|^{2-\nu} \Phi_1 \]
\[ \leq \left| w_1 \right|^{2-\nu} \left( 1 + \sigma b_1 (x_1) \right) + \kappa_{1}^{(2-\nu)/(1-\nu)} + \gamma_1 \left| w_1 \right| \left| w_2 \right|^{1/p_1} + \gamma_1 \left| w_1 \right| \left| w_1 \right|^{2-\nu} \Phi_1, \]
(19)
where \( b_1 (x_1) = \kappa_{12} (x_1) = |x_1|^{p_1} \kappa_{12} (x_1) \) is a \( \mathcal{C} \) nonnegative function and \( b_1 (0) = 0. \) By Lemma 8, we choose a \( \mathcal{C}^1 \) positive function \( \Phi_1 (x_1, \sigma) \) dominating the following function:
\[ \Phi_1^0 = \frac{1}{1 - y_1} \left( 1 + L_1 + \frac{\gamma_0 \left( |x_1| \right)}{|x_1|^{2-\nu}} \left( \mu (x_1) + 1 \right) \right. \]
\[ + \sigma b_1 (x_1), \quad L_1 > 0. \]
(20)
where \( \mu (x_1) \) is a \( \mathcal{C}^0 \) function to be determined. Because of \( \nu > 0 \) and Assumption 3, \( \Phi_1 \) is a continuous function. After some manipulations, we have
\[ \dot{V}_1 (x_1) \leq -y_0 \left( |x_1| \right) \left( \mu (x_1) + 1 \right) - L_1 |w_1|^{2-\nu} \]
\[ + \kappa_{1}^{(2-\nu)/(1-\nu)} + |w_1| \left| x_1 \right|^{p_1 - v_1^{p_1}} \]
\[ + \left( \sigma + \eta_1 \right) \phi_1 + \gamma_1 |w_1| |w_2|^{1/p_1}, \]
(21)
where \( \eta_1 = 0, \) \( \phi_1 (x_1) = |x_1|^{2-\nu} b_1 (x_1) \geq 0, \) and \( \sigma \) will be determined later.

**Inductive Step.** We give this step by the following proposition.

**Proposition 6.** Suppose at Step \( i - 1, \) for system
\[ x_j = x_{j+1}^p + \phi_j \left( x_1, \ldots, x_j, z, d \right), \quad j = 1, \ldots, i - 1, \]
(22)
there is a \( \mathcal{C}^1 \) function \( V_{i-1} (x_{i-1}, \sigma) \) which is positive-definite with respect to \( x_1, \ldots, x_{i-1} \) and \( \mathcal{C}^1 \) positive function \( \Phi_{i-1} \) such that
\[ \dot{V}_{i-1} \left( x_{i-1}, \sigma \right) \leq -y_i \left( |x_i| \right) \left( \mu (x_1) + 1 \right) - L_{i-1} Q_{i-1}^{2-\nu} \]
\[ + |w_{i-1}| \left| w_1 \right|^{2-\nu} \left| x_1 \right|^{p_1 - v_1^{p_1}} \]
\[ + \left( \sigma + \eta_{i-1} \right) \phi_{i-1} + \gamma_{i-1} |w_{i-1}| |w_2|^{1/p_1}, \]
\[ + \sum_{j=1}^{i-1} \kappa_{i}^{(2-\nu)/(1-\nu)} + \sum_{j=1}^{i-1} \kappa_{i}^{(2-\nu)/(1-\nu)} \phi_{j-1} \sigma_{j-1}. \]
(23)
where \( Q_{i-1} \left( x_{i-1}, \sigma \right) = \left( |w_{i-1}|^{2-\nu} + |w_2|^{(2-\nu)/(1-\nu)} \right)^{1/(2-\nu)}, \)
\[ \eta_{i-1} \left( x_{i-1}, \sigma \right) = \frac{i-1}{\partial W_{i-1}} \partial \sigma, \]
(24)
\[ \phi_{i-1} (x_{i-1}, \sigma) \geq 0 \text{ is } \mathcal{C}^1. \]
Then, for system
\[ x_j = x_{j+1}^p + \phi_j \left( x_1, \ldots, x_j, z, d \right), \quad j = 1, \ldots, i, \]
(26)
the $i$th function $V_i(x_i, \hat{\sigma}) = V_{i-1}(x_{i-1}, \hat{\sigma}) + W_i(x_i, \hat{\sigma})$ is $C^3$ and positive-definite with respect to $x_i$, $\ldots$, $x_n$, and one can find a $C^3$ positive function $\Phi_i$ (see (A.14) in the Appendix) such that

$$
\begin{align*}
\dot{V}_i(x_i, \hat{\sigma}) & \leq -\gamma_0 \left( \int_{\chi} \right) \left( \mu(x_i) + 1 \right) - L_n Q_n^{-\gamma} + \\
& \quad + \sum_{j=1}^{i-1} \left( \int_{\chi} \right) \left( \mu(x_j) + 1 \right) - L_n Q_n^{-\gamma} + \\
& \quad + \sum_{j=i+1}^{n} \left( \int_{\chi} \right) \left( \mu(x_j) + 1 \right) - L_n Q_n^{-\gamma} + \\
& \quad \dot{\sigma} + \eta_n \Phi_i - \eta_n \hat{\sigma} + \\
& \quad \sum_{j=1}^{i-1} \kappa_i^{(\gamma+1)/2(2-\gamma)} b_i(x_i, \hat{\sigma}) + \gamma_i \left( \int_{\chi} \right) |w_{i+1}|^{1/\beta},
\end{align*}
$$
(27)

where $L_i = (n-1)/n L_{i-1} > 0$, $\phi_i(\chi, \hat{\sigma}) = \phi_{i-1}(\chi_{i-1}, \hat{\sigma}) + |w_{i+1}|^{(\gamma+1)/2\beta_i} b_i(\chi_i, \hat{\sigma})$ (see (A.12) in the Appendix) is continuous, and

$$
W_i(\chi_i, \hat{\sigma}) = \int_{\chi_{i-1}} \left( \int_{\chi} \right) \left( \mu(x_i) + 1 \right) - L_n Q_n^{-\gamma} + 0 \leq i \leq n.
$$
(28)

Proof. See the Appendix.

At Step $n$, (27) holds with $w_{n+1} = 0$. Hence, by choosing $V_n(x, \hat{\sigma}) = V_{n-1}(\chi_{n-1}, \hat{\sigma}) + W_n(x, \hat{\sigma})$, we can get

$$
\dot{u}^n = v^n = -\left( \int_{\chi} \right) \left( \mu(x_i) + 1 \right) - L_n Q_n^{-\gamma} + \Phi_n(x, \hat{\sigma})
$$
(29)

with an appropriate function $\Phi_n$ and

$$
\dot{\hat{\sigma}} = -\Phi_n = \Phi_n
$$
(30)

such that

$$
\begin{align*}
\dot{V}_n(x, \hat{\sigma}) & \leq -\gamma_0 \left( \int_{\chi} \right) \left( \mu(x_1) + 1 \right) - L_n Q_n^{-\gamma} + \Phi_n + \\
& \quad + \sum_{j=1}^{n-1} \left( \int_{\chi} \right) \left( \mu(x_j) + 1 \right) - L_n Q_n^{-\gamma} + \\
& \quad + \sum_{j=1}^{n} \left( \int_{\chi} \right) \left( \mu(x_j) + 1 \right) - L_n Q_n^{-\gamma} + \Phi_n,
\end{align*}
$$
(31)

From Assumption 3, $\kappa_1(s)$ is $C^3$, $(2-\gamma)/(2-\gamma-\beta_i)$, $1 + 1/(1- \gamma) > 2$, and, for $i = 2, \ldots, n$, $(2-\gamma)/(2-\gamma-\beta_i, \beta_{i+1}) > 2$, $(2-\gamma)/(2-\gamma-\beta_i) > 2 \alpha_0$, it follows that

$$
\limsup_{s \to \infty} \tilde{V}_i(s) < +\infty.
$$
(33)

The proof is divided into two parts.

**Step 1.** We show that the trajectories $(z(t), x(t), \hat{\sigma}(t))$ of the closed-loop system (2), (29), and (30) are bounded.

Take a Lyapunov function as

$$
V_c(x, z, \hat{\sigma}) = V_n(x, \hat{\sigma}) + \frac{\gamma^3}{2} \gamma^3 + \\
\int_{0}^{V_c(z)} \rho(s) \, ds,
$$
(34)

where $\rho : R^+ \to R^+$ is a continuous nondecreasing function with $\rho(0) > 0$. It follows from (7) that when $\|z\| > (V_0(\|z\|)/c_0^2)^{1/\alpha_0}$, $-c_0 e^\rho(V_0(z)V_0^{\alpha_0}(z) + \rho(V_0(z))y_0(\|z\|) < 0; \text{ when } \|z\| \leq (V_0(\|z\|)/c_0^2)^{1/\alpha_0}$, $\rho(V_0(z))y_0(\|z\|) \leq \rho(\pi_2(\pi_1^{-1}((V_0(\|z\|)/c_0^2)^{1/\alpha_0}))y_0(\|z\|)$. Then we observe from (30), (31), and (34) that

$$
\dot{V}_c
$$

$$
\leq -\gamma_0 \left( \int_{\chi} \right) \left( \mu(x_1) + 1 \right) - L_n Q_n^{-\gamma} + \Phi_n + \\
\quad \quad \quad - c_0 \rho(V_0(z))V_0^{\alpha_0}(z) + \rho(V_0(z))y_0(\|z\|)
$$
(35)

$$
\leq -\gamma_0 \left( \int_{\chi} \right) \left( \mu(x_1) + 1 \right) - L_n Q_n^{-\gamma} + \Phi_n + \\
\quad \quad \quad - c_0 (1-\epsilon) \rho(V_0(z))V_0^{\alpha_0}
$$
(35)

$$
\quad + \rho \left( \frac{\pi_2 \left( \frac{1}{\epsilon_0} \left( \frac{\gamma_0 (\|z\|)}{c_0^2} \right)^{1/\alpha_0} \right) \right) y_0(\|z\|),
$$

0 < \epsilon < 1.

According to (33) and Lemma 8, we can find a desired function $\rho$ and a $C^3$ function $\mu$ such that

$$
\mu(x_1) + 1 \geq \rho \left( \frac{1}{\epsilon_0} \left( \frac{\gamma_0 (\|z\|)}{c_0^2} \right)^{1/\alpha_0} \right).
$$
(36)

Substituting (36) into (35) leads to

$$
\dot{V}_c \leq -L_n Q_n^{-\gamma} - \frac{\gamma_0 (1-\epsilon)}{2} \rho(V_0(z))V_0^{\alpha_0}(z) \leq 0,
$$
(37)

which implies the boundedness of $z(t), x(t)$, and $\hat{\sigma}(t)$.

**Step 2.** We show that the trajectories $(z(t), x(t))$ are finite-time convergent.

Consider $\tilde{V}_c(x, z, \hat{\sigma}) = V_n(x, \hat{\sigma}) + V_c(z)$, which is positive-definite with respect to $(x, z)$. Then

$$
\dot{V}_c \leq -L_n Q_n^{-\gamma} - \frac{\gamma_0 (1-\epsilon)}{2} \rho(V_0(z))V_0^{\alpha_0}(z) + c\Phi_n
$$
(38)
At first, we consider local finite-time convergence in a small neighborhood around \((z, x) = (0, 0)\). It is easy to see that, locally around \(z = 0\), \(\lim_{t\to0} (V_n^{\alpha}(z)/\rho(V_n(0)))V_n^{\alpha}(z) = 1/\rho(0) < \infty\), \(\lim_{t\to0} (V_n(z)/\rho(V_n(z))) = \rho(0) < \infty\), and then \(\lim_{t\to0} (V_n^{\alpha}(z)/\rho(V_n(0)))V_n^{\alpha}(z) = 1\), which implies that when \(\|z\|\) is small enough, there is a constant \(C_1\) such that \(V_n(\alpha) \leq (C_1(1-\epsilon)/C_1)\rho(V_n(z))V_n^{\alpha}(z)\). Thus, locally around \((z, x) = (0, 0)\), by (38), one has

\[
\dot{V}_c = -L_nQ_n^{2}\gamma - C_1V_n^{\alpha} + \alpha \Phi_n.
\] (39)

Note that

\[
\Phi_n(x, \tilde{\sigma}) = \sum_{j=1}^{n} w_j(x_j^{(2-\gamma)/\gamma}, \beta_j, \gamma, \gamma_j, p_j) = \frac{1}{2} - \frac{1}{L_n}(\sigma + \mathcal{C} \phi_n),
\] (40)

where \(\Phi_n = \sum_{j=1}^{n} b_j\) is continuous with \(\Phi_n(0, \tilde{\sigma}) = 0\) because \(b_j(0, \tilde{\sigma}) = 0\). Substituting (40) into (39), it is easy to obtain

\[
\dot{V}_c = -L_nQ_n^{2}\gamma - C_1V_n^{\alpha} - L_nQ_n^{2}\gamma \left(1 - \frac{1}{2} - \frac{1}{L_n}(\sigma + \mathcal{C} \phi_n)\right),
\] (41)

where \(\mathcal{C}\) is a positive constant. By (15) and Lemma 9, then \(V_n = \sum_{j=1}^{n} \int_{s_{j-1}}^{s_j} (s|s_j^{(2-\gamma)/\gamma}, \beta_j, \gamma, \gamma_j, p_j) ds = \sum_{j=1}^{n} 2|w_j(x_j^{(2-\gamma)/\gamma}, \beta_j, \gamma, \gamma_j, p_j)|\), which together with (24) and Lemma 5 mean that \(V_n^{(2-\gamma)/\gamma} \leq 2Q_n^{2}\gamma\). Hence, (41) becomes

\[
\dot{V}_c \leq -\frac{L_n}{4}V_n^{(2-\gamma)/2} - C_1V_n^{\alpha} - L_nQ_n^{2}\gamma \left(1 - \frac{1}{2} - \frac{1}{L_n}(\sigma + \mathcal{C} \phi_n)\right),
\] (42)

where \(\dot{V}(x, \tilde{\sigma}) = (1/L_n)(\sigma + \mathcal{C} \phi_n)\) is continuous with \(\dot{V}(0, \tilde{\sigma}) = 0\). Because \(\dot{V}\) is continuous and \(\dot{V}(0, \tilde{\sigma}) = 0\), it is obvious that there is a constant \(0 < \varphi \leq 1\) such that \(\dot{V} \leq 1/2\) if \(\dot{V}_c \leq \varphi\). By Lemma 5 and \(\dot{V}_c \leq \varphi \leq 1\), choosing \(c_\varphi = \min\{L_n/4, C_1/2\}\), \(\alpha = \max\{(2-\gamma)/2, \alpha_0\}\), we have \((L_n/4)V_n^{(2-\gamma)/\gamma} + (C_1/2)\mathcal{C}_n^{\alpha} \geq (C_1(1-\epsilon)/C_1)\rho(V_n(z))V_n^{\alpha}(z)\). Thus, in a neighborhood \(\Omega = \{(z, x, \tilde{\sigma}) : \dot{V}_c \leq \varphi\}\), (42) becomes

\[
\dot{V}_c \leq -c_\varphi V_n^{\alpha},
\] (43)

which implies the local finite-time convergence in \(\Omega\) by Lemma 3.

We consider the global finite-time convergence. The finite-time convergence in \(\Omega\) has been achieved. We now study the situation outside \(\Omega\). When the initial condition is outside \(\Omega\), one has \(\dot{V}_c \geq \varphi\). It is easy to see that there is a constant \(c_\varphi\) such that \((L_n/2)V_n^{(2-\gamma)/\gamma} + (C_1/2)\mathcal{C}_n^{\alpha} \geq C_2\varphi^\alpha\). Let \(T_1\) be the first time that \((z(t), x(t), \tilde{\sigma}(t))\) intersects the boundary of \(\Omega\). Using (37) and the above-mentioned argument, for any \(t \in [0, T_1]\), one has

\[
\dot{V}_c(z(0), x(0), \tilde{\sigma}(0)) \geq \dot{V}_c(z(r), x(r), \tilde{\sigma}(r)) - V_c(z(r), x(r), \tilde{\sigma}(r))
\] (44)

\[
= \int_0^r -\dot{V}_c(z(s), x(s), \tilde{\sigma}(s)) ds
\]

\[
= \int_0^r (-\dot{V}_c + \alpha \Phi_n) ds
\]

\[
\geq \int_0^r \left(\frac{L_n}{2}V_n^{(2-\gamma)/2} + C_1V_n^{\alpha}\right) ds \geq C_2\varphi^\alpha.
\]

If \((z(t), x(t), \tilde{\sigma}(t))\) do not enter in finite time, then \(T_1 = \infty\). When \(t \to T_1\), \(C_2\varphi^\alpha \to \infty\), while \(V_n(z(0), x(0), \tilde{\sigma}(0))\) is a finite constant, then (44) leads to a contradiction. So \((z(t), x(t), \tilde{\sigma}(t))\) will reach \(\Omega\) in finite time.

**Remark 8.** We estimate the settling time. In practice, although we do not know the real value of \(\sigma\) (or \(\tilde{\sigma}\)), we always have its range. Without loss of generality, we assume that \(0 < \sigma < \Sigma\), where \(\Sigma > 0\) is a constant. We consider two cases.

**Case 1** \((z(0), x(0), \tilde{\sigma}(0)) \in \Omega\). By (43), we have

\[
T \leq \frac{\dot{V}_c(z(0), x(0), \tilde{\sigma}(0))^{1-\max((2-\gamma)/2, \alpha_0)}}{c_\varphi \left(1 - \max\{(2-\gamma)/2, \alpha_0\}\right)}.
\] (45)

**Case 2** \((z(0), x(0), \tilde{\sigma}(0)) \notin \Omega\). Before the state reaches \(\Omega\), from (34) and (37), it follows that \(\tilde{\sigma}^2 \leq 2V_n(z(0), x(0), \tilde{\sigma}(0)) \leq 2\tilde{\sigma}(z(0), x(0), \tilde{\sigma}(0)) + (|\tilde{\sigma}(0)| + \Sigma)^2\), then \(V_n(z(0), x(0), \tilde{\sigma}(0)) \leq \dot{V}_c(z(0), x(0), \tilde{\sigma}(0)) + (1/2)\tilde{\sigma}(0)^2 \leq 2V_n(z(0), x(0), \tilde{\sigma}(0)) + (1/2)(|\tilde{\sigma}(0)| + \Sigma)^2\). Therefore, by (44), \((x, \tilde{\sigma})\) will reach \(\Omega\) within \(T_1\):

\[
T_1 \leq \frac{\dot{V}_c(z(0), x(0), \tilde{\sigma}(0)) + (|\tilde{\sigma}(0)| + \Sigma)^2}{C_2\tilde{\sigma}^{\max((2-\gamma)/2, \alpha_0)}}.
\] (46)

After \(T_1\), it will stay in \(\Omega\), and it will take

\[
T_2 \leq \frac{e^{1-\max((2-\gamma)/2, \alpha_0)}}{c_\varphi \left(1 - \max\{(2-\gamma)/2, \alpha_0\}\right)}.
\] (47)

to reach the origin. Therefore, in this case, the settling time can be estimated as \(T_1 + T_2\).

**Remark 9.** Compared with [14], due to the appearance of dynamic and parametric uncertainties, and the weaker condition on nonlinear functions, the main difficulty in this paper is how to skillfully combine Lyapunov function, sign function, backstepping, and FTISS approaches to give the design and rigorous analysis of finite-time controller.
5. A Simulation Example

Consider a simple system

$$\begin{align*}
\dot{z} &= -z^{1/3} + x_1, \\
\dot{x}_1 &= x_2^{9/7} + \theta \sin x_1 \cos z, \\
\dot{x}_2 &= u - z^2 \cos x_2,
\end{align*}$$

where $\theta$ is an unknown parameter and $z$ is the unmeasurable dynamic uncertainty.

Choose $\nu = 2/11$; then $r_1 = 1, r_2 = 7/11$. By (48), we have $\gamma_1 = \gamma_2 = 0, |\phi_1| = |\theta \sin x_1 \cos z| \leq \theta|\sin x_1| \leq \theta|x_1|^{9/11}, |\phi_2| = |z^2 \cos x_2| \leq |z|^2, \kappa_i(|z|) = |z|^2, \kappa_{12}(x_1) = 1, \kappa_{12}(x_1, x_2) = 0, \text{ and } \mu_{11} = \mu_{21} = \mu_{22} = 0$. Choose $V_0(z) = z^2$; then $\dot{V}_0 \leq -(1/2)z^{4/3} + (1/2)x_1^4 \leq -(1/2)V_0^{2/3} + (1/2)x_1^4$; $z$-subsystem is obviously finite-time input-to-state stable with $\gamma_0(x_1) = (1/2)x_1^4$. Hence, Assumptions 1–3 hold.

Take $\beta_0 = 1$; then $r_2 \rho_1 + r_1 \beta_0 \rho_0 = 2 - \nu = 20/11$. Following the design procedure in Section 3, one can obtain the adaptive finite-time controller

$$\begin{align*}
u &= -0.2 \left( x_2^{15/7} + 2x_1^{15/11} \Phi_1^{5/3} \right)^{1/3} \\
\cdot \left( 4 + 2\Phi_1 + 3.2\Phi_2 + 2\sigma x_1^{4/9} \right), \quad \Phi_1 = 4\sigma^2 x_1^4 + 2,
\end{align*}$$

Choosing the initial conditions $z(0) = 0.1, x_1(0) = 0.1, x_2(0) = -0.2$, and $\sigma(0) = 0$, we estimate the settling time. Since $\Omega = \{(z, x, \sigma) : \nabla_e \leq 0.2\}$, $\rho(s) = 8(s^{14/9} + s^{3} + s^{41/78})$, one has $\nabla_e(z(0), x(0), \sigma(0)) = 0.0338$, which is in $\Omega$. According to Case 1 in Remark 8, by $\max(2 - \nu)/2, \alpha \approx 10/11, c_s = \min(0.25, 0.1813) = 0.1813$, one has $T \leq 44.5923$.

With $\theta = 1$, Figure 1 shows the trajectories of $z, x_1, x_2, u, \sigma$, and one can see that $z, x_1, x_2$ reach the origin within $T$. 

![Figure 1: Trajectories of $z, x_1, x_2, u, \sigma$.](image-url)
6. Conclusions

By characterizing the dynamic uncertainty with FTISS, under the weaker assumption on nonlinear functions, the problem of adaptive finite-time stabilization for more general high-order nonlinear systems with dynamic and parametric uncertainties is solved.

Some interesting problems are still remaining: (1) For system (2) with possibly nonvanishing disturbances and more general dynamic uncertainty, can a finite-time convergent controller be given? (2) How can we construct output feedback to stabilize system (2) in finite time? (3) In recent years, some results on stochastic nonlinear systems with SISS/SISS dynamic uncertainty have been obtained, for example, [25–36] and the references therein, but these papers only consider the global asymptotic stabilization. An important problem is whether finite-time stabilization can be obtained for stochastic nonlinear systems with dynamic and parametric uncertainties.

Appendix

Proof of Proposition 6. We first prove that $V_j$ is $C^4$. From (15) and $\Phi_{j-1}(\bar{x}_{j-1}, \tilde{\sigma})$ being positive and $C^4$ and $[v_{j-1}]^{\beta_{j-1}r_{j-1}}$ being $C^3$, then $W_j$ is $C^3$, so is $V_j$.

Secondly, we prove that $V_j$ is positive-definite with respect to $x_1, \ldots, x_i$. When $x_i > v_{j-1}$, by (15), (28), and Lemmas 7 and 9, one has

$$W_j \geq 2^{1-\beta_{j-1}r_{j-1}} \int_{v_{j-1}}^{x_i} (s - v_{j-1})^{\beta_{j-1}r_{j-1}} \, ds \geq 2^{1-\beta_{j-1}r_{j-1}} \left[ x_i - v_{j-1} \right]^{\beta_{j-1}r_{j-1}+1} \geq 0. \quad \text{(A.1)}$$

When $x_i \leq v_{j-1}$, it can be shown that (A.1) also holds in a similar way.

From (A.1), the definition of $V_j$, and the positive definiteness of $V_{j-1}$ with respect to $x_1, \ldots, x_i$, it is easy to see that $V_j \geq 0$ and, for fixed $\tilde{\sigma}$, $V_j(\bar{x}, \tilde{\sigma}) = 0$ if and only if $\bar{x} = 0$, which implies that $V_j$ is positive-definite with respect to $x_1, \ldots, x_i$.

Finally, from the definition of $V_j$ and (23), it follows that

$$\dot{V}_j(\bar{x}_i, \tilde{\sigma}) \leq -y_0 \left( |x_1| \right) \mu(x_1) + L_{j-1}Q_{j-1}^{2-v}$$

$$+ \sum_{j=1}^{i-1} k_1 \left[ v_{j-1} - v_{j-1} \right]^{2-v} \sum_{j=2}^{i-1} k_1 \left[ v_{j-1} - v_{j-1} \right]^{2-v}$$

$$+ \left( |w_{j-1}| \left[ v_{j-1} - v_{j-1} \right] + y_{j-1} |w_{j-1}| \right)^{\beta_{j-1}} \quad \text{(A.2)}$$

We estimate the last three terms on the right-hand side of (A.2).

(i) By (15), (16), (17), (24), and Lemmas 4 and 9, one has

$$|w_{j-1}| \left[ x_j^{p_{j-1}} - v_{j-1}^{p_{j-1}} \right] + y_{j-1} |w_{j-1}| \left| w_{j-1} \right|^{\beta_{j-1}} \leq (2 + y_{j-1}) |w_{j-1}| \left| w_{j-1} \right|^{\beta_{j-1}} \quad \text{(A.3)}$$

where $l_i > 0$ is a constant dependent on $v, L_{j-1}, y_{j-1}, p_1, \ldots, p_i$.

(ii) For $j \leq i-1$, from (15), (17), (24), and Lemmas 4, 8, and 10, the following holds:

$$\left| \frac{\partial [v_{j-1}]^{\beta_{j-1}r_{j-1}}}{\partial x_j} \right| \leq |w_{j-1}| \left[ r_{j-1} + r_{j-1} \right] g(\bar{x}_{j-1}, \tilde{\sigma})$$

$$+ \sum_{j=1}^{i-2} \left( \prod_{k=j+1}^{i-1} |w_j|^{r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}} \right) \cdot \left( \prod_{j=1}^{i-1} |w_j|^{r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}r_{j-1}} \right)$$

$$\cdot g(\bar{x}_{j-1}, \tilde{\sigma}) \leq |w_{j-1}| \left[ r_{j-1} + r_{j-1} \right] g(\bar{x}_{j-1}, \tilde{\sigma}) \quad \text{(A.4)}$$

where $g(\bar{x}_{j-1}, \tilde{\sigma})$ is a general $C^4$ nonnegative function that may be implicitly changed in various places and $h_{j-1}(\bar{x}_{j-1}, \tilde{\sigma})$ is a $C^1$ nonnegative function. From (15), (17), (28), (A.4), and Lemma 9, it follows that

$$\frac{\partial W_j}{\partial x_j} \leq |x_i - v_{j-1}| \left| \frac{\partial [v_{j-1}]^{\beta_{j-1}r_{j-1}}}{\partial x_j} \right| \quad \text{(A.5)}$$

$$\leq 2 |w_j|^{1/\beta_{j-1}r_{j-1}} Q_{i-1}^{\beta_{j-1}r_{j-1}r_{j-1}} h_{j-1}, \quad j \leq i-1.$$
For $i = 1, \ldots, n$, by Assumption 2, one has

$$\left| \phi_i(x_1, \ldots, x_i, z, d) \right| \leq \kappa_1(\|z\|) + \gamma_i \left| x_{i+1} \right|^{p_i}$$

(A.6)

where $\kappa_2 = \sum_{j=1}^{i} |x_j|^{p_j}/\kappa_2$ is a $C$ nonnegative function vanishing at the origin. For $j < i$, by (15), (17), (24), (A.6), and Lemmas 6 and 9,

$$x_{j+1}^{p_j} + \phi_j \leq \left| x_{j+1}^{p_j} - v_j^{p_j} \right| + \left| v_j^{p_j} \right| + \left| \phi_j \right| \leq 2 \left| w_{j+1} \right|^{1/\beta_j}$$

(A.7)

From (16), (A.5), (A.7), Lemma 4, and $(2 - \nu)/\nu \leq (2 - \nu)/r_n$, where $\psi_{j+1}(\xi_{i+1}, \sigma)$ and $b_{j+1}(\xi_{i+1}, \sigma)$ are $C$ nonnegative functions with $b_j(0, \sigma) = 0$. It follows from (15), (16), (17), (24), (28), (A.6), and Lemmas 4 and 6 that

$$\frac{\partial W_i}{\partial x_j} (x_{i+1}^{p_j} + \phi_j) \leq \left| w_{i+1} \right| \left| x_{i+1}^{p_i} - v_i^{p_i} \right| + \left| x_{i+1}^{p_i} - v_i^{p_i} \right| + \left| \phi_i \right| \left( \kappa_1 + \gamma_i \right) \left( 2 + \Phi_i Q_{j+1}^{\nu-\gamma} \right)$$

(A.8)

where $\psi_{j+1}(\xi_{i+1}, \sigma)$ and $b_{j+1}(\xi_{i+1}, \sigma)$ are $C$ nonnegative functions with $b_{j+1}(0, \sigma) = 0$. Combining (A.8) with (A.9) leads to

$$\sum_{j=1}^{i} \frac{\partial W_i}{\partial x_j} (x_{i+1}^{p_j} + \phi_j) \leq \left| w_{i+1} \right| \left| x_{i+1}^{p_i} - v_i^{p_i} \right| + \left| x_{i+1}^{p_i} - v_i^{p_i} \right| + \left| \phi_i \right| \left( \kappa_1 + \gamma_i \right) \left( 2 + \Phi_i Q_{j+1}^{\nu-\gamma} \right)$$

(A.9)

where $\psi_{j+1}(\xi_{i+1}, \sigma)$ and $b_{j+1}(\xi_{i+1}, \sigma)$ are $C$ nonnegative functions with $b_{j+1}(0, \sigma) = 0$. Combining (A.8) with (A.9) leads to

(iii) For the last term of (A.2), from (17), (28), and Lemma 8, there is a $C$ nonnegative function $\delta_j(\xi_j, \sigma)$ such that $|\partial \psi_j(v_j^{p_j})/\partial \sigma| \leq \delta_j(\xi_j, \sigma)$ ($j \leq i - 1$). By (24), (25), and Lemma 4,

$$\left( \sigma + \eta_{i-1} \right) \psi_{i+1} - \eta_{i-1} \hat{\sigma} + \frac{\partial W_i}{\partial \sigma} \hat{\sigma} = \left( \sigma + \eta_{i-1} + \frac{\partial W_i}{\partial \sigma} \right) \left( \psi_i - \left| w_i \right|^{2/\nu} r_i \beta_i \phi_i \right) - \eta_{i-1} \hat{\sigma}$$
where $\psi_i^0(\bar{x}_i, \bar{\sigma})$ is $\mathcal{C}'$ nonnegative function;

$$\psi_i(\bar{x}_i, \bar{\sigma}) = \psi_{i-1}(\bar{x}_{i-1}, \bar{\sigma}) + \left| w_i \right|^{(2-\eta)/(2-\gamma)} b_i \left( \bar{\sigma} + \eta_i \right), \quad (A.12)$$

Substituting (A.3), (A.10), and (A.11) into (A.2), one has

$$V_i(\bar{x}_i, \bar{\sigma}) \leq \frac{L_{i-1} Q_i^{2-\eta}}{4n} + \left| w_i \right|^{(2-\eta)/(2-\gamma)} b_i \left( \bar{\sigma} + \eta_i \right), \quad (A.13)$$

Using Lemma 8, one can choose a $\mathcal{C}'$ positive function

$$\Phi_i \geq \frac{1}{1 - \gamma_i} \left( \frac{n-1}{n} L_{i-1} + l_i + \psi_i + \phi_i^0 + (1 + \bar{\sigma}^2) b_i \right), \quad (A.14)$$

such that

$$w_i v_i^p + \left| w_i \right|^{(2-\eta)/(2-\gamma)} b_i \left( l_i + \psi_i + \phi_i^0 + \bar{\sigma} b_i \right) + \gamma_i \left| w_i \right|^{(2-\eta)/(2-\gamma)} \Phi_i \leq -L_i \left| w_i \right|^{(2-\eta)/(2-\gamma)} b_i, \quad (A.15)$$

where $L_i = ((n - 1)/n) L_{i-1}$. Substituting (A.15) into (A.13) leads to (27).


Submit your manuscripts at http://www.hindawi.com