Research Article

Fractional-Order Two-Port Networks

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We introduce the concept of fractional-order two-port networks with particular focus on impedance and admittance parameters. We show how to transform a 2×2 impedance matrix with fractional-order impedance elements into an equivalent matrix with all elements represented by integer-order impedances; yet the matrix rose to a fractional-order power. Some examples are given.

1. Introduction

Two-port networks are widely used in linear circuit analysis and design [1, 2]. The system under consideration is represented by a describing matrix which relates its input and output variables (voltages and currents). Such a representation enables the treatment of the system as a black box where the internal details become irrelevant. It also offers an extremely efficient computational technique which can be used to model series, parallel, or cascade interconnects of several systems. Standard Network Analyzers can be configured to measure several types of two-port network parameters including impedance, admittance, transmission, and scattering parameters.

Consider, for example, the impedance matrix representation of a system in which case we have

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = (Z) \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad (1)$$

where $V_{1,2}$ ($I_{1,2}$) are the voltages (currents) at the input port and output port, respectively, as shown in Figure 1. All elements in the 2×2 impedance matrix are measured in Ω and if $Z_{11} = Z_{22}$ the network is known to be symmetrical while if $Z_{12} = Z_{21}$ it is known to be reciprocal. However, with the increasing use of fractional-order impedance models, particularly in representing supercapacitors [3, 4], energy storage devices [5], oscillators [6], filters [7], and new

electromagnetic charts [8], it is possible that the elements of (Z) are of fractional order. Consider the simple case of the grounded impedance Z , shown in Figure 2(a). Treated as a two-port network, this impedance is described by the impedance matrix

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} Z & Z \\ Z & Z \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}. \quad (2)$$

Let Z be a supercapacitor operating in its Warburg mode, where $Z = 1/Q\sqrt{s}$; $s = j\omega$. In this region of operation, the magnitude is proportional to $1/\sqrt{\omega}$, the phase angle is fixed at -45° , and Q is the pseudocapacitance of the device [9, 10]. As a two-port network, this device would be described as

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{Q} \begin{pmatrix} \frac{1}{\sqrt{s}} & \frac{1}{\sqrt{s}} \\ \frac{1}{\sqrt{s}} & \frac{1}{\sqrt{s}} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{Q} (Z_F) \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}. \quad (3)$$

Therefore all elements of the (Z_F) matrix are of fractional order. However, we can rewrite the above equation in the alternative form

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \frac{1}{Q} \begin{pmatrix} \frac{2}{s} & \frac{2}{s} \\ \frac{2}{s} & \frac{2}{s} \end{pmatrix}^{1/2} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \frac{1}{Q} (Z_I)^{1/2} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad (4)$$

FIGURE 1: Two-port network variables.

since

$$\begin{pmatrix} \frac{1}{\sqrt{s}} & \frac{1}{\sqrt{s}} \\ \frac{1}{\sqrt{s}} & \frac{1}{\sqrt{s}} \end{pmatrix}^2 = \begin{pmatrix} \frac{2}{s} & \frac{2}{s} \\ \frac{2}{s} & \frac{2}{s} \end{pmatrix}. \quad (5)$$

It is clear that the elements $Z_{11} \rightarrow Z_{22}$ of (Z_I) are all integer-order impedances, each representing a capacitor of $(1/2)$ Farad whereas the power of the matrix is fractional; that is, $\sqrt{(Z_I)} = (Z_F)$. In this paper we seek to generalize this procedure by obtaining the equivalent matrix (Z_I) and its fractional exponent such that $(Z_I)^n = (Z_F)$. The procedure is not restricted to the impedance matrix and can be applied

to any other type of two-port network parameters. The main advantage of this conversion is that an equivalent circuit of (Z_I) can be easily obtained with integer-order components. For example, if (Z_I) is reciprocal, then its equivalent circuit is that shown in Figure 2(b). However, it is not yet known how to use this equivalent circuit in association with the fractional exponent of the matrix to construct an overall equivalent model of the originally fractional-order two-port network.

2. Power of a Matrix

A matrix M^n for a nonnegative exponent n is defined as the matrix product of n copies of M . However, if n is a noninteger, then we need to revert to the Cayley-Hamilton theorem. In particular, if M is a 2×2 matrix and I is an identity matrix, then

$$M^n = \alpha_0 I + \alpha_1 M, \quad (6)$$

where

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 \lambda_1^n - \lambda_2 \lambda_1^n \\ \lambda_2^n - \lambda_1^n \end{pmatrix} \quad (7)$$

and $\lambda_{1,2}$ are nonrepeated eigenvalues. Hence, for a given impedance matrix Z_I , we have

$$(Z_I)^n = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_1^n (\lambda_2 - Z_{11}) + \lambda_2^n (Z_{11} - \lambda_1) & Z_{12} (\lambda_2^n - \lambda_1^n) \\ Z_{21} (\lambda_2^n - \lambda_1^n) & \lambda_1^n (\lambda_2 - Z_{22}) + \lambda_2^n (Z_{22} - \lambda_1) \end{pmatrix}, \quad (8)$$

where $\lambda_{1,2} = (1/2)(Z_{11} + Z_{22} \pm \sqrt{(Z_{11} - Z_{22})^2 + 4Z_{12}Z_{21}})$. For the case of repeated eigenvalues ($\lambda_1 = \lambda_2 = \lambda$) we obtain

$$(Z_I)^n = \frac{(Z_{11} + Z_{22})^{n-1}}{2^n} \cdot \begin{pmatrix} Z_{11} + Z_{22} + n(Z_{11} - Z_{22}) & 2nZ_{12} \\ 2nZ_{21} & Z_{11} + Z_{22} + n(Z_{11} - Z_{22}) \end{pmatrix}. \quad (9)$$

If the two-port network is symmetrical, that is, $Z_{11} = Z_{22}$, then for nonrepeated eigenvalues

$$(Z_I)^n = \frac{1}{2} \begin{pmatrix} x & \sqrt{\frac{Z_{12}}{Z_{21}}} y \\ \sqrt{\frac{Z_{21}}{Z_{12}}} y & x \end{pmatrix}, \quad (10)$$

where $x = (Z_{11} - \sqrt{Z_{12}Z_{21}})^n + (Z_{11} + \sqrt{Z_{12}Z_{21}})^n$ and $y = (Z_{11} + \sqrt{Z_{12}Z_{21}})^n - (Z_{11} - \sqrt{Z_{12}Z_{21}})^n$ while for repeated eigenvalues we obtain

$$(Z_I)^n = Z_{11}^{n-1} \begin{pmatrix} Z_{11} & nZ_{12} \\ nZ_{21} & Z_{11} \end{pmatrix}. \quad (11)$$

If, in addition to being symmetrical, the network is also reciprocal (i.e., $Z_{12} = Z_{21}$), we then obtain

$$(Z_I)^n = \frac{1}{2} \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \quad (12)$$

where $x = (Z_{11} + Z_{12})^n + (Z_{11} - Z_{12})^n$ and $y = (Z_{11} + Z_{12})^n - (Z_{11} - Z_{12})^n$. Noting the semigeneral case of a matrix M given by $(M) = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$, then it can be easily shown that

$$(M)^n = \frac{1}{2} \cdot \begin{pmatrix} (a - \sqrt{bc})^n + (a + \sqrt{bc})^n & \frac{(a - \sqrt{bc})^n - (a + \sqrt{bc})^n}{\sqrt{c/b}} \\ \frac{(a - \sqrt{bc})^n - (a + \sqrt{bc})^n}{\sqrt{b/c}} & (a - \sqrt{bc})^n + (a + \sqrt{bc})^n \end{pmatrix}. \quad (13)$$

In Section 3 a number of circuit applications are considered.

3. Applications

Case 1. Consider the case of floating impedance, as shown in Figure 3(a), and assume that this impedance represents a fractional-order inductor with impedance $Z = Ls^\alpha$; $s = j\omega$ and L is the pseudoinductance. This single impedance cannot

FIGURE 2: (a) Single grounded impedance as a two-port network and (b) general equivalent circuit from a reciprocal impedance matrix.

FIGURE 3: (a) Single floating impedance as a two-port network, (b) π -model of a transmission line, and (c) equivalent circuit of (Y_I) in (21).

be described by an impedance matrix but can be described by an admittance matrix in the form

$$(Y_F) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{Z} & \frac{-1}{Z} \\ \frac{-1}{Z} & \frac{1}{Z} \end{pmatrix} = \begin{pmatrix} \frac{1}{Ls^\alpha} & \frac{-1}{Ls^\alpha} \\ \frac{-1}{Ls^\alpha} & \frac{1}{Ls^\alpha} \end{pmatrix} \quad (14)$$

which is both symmetrical and reciprocal since $Y_{11} = Y_{22}$ and $Y_{12} = Y_{21}$. Using (12) we can write

$$(Y_I)^n = 2^{n-1} \begin{pmatrix} \frac{1}{(Ls^\alpha)^n} & \frac{-1}{(Ls^\alpha)^n} \\ \frac{-1}{(Ls^\alpha)^n} & \frac{1}{(Ls^\alpha)^n} \end{pmatrix}. \quad (15)$$

Choosing $n = 1/\alpha$ we obtain

$$(Y_F) = (Y_I)^\alpha = \left(\frac{2^{1/\alpha-1}}{L^{1/\alpha}} \begin{pmatrix} \frac{1}{s} & \frac{-1}{s} \\ \frac{-1}{s} & \frac{1}{s} \end{pmatrix} \right)^\alpha, \quad (16)$$

where the elements inside (Y_I) represent an integer-order inductor with inductance $(L/2^{\alpha-1})^{1/\alpha}$.

Case 2. Consider the transmission line π -model shown in Figure 3(b). The admittance matrix for this section is

$$(Y_F) = \begin{pmatrix} Y + \frac{1}{Z} & \frac{-1}{Z} \\ \frac{-1}{Z} & Y + \frac{1}{Z} \end{pmatrix} \quad (17)$$

which is both symmetrical and reciprocal. Assume that Z is a fractional-order inductor ($Z = Ls^\alpha$) and that Y is a fractional-order capacitor ($Y = Cs^\beta$); then

$$(Y_I)^n = \frac{1}{2} \cdot \begin{pmatrix} (Cs^\beta)^n + \left(\frac{2}{Ls^\alpha} + Cs^\beta \right)^n & (Cs^\beta)^n - \left(\frac{2}{Ls^\alpha} + Cs^\beta \right)^n \\ (Cs^\beta)^n - \left(\frac{2}{Ls^\alpha} + Cs^\beta \right)^n & (Cs^\beta)^n + \left(\frac{2}{Ls^\alpha} + Cs^\beta \right)^n \end{pmatrix}. \quad (18)$$

Selecting $\alpha = l/n$ and $\beta = k/n$ such that $l + k = n$ will guarantee the existence of integer-order elements in (Y_I) which is then given by

$$(Y_I)^n = \frac{1}{2} \begin{pmatrix} s^k C^n + \frac{1}{L^n s^l} (2 + LCs^{(l+k)/n})^n & s^k C^n - \frac{1}{L^n s^l} (2 + LCs^{(l+k)/n})^n \\ s^k C^n - \frac{1}{L^n s^l} (2 + LCs^{(l+k)/n})^n & s^k C^n + \frac{1}{L^n s^l} (2 + LCs^{(l+k)/n})^n \end{pmatrix}. \quad (19)$$

FIGURE 4: (a) T-model and (b) equivalent circuit of (Y_I) in (25).

Note that $(Y_I)^n$ is symmetrical and reciprocal where its elements can be expanded to

$$Y_{11} = Y_{22} = \frac{(2^{n-1})}{s^l L^n} + s^k C^n + \sum_{j=1}^{n-1} C_j^n \frac{2^{n-j-1} C^j s^{j(l+k)/n-l}}{L^{n-j}}, \quad (20)$$

$$Y_{12} = Y_{21} = -\frac{(2^{n-1})}{s^l L^n} - \sum_{j=1}^{n-1} C_j^n \frac{2^{n-j-1} C^j s^{j(l+k)/n-l}}{L^{n-j}},$$

where $C_j^n = n!/j!(n-j)!$. If the condition $l+k=n$ is not satisfied, fractional elements will still exist inside the two-port network. Now consider, for example, the case $\alpha = \beta = 0.5$; then choosing $l = k = 1$ and $n = 2$ yields

$$(Y_F) = (Y_I)^{1/2} = \begin{pmatrix} C^2 s + \frac{2C}{L} + \frac{2}{L^2 s} & -\frac{2C}{L} - \frac{2}{L^2 s} \\ -\frac{2C}{L} - \frac{2}{L^2 s} & C^2 s + \frac{2C}{L} + \frac{2}{L^2 s} \end{pmatrix}^{1/2}. \quad (21)$$

The elements inside (Y_I) can be represented by the equivalent circuit in Figure 3(c), all of which are integer-order elements. Alternatively, as an example for nonequal fractional-order elements, let $\alpha = 1/3$ and $\beta = 2/3$; then $l = 1$, $k = 2$, and $n = 3$; the corresponding fractional matrix is given as follows:

$$(Y_F) = (Y_I)^{1/3} = \begin{pmatrix} \frac{4}{L^3 s} + C^3 s^2 + \frac{3C^2}{L} s + \frac{6C}{L^2} & -\frac{4}{L^3 s} - \frac{3C^2}{L} s - \frac{6C}{L^2} \\ -\frac{4}{L^3 s} - \frac{3C^2}{L} s - \frac{6C}{L^2} & \frac{4}{L^3 s} + C^3 s^2 + \frac{3C^2}{L} s + \frac{6C}{L^2} \end{pmatrix}^{1/3}, \quad (22)$$

where all elements of (Y_I) can also be easily realized. It is worth noting that the restriction $\alpha + \beta = 1$ imposed above also guarantees that the equivalent circuit of (Y_I) is a π -model (see Figure 3(c)). However, lifting this restriction is possible.

Case 3. Consider the transmission-line T-model shown in Figure 4(a) which has the admittance matrix

$$(Y_F) = \begin{pmatrix} Z + \frac{1}{Y} & -\frac{1}{Y} \\ -\frac{1}{Y} & Z + \frac{1}{Y} \end{pmatrix} \quad (23)$$

and assume that Z is a fractional-order inductor ($Z = Ls^\alpha$) while Y is a fractional-order capacitor ($Y = Cs^\beta$); then in this case

$$(Y_I)^n = \frac{1}{2} \cdot \begin{pmatrix} (Ls^\alpha)^n + \left(\frac{2}{Cs^\beta} + Ls^\alpha\right)^n & \left(\frac{2}{Cs^\beta} + Ls^\alpha\right)^n - (Ls^\alpha)^n \\ \left(\frac{2}{Cs^\beta} + Ls^\alpha\right)^n - (Ls^\alpha)^n & (Ls^\alpha)^n + \left(\frac{2}{Cs^\beta} + Ls^\alpha\right)^n \end{pmatrix}. \quad (24)$$

Following a similar procedure to that of the π -model for the case $\alpha = \beta = 0.5$ we can show that

$$(Y_F) = (Y_I)^{1/2} = \begin{pmatrix} L^2 s + \frac{2L}{C} + \frac{2}{C^2 s} & \frac{2L}{C} + \frac{2}{C^2 s} \\ \frac{2L}{C} + \frac{2}{C^2 s} & L^2 s + \frac{2L}{C} + \frac{2}{C^2 s} \end{pmatrix}^{1/2} \quad (25)$$

and hence (Y_I) has the equivalent T-model given in Figure 4(b).

4. Conclusion

We attempted to introduce the idea of fractional-order two-port networks and its application to impedance and admittance parameters of fractional-order elements. The topic is still in its early stages [11] and much more work needs to be done both theoretically and experimentally. In particular, we considered the following.

- (i) We demonstrated here application to the impedance and admittance matrices; however there are other important two-port network parameters such as the transmission and scattering matrices for which some of the elements inside the matrix are unitless and obtaining an equivalent circuit requires transformation from these types of parameters back to impedance or admittance parameters. Therefore matrix transformations (conversions) need to be studied in the context of matrices raised to noninteger power and the table of conversions updated.
- (ii) Network interconnects (series, parallel and cascade interconnects) with matrices raised to a fractional-order power need to be studied. Such interconnects

require the addition and multiplication of matrices in the classical integer-order context.

- (iii) The restriction imposed in our analysis ($l + k = n$) if not possible to satisfy would imply the existence of fractional-order matrix parameters in addition to the fractional-order matrix exponent. A solution to this problem is required.
- (iv) It should be somehow possible to define a noninteger square matrix of dimension $\alpha \times \alpha$, $\alpha \leq 1$, equivalent to an $n \times n$ square matrix raised to the non-integer-order α . Such a definition and relationship require further investigation.

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Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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