Research Article

Consensus of Third-Order Multiagent Systems with Time Delay in Undirected Networks

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We consider consensus of a class of third-order continuous-time multiagent systems with time delay in undirected networks. By using matrix analysis and a frequency domain approach, a necessary and sufficient condition for consensus is established. A simulation result is also given to illustrate the main theoretical result.

1. Introduction

In recent years, more and more people are interested in multiagent systems due to their extensive applications in real world, such as cooperative control of unmanned aerial vehicle, autonomous underwater submarine, flocking, and network control in information. Among the corresponding theories for multiagent systems, consensus problem is a critical issue which has attracted interdisciplinary researchers from the fields of control, mathematics, biology, physics, computer, robot, communication and artificial intelligence, and so forth.

Most work on consensus focuses on protocols taking the form of first-order dynamics [1–5]. Extensions of consensus protocols to second-order dynamics are reported in [6–11]. Just as is shown in [9], the convergence of second-order consensus algorithms relies on not only information exchange topology but also the coupling strength between the information states in contrast to first-order consensus algorithms. There is also some recent work focusing on consensus protocols with both first-order and second-order dynamics [12, 13].

It is well known that the dynamic equation given by second-order consensus protocols contains the derivatives of position and velocity. In practice, the dynamics of agents may involve the derivative of accelerated speed. The derivative of accelerated speed is usually called Jerk in engineering.

Jerk is a physical quantity to describe the changing rate of acceleration. A Jerk is often required in engineering, especially in transportation design and materials, and so forth. Therefore, it is necessary and significant to design a class of third-order consensus protocols by using the information of position, velocity, and accelerated speed of agents, which is the main purpose of this paper.

Recently, some people have studied some higher-order consensus protocols [14, 15]. In [14], the authors presented some necessary and sufficient conditions for consensus of high-order multiagent systems. Consensus of multiagent systems with linear higher-order agents was considered in [15], where the authors answered whether the group converges to consensus and what consensus value it eventually reaches. However, necessary and sufficient conditions for consensus analysis of higher-order protocols in [14, 15] are given based on the distribution of eigenvalues of a complex matrix, which do not clearly illustrate the relationship among the eigenvalues of the involved Laplacian matrix, coupling strengths, and time delay when the system achieves consensus. On the other hand, higher-order consensus protocols with time delay receive less attention in the literature. Therefore, in this paper we will introduce a third-order consensus protocol with time delay and provide a convergence analysis for the given third-order consensus protocol in undirected networks. A necessary and sufficient condition for consensus is established, which clearly shows the relationship among

the nonzero eigenvalues of the Laplacian matrix, coupling strengths, and time delay when the system achieves consensus asymptotically.

An outline of this paper is as follows. In Section 2, some preliminaries about graph and a class of third-order protocol are introduced. The main result of this paper is proposed in Section 3. In Section 4, a simulation result is given to illustrate the main result. Section 5 concludes the whole paper.

Notations. The following notations will be used throughout this paper. $\|\cdot\|$ denotes Euclidean norm. $\mathbf{j}$ is the imaginary unit. $\otimes$ denotes the Kronecker product. R^n and $R^{N \times N}$ denote the real space of n -dimensional column vectors and $N \times N$ -dimensional matrices, respectively. $I_N(O_N) \in R^{N \times N}$ is an $N \times N$ -dimensional identity (zero) matrix and $\mathbf{1}_N = (1, \dots, 1)^T$ ($\mathbf{0}_N = (0, \dots, 0)^T$) denotes the N -dimensional column vector with entries 1 (0).

2. Preliminaries

2.1. Graph Theory. Let $G = (\mathcal{V}, \mathcal{E}, A)$ be an N -order weighted undirected graph with set of nodes $\mathcal{V} = \{\vartheta_1, \vartheta_2, \dots, \vartheta_N\}$, set of edges $\mathcal{E} = \{(\vartheta_i, \vartheta_j) : \vartheta_i, \vartheta_j \in \mathcal{V}\} \subseteq \mathcal{V} \times \mathcal{V}$, and weighted adjacent matrix $A = (a_{ij})_{N \times N}$. An edge of G is denoted by $e_{ij} = (\vartheta_i, \vartheta_j)$. In the undirected graph, $e_{ij} \in \mathcal{E} \Leftrightarrow e_{ji} \in \mathcal{E}$. The adjacency matrix is defined as $a_{ii} = 0$ and $a_{ij} = a_{ji} \geq 0$ for $i \neq j$ and $i, j \in \{1, 2, \dots, N\}$, where $a_{ij} > 0 \Leftrightarrow e_{ij} \in \mathcal{E}$. A path is a sequence of edges of the form $(\vartheta_{i1}, \vartheta_{i2}), (\vartheta_{i2}, \vartheta_{i3}), \dots$, where $\vartheta_{ij} \in \mathcal{V}$. If there is a path for arbitrary two nodes in the graph, we call that the graph is connected.

The Laplacian matrix of the graph G is defined as $L = [l_{ij}] \in R^{N \times N}$, where

$$l_{ii} = - \sum_{j=1, j \neq i}^N l_{ij}, \quad l_{ij} = -a_{ij}, \quad i \neq j. \quad (1)$$

Obviously, the Laplacian matrix L is a positive semidefinite matrix and satisfies $l_{ii} \geq 0$, $l_{ij} \leq 0$ for $i \neq j$ and $\sum_{j=1}^N l_{ij} = 0$ for $i = 1, \dots, N$.

The following lemma is well known.

Lemma 1 (see [2]). *If the undirected graph G is connected, its Laplacian L has a simple zero eigenvalue with the associated eigenvector $\mathbf{1}_N$ and all the other eigenvalues are positive real numbers.*

2.2. Third-Order Consensus Protocol. Suppose the dynamics of each node is described by the following third-order system:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= z_i(t), \\ \dot{z}_i(t) &= u_i(t), \end{aligned} \quad (2)$$

$$i = 1, 2, \dots, N,$$

where $t \geq 0$, $x_i(t) \in R^n$, $v_i(t) \in R^n$, and $z_i(t) \in R^n$ represent the position, velocity, and accelerated speed of node

ϑ_i , respectively, and $u_i(t) \in R^n$ is the control input (or protocol) to be designed.

In this paper we consider the following third-order protocol:

$$\begin{aligned} u_i(t) &= k_1 \sum_{j=1}^N a_{ij} (x_j(t-\tau) - x_i(t-\tau)) \\ &\quad + k_2 \sum_{j=1}^N a_{ij} (v_j(t-\tau) - v_i(t-\tau)) \\ &\quad + k_3 \sum_{j=1}^N a_{ij} (z_j(t-\tau) - z_i(t-\tau)), \end{aligned} \quad (3)$$

where $i = 1, 2, \dots, N$, $k_1 > 0$, $k_2 > 0$, and $k_3 > 0$ are appropriate coupling strengths.

Definition 2. Say system (2) with protocol (3) achieves consensus asymptotically, if $\lim_{t \rightarrow \infty} \|x_i(t) - x_j(t)\| = 0$, $\lim_{t \rightarrow \infty} \|v_i(t) - v_j(t)\| = 0$, and $\lim_{t \rightarrow \infty} \|z_i(t) - z_j(t)\| = 0$ for any $i, j = 1, 2, \dots, N$ and any initial condition $x(t) = \phi(t)$, where ϕ is a continuous function defined on $[-\tau, 0]$.

We first have the following lemma for system (2) with protocol (3).

Lemma 3. *If system (2) with protocol (3) achieves consensus asymptotically, one has*

$$\begin{aligned} \lim_{t \rightarrow \infty} \left\| x_i(t) - \frac{1}{N} \sum_{j=1}^N \left[x_j(0) + v_j(0)t + \frac{1}{2} z_j(0)t^2 \right] \right\| &= 0, \\ \lim_{t \rightarrow \infty} \left\| v_i(t) - \frac{1}{N} \sum_{j=1}^N [v_j(0) + z_j(0)t] \right\| &= 0, \\ \lim_{t \rightarrow \infty} \left\| z_i(t) - \frac{1}{N} \sum_{j=1}^N z_j(0) \right\| &= 0. \end{aligned} \quad (4)$$

Proof. (i) Since the graph is undirected, by (2) and (3), we have $\sum_{j=1}^N \dot{z}_j(t) = \mathbf{0}$. Integrating it on both sides from 0 to t , we have

$$\sum_{j=1}^N z_j(t) = \sum_{j=1}^N z_j(0). \quad (5)$$

Denote $e_{ij}(t) = z_j(t) - z_i(t)$ ($i, j = 1, \dots, N$, $j \neq i$). Thus, (5) will be rewritten into the following form:

$$N z_i(t) + \sum_{j=1, j \neq i}^N e_{ij}(t) = \sum_{j=1}^N z_j(0). \quad (6)$$

Namely,

$$z_i(t) - \frac{1}{N} \sum_{j=1}^N z_j(0) = -\frac{1}{N} \sum_{j=1, j \neq i}^N e_{ij}(t). \quad (7)$$

Note that the system achieves consensus; we have $\lim_{t \rightarrow \infty} \|e_{ij}(t)\| = 0$ for $i, j = 1, \dots, N, j \neq i$. Therefore, we get from (7) that

$$\lim_{t \rightarrow \infty} \left\| z_i(t) - \frac{1}{N} \sum_{j=1}^N z_j(0) \right\| = 0. \quad (8)$$

(ii) By (2) and (5), we have

$$\sum_{j=1}^N \dot{v}_j(t) = \sum_{j=1}^N z_j(t) = \sum_{j=1}^N z_j(0). \quad (9)$$

Integrating the above equality from 0 to t yields

$$\sum_{j=1}^N v_j(t) = \sum_{j=1}^N (v_j(0) + z_j(0)t). \quad (10)$$

Similar to the discussion in (i), we can easily get the second equality of (4).

(iii) By (2) and (10), we have

$$\sum_{j=1}^N \dot{x}_j(t) = \sum_{j=1}^N v_j(t) = \sum_{j=1}^N (v_j(0) + z_j(0)t). \quad (11)$$

By integrating it from 0 to t , we get

$$\sum_{j=1}^N x_j(t) = \sum_{j=1}^N \left(x_j(0) + v_j(0)t + \frac{1}{2} z_j(0)t^2 \right). \quad (12)$$

We can similarly derive the first equality of (4) as above. The proof of Lemma 3 is complete. $\square$

3. Main Result

In the sequel, set

$$\begin{aligned} \xi(t) &= (x_1^T(t), v_1^T(t), z_1^T(t), \dots, x_N^T(t), v_N^T(t), z_N^T(t))^T. \end{aligned} \quad (13)$$

System (2) with protocol (3) can be rewritten into

$$\dot{\xi}(t) = (I_N \otimes A) \xi(t) - (L \otimes B) \xi(t - \tau), \quad (14)$$

where

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \otimes I_n, \\ B &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{pmatrix} \otimes I_n. \end{aligned} \quad (15)$$

By Lemma 3, let

$$\bar{\delta}(t) = \xi(t) - \mathbf{1}_N \otimes (\alpha^T(t), \beta^T(t), \gamma^T)^T, \quad (16)$$

where

$$\begin{aligned} \alpha(t) &= \frac{1}{N} \sum_{j=1}^N \left(x_j(0) + v_j(0)t + \frac{z_j(0)t^2}{2} \right), \\ \beta(t) &= \frac{1}{N} \sum_{j=1}^N (v_j(0) + z_j(0)t), \\ \gamma &= \frac{1}{N} \sum_{j=1}^N z_j(0). \end{aligned} \quad (17)$$

A straightforward computation yields that system (14) is equivalent to

$$\dot{\bar{\delta}} = (I_N \otimes A) \bar{\delta}(t) - (L \otimes B) \bar{\delta}(t - \tau). \quad (18)$$

By Lemma 1, we can denote the nonzero eigenvalues of L by $\lambda_2, \lambda_3, \dots, \lambda_N$ and set $0 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_N$. Then there exists an orthogonal matrix W such that

$$\begin{aligned} \bar{\Lambda} &= W^T L W = \text{diag}\{0, \lambda_2, \dots, \lambda_N\}, \\ (W^T \otimes I_3) \bar{\delta}(t) &= [\mathbf{0}_n^T, \mathbf{0}_n^T, \mathbf{0}_n^T, \delta^T(t)]^T, \end{aligned} \quad (19)$$

where $\delta(t)$ is a column vector of appropriate dimension. Premultiplying both sides of (18) by $W^T \otimes I_3$, we have

$$\dot{\delta}(t) = (I_{N-1} \otimes A) \delta(t) - (\Lambda \otimes B) \delta(t - \tau), \quad (20)$$

where $\Lambda = \text{diag}\{\lambda_2, \dots, \lambda_N\}$.

Theorem 4. System (2) with protocol (3) achieves consensus asymptotically if and only if

$$\frac{\lambda_i (k_2 \omega \cos(\omega\tau) + k_3 \omega^2 \sin(\omega\tau) - k_1 \sin(\omega\tau))}{\omega^3} < 1, \quad (21)$$

where $i = 2, 3, \dots, N$; ω satisfies the following equation:

$$k_2 \omega \tan(\omega\tau) = k_3 \omega^2 - k_1. \quad (22)$$

Proof. Based on the above analysis, consensus of system (14) is equivalent to the asymptotic stability of system (20). On the other hand, system (20) is asymptotically stable if and only if all its eigenvalues lie on the open left half plane (LHP). Thus, it only requires showing that all the eigenvalues of system (20) lie on the open LHP if and only if (21) and (22) hold. By using Laplace transform on both sides of (20), we have $\delta(s) = G_\tau^{-1}(s) \delta(0)$, where

$$G_\tau(s) = s I_{3m(N-1)} - I_{N-1} \otimes A + (\Lambda \otimes B) e^{-\tau s}. \quad (23)$$

Note that

$$\begin{aligned} I_{N-1} \otimes A - (\Lambda \otimes B) e^{\tau s} \\ = \text{diag}\{A - \lambda_2 e^{\tau s} B, \dots, A - \lambda_n e^{\tau s} B\}. \end{aligned} \quad (24)$$

By a direct calculation, we get the determinant of $G_\tau(s)$:

$$\begin{aligned} \det(G_\tau(s)) &= \prod_{i=2}^N \det(sI_{3n} - A + \lambda_i e^{-\tau s} B) \\ &= \prod_{i=2}^N \det((sI_3 - M_i) \otimes I_n) \\ &= \prod_{i=2}^N (s^3 + \lambda_i k_3 e^{-\tau s} s^2 + \lambda_i k_2 e^{-\tau s} s + \lambda_i k_1 e^{-\tau s})^n, \end{aligned} \quad (25)$$

where

$$M_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda_i k_1 & -\lambda_i k_2 & -\lambda_i k_3 \end{pmatrix}, \quad i = 2, 3, \dots, N. \quad (26)$$

By (25), all the zeros of $G_\tau(s)$ lie on the open LHP if and only if all the roots of equations

$$s^3 + \lambda_i k_3 e^{-\tau s} s^2 + \lambda_i k_2 e^{-\tau s} s + \lambda_i k_1 e^{-\tau s} = 0, \quad i = 2, \dots, N, \quad (27)$$

are all located on the open LHP. Since (27) is equivalent to

$$1 + \frac{\lambda_i k_3 e^{-\tau s} s^2 + \lambda_i k_2 e^{-\tau s} s + \lambda_i k_1 e^{-\tau s}}{s^3} = 0, \quad (28)$$

for $i = 2, \dots, N$, all the roots of (27) lie on the open LHP if and only if all the roots of (28) are located on the open LHP. By the Nyquist stability criterion, all the roots of (28) lie on the open LHP if and only if the Nyquist curve

$$z = \frac{\lambda_i k_3 e^{-\tau j\omega} (j\omega)^2 + \lambda_i k_2 e^{-\tau j\omega} j\omega + \lambda_i k_1 e^{-\tau j\omega}}{(j\omega)^3} \quad (29)$$

does not enclose the point $(-1, 0j)$, where j is an imaginary number unit and $i = 2, \dots, N$. Using Euler's formula for (29), we have

$$\begin{aligned} z &= \frac{-\lambda_i (k_2 \omega \cos(\omega\tau) + k_3 \omega^2 \sin(\omega\tau) - k_1 \sin(\omega\tau))}{\omega^3} \\ &+ \frac{(-k_3 \omega^2 \cos(\omega\tau) + k_2 \omega \sin(\omega) + k_1 \cos(\omega\tau)) \lambda_i}{\omega^3} \quad (30) \end{aligned}$$

$\cdot j$.

Hence, all the Nyquist curves (29) do not enclose the point $(-1, 0j)$ if and only if (21) holds, where ω is the solution of equation

$$(-k_3 \omega^2 \cos(\omega\tau) + k_2 \omega \sin(\omega) + k_1 \cos(\omega\tau)) \lambda_i = 0. \quad (31)$$

Namely, (22) holds. The proof of Theorem 4 is complete. $\square$

For the particular case when $\tau = 0$, Theorem 4 reduces to Corollary 5 in [16].

FIGURE 1: An undirected graph.

Corollary 5. System (2) with protocol (3) and $\tau = 0$ achieves consensus asymptotically if and only if the coupling strengths k_1 , k_2 , and k_3 and the eigenvalue λ_2 of the Laplacian matrix L satisfy

$$\lambda_2 > \frac{k_1}{k_2 k_3}. \quad (32)$$

4. Simulation

In this section, we will give a simulation to illustrate the main result. The topology graph in the simulation is shown in Figure 1 which has four agents and 0-1 weights.

It is easy to see the corresponding Laplacian matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}. \quad (33)$$

If we consider $k_1 = k_2 = k_3 = 1$ and $\tau = 0.3$, conditions (21) and (22) are satisfied. By Theorem 4, consensus of system (2) with protocol (3) is achieved asymptotically. However, when $\tau = 0.356$, conditions (21) and (22) are not satisfied. Therefore, system (2) with protocol (3) does not achieve consensus asymptotically. The states of position, velocity, and accelerated speed of agents with $\tau = 0.3$ and $\tau = 0.356$ are shown in Figures 2 and 3, respectively.

5. Conclusions

In this paper, we study the consensus problem of the third-order dynamic multiagent system with time delay in undirected graphs. A necessary and sufficient condition for consensus of the system has been established. A simulation result illustrates the effectiveness of the theoretical result. Consensus of third-order multiagent systems with time delay in directed graphs will be further studied in the future.

Competing Interests

The authors declare that they have no competing interests.

FIGURE 2: Trajectories of system (2) with $k_1 = k_2 = k_3 = 1$, $\tau = 0.3$.

FIGURE 3: Trajectories of system (2) with $k_1 = k_2 = k_3 = 1$, $\tau = 0.356$.

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