Research Article

Nonlinear Dynamics and Exact Traveling Wave Solutions of the Higher-Order Nonlinear Schrödinger Equation with Derivative Non-Kerr Nonlinear Terms

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Received 9 July 2015; Revised 8 November 2015; Accepted 10 December 2015

Academic Editor: Giacomo Innocenti

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By using the method of dynamical system, the exact travelling wave solutions of the higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms are studied. Based on this method, all phase portraits of the system in the parametric space are given with the aid of the Maple software. All possible bounded travelling wave solutions, such as solitary wave solutions, kink and anti-kink wave solutions, and periodic travelling wave solutions, are obtained, respectively. The results presented in this paper improve the related previous conclusions.

1. Introduction

Nonlinear Schrödinger (NLS) equation is one of the most important nonlinear models in mathematical physics and has many applications in nonlinear optics, plasma physics, condensed matter physics, photonics, and Bose-Einstein condensates. In particular, in the studies on optical fibers, the NLS equation is very important. As is well known, the higher-order nonlinear Schrödinger (HNLS) equation describes the propagation of picosecond or femtosecond optical pulse in fibers. Therefore, the study of the higher-order nonlinear Schrödinger (HNLS) equation is the hotspot in the study of nonlinear scientific fields.

In this paper, we consider the following higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms [1]:

$$\begin{aligned} E_z = & i(a_1 E_{tt} + a_2 |E|^2 E) + a_3 E_{ttt} + a_4 (|E|^2 E)_t \\ & + a_5 E (|E|^2)_t + ia_6 |E|^4 E + a_7 (|E|^4 E)_t \\ & + a_8 E (|E|^4)_t, \end{aligned} \quad (1)$$

where $E(z, t)$ is the slowly varying envelope of the electric field, the subscripts z, t are the spatial and temporal partial derivatives in related time coordinates, and a_1, a_2, a_3, a_4 , and a_5 are the real parameters related to the group velocity, self-phase modulation, third-order dispersion, self-steeping, and self-frequency shift arising from stimulated Raman scattering, respectively, [2]. The terms related to coefficients a_6, a_7, a_8 in (1) represent the quintic non-Kerr nonlinearities. The investigation of this equation has raised great interest due to its wide range of applications. In [1], Choudhuri and Porsezian investigated the periodic wave solutions, the bright and dark solitary wave solutions of (1). In [3], Choudhuri and Porsezian investigated the Dark-in-the-Bright (DITB) solitary wave solution of (1). In [4], Choudhuri and Porsezian have studied the modulational instability (MI) of (1) with forth-order dispersion in context of optics and presented an analytical expression for MI gain to show the effects of non-Kerr nonlinearities and higher-order dispersions on MI gain spectra, and so on. However, we notice that the dynamics of the traveling wave solutions of (1) have not been studied. It is meaningful and necessary to consider the dynamical behavior of (1) and to find all possible exact solutions of (1). In the present paper, we will use the dynamical system method

to investigate the travelling wave solutions of the higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms.

Firstly, to investigate the existence of travelling wave solution of HNLS equation in presence of non-Kerr terms, we begin with scaling the variables of (1) in the form

$$\begin{aligned} E &= b_1 \psi, \\ z &= b_2 x, \\ t &= b_3 \tau \end{aligned} \quad (2)$$

and choosing b_1 , b_2 , and b_3 such that the coefficients corresponding to group velocity dispersion (GVD), self-phase modulation (SPM), and third-order dispersion (TOD) become unity. Thus (1) becomes

$$\begin{aligned} \psi_x &= i(\psi_{\tau\tau} + |\psi|^2 \psi) + \psi_{\tau\tau\tau} + c_1(|\psi|^2 \psi)_\tau \\ &+ c_2 \psi(|\psi|^2)_\tau + ic_3 |\psi|^4 \psi + c_4(|\psi|^4 \psi)_\tau \\ &+ c_5 \psi(|\psi|^4)_\tau, \end{aligned} \quad (3)$$

where $c_1 = b_1^2 b_2 a_4 / b_3 = a_4 a_1 / a_2 a_3$, $c_2 = b_1^2 b_2 a_5 / b_3 = a_5 a_1 / a_2 a_3$, $c_3 = b_1^4 b_2 a_6 / a_2^2 a_3^2$, $c_4 = b_1^4 b_2 a_7 / b_3 = a_7 a_1^4 / a_2^2 a_3^3$, and $c_5 = b_1^4 b_2 a_8 / b_3 = a_8 a_1^4 / a_2^2 a_3^3$. We have chosen $b_1 = \sqrt{a_1^3 / a_2^2 a_3^3}$, $b_2 = a_2^2 / a_1^3$ and $b_3 = a_3 / a_1$ in writing (3).

To obtain the exact travelling wave solutions of (3), we consider the travelling wave solutions of the following form:

$$\psi(x, \tau) = P(\xi) e^{i\eta}, \quad \xi = vx + \tau, \quad \eta = kx - c\tau, \quad (4)$$

where k , c , and v are travelling wave parameters. Substituting (4) into (3), canceling $e^{i\eta}$, and separating the real and imaginary parts, we have

$$P'' = (v - 2c + 3c^2)P - \frac{3c_1 + 2c_2}{3}P^3 - \frac{5c_4 + 4c_5}{5}P^5, \quad (5)$$

$$P'' = \frac{k + c^2 - c^3}{1 - 3c}P - \frac{1 - cc_1}{1 - 3c}P^3 - \frac{c_3 - cc_4}{1 - 3c}P^5.$$

Equating the two equations, we get the following conditions:

$$\begin{aligned} c &= \frac{3c_1 + 2c_2 - 3}{6(c_1 + c_2)} = \frac{5c_4 + 4c_5 - 5c_3}{10c_4 + 12c_5}, \\ k &= (1 - 3c)(v - 2c + 3c^2) - c^2 + c^3, \end{aligned} \quad (6)$$

with constraint relations

$$\begin{aligned} c_3 &= \frac{3}{5}, \\ c_4 &= \frac{3c_1}{5}, \\ c_5 &= \frac{c_2}{2}. \end{aligned} \quad (7)$$

As a result of the freedom of these parameters which is consistency, under condition (6) and (7), (5) is simplified to the following equation:

$$P'' - (v - 2c + 3c^2)P + \frac{3c_1 + 2c_2}{3}P^3 + \frac{5c_4 + 4c_5}{5}P^5 = 0. \quad (8)$$

Denote that $d_1 = v - 2c + 3c^2$, $d_2 = -(3c_1 + 2c_2)/6$, and $d_3 = -(5c_4 + 4c_5)/15$. Thus, (8) has the following form:

$$P'' - d_1 P - 2d_2 P^3 - 3d_3 P^5 = 0, \quad (9)$$

which corresponds to the two-dimensional Hamiltonian system:

$$\begin{aligned} \frac{dP}{d\xi} &= y, \\ \frac{dy}{d\xi} &= d_1 P + 2d_2 P^3 + 3d_3 P^5 \end{aligned} \quad (10)$$

with the Hamiltonian

$$H(P, y) = \frac{1}{2}y^2 - \frac{d_1}{2}P^2 - \frac{d_2}{2}P^4 - \frac{d_3}{2}P^6 = h. \quad (11)$$

According to the Hamiltonian, we can get all kinds of phase portraits in the parametric space. Because the phase orbits defined the vector fields of system (10) and determined all their travelling wave solutions of (3), by investigating the bifurcations of phase portraits of system (10), we can seek the travelling wave solutions of (3) [5–11]. The detailed calculation procedure can be found in the technical appendix. The rest of this paper is built up as follows. In Section 2, we give all phase portraits of system (10) and discuss the dynamics of phase portraits of system (10). In Section 3, according to the dynamics of the phase orbits of system (10) given by Section 2, we obtain all possible bounded travelling wave solutions of (3). Finally, a conclusion is given in Section 4.

2. Nonlinear Dynamics of Phase Portraits of System (10)

In this section, we consider the bifurcations of the phase orbits of system (10) on the phase plane (P, y) as the parameters d_1, d_2, d_3 are changed.

Firstly, we consider the distribution of the equilibrium points of system (10). Obviously, the zeros of the function $f(P) = d_1 P + 2d_2 P^3 + 3d_3 P^5$ appeared and the second term of system (10) is the abscissas of equilibrium points of system (10) on the phase plane (P, y) . We write that $\Delta = d_2^2 - 3d_1 d_3$, $f_1 = (-d_2 + \sqrt{\Delta})/3d_3$, and $f_2 = (-d_2 - \sqrt{\Delta})/3d_3$. For $c_4 = 3c_1/5$ and $c_5 = c_2/2$, $d_2 = (-3c_1 - 2c_2)/6$ and $d_3 = (-3c_1 - 2c_2)/15$ are the same sign, according to Vieta theorem, that is, the relationship between the root and coefficient of quadratic equation with one unknown; we have the following proposition.

Proposition 1. (1) For the case of $d_3 = 0$, because $d_2 = (-3c_1 - 2c_2)/6$ and $d_3 = (-3c_1 - 2c_2)/15$ are the same sign, $d_2 = d_3 = 0$. system (10) has only one equilibrium point $U_0(0, 0)$. (2) For the case of $d_3 \neq 0$, when $\Delta < 0$ or $d_1 d_3 \geq 0$, system (10) has only one equilibrium point $U_0(0, 0)$. (3) For the case of $d_3 \neq 0$, when $\Delta > 0$ and $d_1 d_3 < 0$, system (10) has three equilibrium points, $U_0(0, 0)$ and $U_1 \pm (\pm\sqrt{f_1}, 0)$ or $U_2 \pm (\pm\sqrt{f_2}, 0)$.

Then, we consider the type of the equilibrium points of system (10). Let $M(U_i, 0)$ ($i = 0, \dots, 2$) be the Jacobin matrix of system (10) at an equilibrium point $(U_i, 0)$ and let $J(U_i, 0)$ ($i = 0, \dots, 2$) be the Jacobin determinant. Thus, we have

$$\begin{aligned} \text{Trace}(U_i, 0) &= 0, \quad (i = 0, \dots, 2), \\ J(0, 0) &= -d_1, \\ J(\pm\sqrt{f_1}, 0) &= -4f_1\sqrt{\Delta}, \\ J(\pm\sqrt{f_2}, 0) &= 4f_2\sqrt{\Delta}. \end{aligned} \quad (12)$$

For $H(P, y) = (1/2)y^2 - (d_1/2)P^2 - (d_2/2)P^4 - (d_3/2)P^6 = h$, we write that

$$\begin{aligned} h_0 &= H(0, 0) = 0, \\ h_1 &= H(\pm\sqrt{f_1}, 0) \\ &= \frac{(d_2 - \sqrt{\Delta})(-d_2^2 + d_2\sqrt{\Delta} + 6d_1d_3)}{54d_3^2}, \\ h_2 &= H(\pm\sqrt{f_2}, 0) \\ &= \frac{(d_2 + \sqrt{\Delta})(-d_2^2 - d_2\sqrt{\Delta} + 6d_1d_3)}{54d_3^2}. \end{aligned} \quad (13)$$

By the bifurcation theory of dynamical system, for an equilibrium point of a planar integrable system, the equilibrium

point is a saddle point when $J < 0$. The equilibrium point is a center point when $J > 0$ and $\text{Trace } M(U_i, 0) = 0$. The equilibrium point is a cusp when $J = 0$ and the Poincare index of the equilibrium point is 0.

Finally, by using the above discussion, the bifurcations of phase portraits of system (10) for the case of $d_3 = 0$, $d_3 > 0$, and $d_3 < 0$ are shown in Figures 1–3 with the aid of Maple.

3. Exact Travelling Wave Solutions of (3)

In this section, corresponding to all phase orbits given by Section 2, through qualitative analysis and the Jacobian elliptic functions [12], we discuss the exact travelling wave solutions of (3). Since only bounded travelling waves are meaningful to a physical model, here we just pay our attention to the bounded solutions of (3).

3.1. The Case of $d_3 = 0$ (See Figure 1). Suppose that $d_1 < 0$ (see Figure 1(b)). When $h \in (0, +\infty)$, there exists a family of periodic travelling wave solutions which correspond to a family of periodic orbits of (10). It follows from (11) that $y^2 = -d_1(-2h/d_1 - P^2)$. By using the first equation of (10), we have

$$P_1(\xi) = \sqrt{-\frac{2h}{d_1}} \cos(\sqrt{-d_1}\xi). \quad (14)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_1(x, \tau) = \sqrt{-\frac{2h}{d_1}} \cos(\sqrt{-d_1}(vx + \tau)) e^{i(kx - c\tau)}. \quad (15)$$

3.2. The Case of $d_3 > 0$ (See Figure 2)

(1) Suppose That $d_1 < 0$ (See Figure 2(b)). When $h \in (0, h_1)$, there exists a family of periodic travelling wave solutions which correspond to a family of periodic orbits of (10). It follows from (11) that $y^2 = d_3(\gamma_1 - P^2)(P^2 - \gamma_2)(\gamma_3 - P^2)$, where $\gamma_1 > \gamma_3 > 0 > \gamma_2$. Therefore, we have

$$P_2(\xi) = \left(\frac{\gamma_2\gamma_3\text{sn}^2\left(\sqrt{d_3\gamma_1(\gamma_3 - \gamma_2)}\xi, \sqrt{\gamma_3(\gamma_1 - \gamma_2)/\gamma_1(\gamma_3 - \gamma_2)}\right)}{\gamma_2 - \gamma_3 + \gamma_3\text{sn}^2\left(\sqrt{d_3\gamma_1(\gamma_3 - \gamma_2)}\xi, \sqrt{\gamma_3(\gamma_1 - \gamma_2)/\gamma_1(\gamma_3 - \gamma_2)}\right)} \right)^{1/2}. \quad (16)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_2(x, \tau) = \left(\frac{\gamma_2\gamma_3\text{sn}^2\left(\sqrt{d_3\gamma_1(\gamma_3 - \gamma_2)}(vx + \tau), \sqrt{\gamma_3(\gamma_1 - \gamma_2)/\gamma_1(\gamma_3 - \gamma_2)}\right)}{\gamma_2 - \gamma_3 + \gamma_3\text{sn}^2\left(\sqrt{d_3\gamma_1(\gamma_3 - \gamma_2)}(vx + \tau), \sqrt{\gamma_3(\gamma_1 - \gamma_2)/\gamma_1(\gamma_3 - \gamma_2)}\right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (17)$$

FIGURE 1: The bifurcations of phase portraits of (10) for $d_3 = 0$. (a) $d_2 = 0, d_1 > 0$. (b) $d_2 = 0, d_1 < 0$.

FIGURE 2: The bifurcations of phase portraits of (10) for $d_3 > 0$. (a) $d_1 \geq 0, d_2 > 0$ or $d_1 > d_2^2/3d_3, d_2 > 0$. (b) $d_1 < 0, d_2 > 0$.

(2) Suppose That $d_1 < 0$ (See Figure 2(b)). When $h = h_1$, there exists a kink wave solution and an anti-kink wave solution which correspond to two heteroclinic orbits of (10). It follows from (11) that $y^2 = d_3(f_1 - P^2)(P^2 - \gamma_4)$, where $\gamma_4 = -(d_2 + 2\sqrt{\Delta})/3d_3, f_1 > 0 > \gamma_4$. Therefore, we have

$$P_3(\xi) = \pm \left(f_1 + \frac{2f_1(f_1 - \gamma_4)}{\gamma_4 - 2f_1 + \gamma_4 \cosh\left(2\sqrt{d_3 f_1(f_1 - \gamma_4)}\xi\right)} \right). \quad (18)$$

Thus, we obtain a kink wave solution and an anti-kink wave solution of (3) as follows:

$$\psi_3(x, \tau) = \pm \left(f_1 + \frac{2f_1(f_1 - \gamma_4)}{\gamma_4 - 2f_1 + \gamma_4 \cosh\left(2\sqrt{d_3 f_1(f_1 - \gamma_4)}(vx + \tau)\right)} \right) \cdot e^{i(kx - c\tau)}. \quad (19)$$

FIGURE 3: The bifurcations of phase portraits of (10) for $d_3 < 0$. (a) $d_1 \leq 0, d_2 < 0$ or $d_1 < d_2^2/3d_3, d_2 < 0$. (b) $d_1 > 0, d_2 < 0$.

3.3. The Case of $d_3 < 0$ (See Figure 3)

3.3.1. The Travelling Wave Solutions Corresponding to Figure 3(a)

(1) Suppose That $d_2 < 0, d_1 < d_2^2/4d_3$ or $d_2 < 0, d_1 < d_2^2/3d_3$. When $h \in (0, +\infty)$, there exists a family of periodic travelling wave solutions which correspond to a family of periodic orbits of (10). Therefore, we have

$$P_4(\xi) = \left(\frac{\gamma_1 B_1 \left(1 - \operatorname{cn} \left(2\sqrt{-d_3 A_1 B_1} \xi, \sqrt{(\gamma_1^2 - (A_1 - B_1)^2)/4A_1 B_1} \right) \right)}{A_1 + B_1 + (A_1 - B_1) \operatorname{cn} \left(2\sqrt{-d_3 A_1 B_1} \xi, \sqrt{(\gamma_1^2 - (A_1 - B_1)^2)/4A_1 B_1} \right)} \right)^{1/2}. \quad (20)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_4(x, \tau) = \left(\frac{\gamma_1 B_1 \left(1 - \operatorname{cn} \left(2\sqrt{-d_3 A_1 B_1} (vx + \tau), \sqrt{(\gamma_1^2 - (A_1 - B_1)^2)/4A_1 B_1} \right) \right)}{A_1 + B_1 + (A_1 - B_1) \operatorname{cn} \left(2\sqrt{-d_3 A_1 B_1} (vx + \tau), \sqrt{(\gamma_1^2 - (A_1 - B_1)^2)/4A_1 B_1} \right)} \right)^{1/2} e^{i(kx - c\tau)}, \quad (21)$$

where $A_1^2 = (\gamma_1 - \alpha_1)^2 + \beta_1^2$, $B_1^2 = \alpha_1^2 + \beta_1^2$, $\alpha_1 = (\gamma_2 + \gamma_3)/2$, and $\beta_1 = -(\gamma_2 - \gamma_3)^2/4$.

(2) Suppose That $d_2 < 0, d_2^2/4d_3 < d_1 \leq 0$. When $h = h_3$, there exists a family of periodic travelling wave solutions which correspond to a family of periodic orbits of (10). It follows from (11) that $y^2 = -d_3(P^2 - f_1)^2(\gamma_4 - P^2)$, where $\gamma_4 = -(d_1 + 2\sqrt{\Delta})/3d_3$, $\gamma_4 > 0 > f_1$, and $h_3 = (\sqrt{\gamma_4}, 0) = (9d_1 d_2 d_3 - 2d_2^3 + 2\Delta^{3/2})/54d_3^2$. Therefore, we have

$$P_5(\xi) = \left(f_1 + \frac{2f_1(\gamma_4 - f_1)}{2f_1 - \gamma_4 - \gamma_4 \cos \left(2\sqrt{d_3 f_1(\gamma_4 - f_1)} \xi \right)} \right)^{1/2}. \quad (22)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_5(x, \tau) = \left(f_1 + \frac{2f_1(\gamma_4 - f_1)}{2f_1 - \gamma_4 - \gamma_4 \cos\left(2\sqrt{d_3 f_1(\gamma_4 - f_1)}(vx + \tau)\right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (23)$$

(3) Suppose That $d_2 < 0$, $d_2^2/4d_3 < d_1 \leq 0$. When $h \in (0, h_3)$, there exists a family of periodic travelling wave solutions which correspond to a family of periodic orbits of (10). It follows from (11) that $y^2 = -d_3(P^2 - \gamma_1)(\gamma_2 - P^2)(P^2 - \gamma_3)$, where $\gamma_2 > 0 > \gamma_3 > \gamma_1$. Therefore, we have

$$P_6(\xi) = \left(\frac{\gamma_2 \gamma_3 \text{sn}^2\left(\sqrt{d_3 \gamma_1(\gamma_2 - \gamma_3)}\xi, \sqrt{\gamma_2(\gamma_3 - \gamma_1)/\gamma_1(\gamma_3 - \gamma_2)}\right)}{\gamma_3 - \gamma_2 + \gamma_2 \text{sn}^2\left(\sqrt{d_3 \gamma_1(\gamma_2 - \gamma_3)}\xi, \sqrt{\gamma_2(\gamma_3 - \gamma_1)/\gamma_1(\gamma_3 - \gamma_2)}\right)} \right)^{1/2}. \quad (24)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_6(x, \tau) = \left(\frac{\gamma_2 \gamma_3 \text{sn}^2\left(\sqrt{d_3 \gamma_1(\gamma_2 - \gamma_3)}(vx + \tau), \sqrt{\gamma_2(\gamma_3 - \gamma_1)/\gamma_1(\gamma_3 - \gamma_2)}\right)}{\gamma_3 - \gamma_2 + \gamma_2 \text{sn}^2\left(\sqrt{d_3 \gamma_1(\gamma_2 - \gamma_3)}(vx + \tau), \sqrt{\gamma_2(\gamma_3 - \gamma_1)/\gamma_1(\gamma_3 - \gamma_2)}\right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (25)$$

(4) Suppose That $d_2 < 0$, $d_2^2/4d_3 < d_1 \leq 0$. When $h \in (h_3, +\infty)$, there exists a family of periodic travelling wave solutions which correspond to a family of periodic orbits of

(10). It follows from (11) that $y^2 = -d_3(P^2 - \gamma_1)(\gamma_2 - P^2)(P^2 - \gamma_3)$, where $\gamma_2 > 0$ and γ_1 and γ_3 are conjugate complex number. Therefore, we have

$$P_7(\xi) = \left(\frac{\gamma_2 B_2 \left(1 - \text{cn}\left(2\sqrt{-d_3 A_2 B_2}\xi, \sqrt{(\gamma_2^2 - (A_2 - B_2)^2)/4A_2 B_2}\right)\right)}{A_2 + B_2 + (A_2 - B_2) \text{cn}\left(2\sqrt{-d_3 A_2 B_2}\xi, \sqrt{(\gamma_2^2 - (A_2 - B_2)^2)/4A_2 B_2}\right)} \right)^{1/2}. \quad (26)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_7(x, \tau) = \left(\frac{\gamma_2 B_2 \left(1 - \text{cn}\left(2\sqrt{-d_3 A_2 B_2}(vx + \tau), \sqrt{(\gamma_2^2 - (A_2 - B_2)^2)/4A_2 B_2}\right)\right)}{A_2 + B_2 + (A_2 - B_2) \text{cn}\left(2\sqrt{-d_3 A_2 B_2}(vx + \tau), \sqrt{(\gamma_2^2 - (A_2 - B_2)^2)/4A_2 B_2}\right)} \right)^{1/2} e^{i(kx - c\tau)}, \quad (27)$$

where $A_2^2 = (\gamma_2 - \alpha_2)^2 + \beta_2^2$, $B_2^2 = \alpha_2^2 + \beta_2^2$, $\alpha_2 = (\gamma_1 + \gamma_3)/2$, and $\beta_2 = -(\gamma_1 - \gamma_3)^2/4$.

(5) Suppose That $d_2 < 0$, $d_1 = d_2^2/4d_3$. Equation (3) has a family of periodic travelling wave solutions with the same solutions as (27).

3.3.2. The Travelling Wave Solutions Corresponding to Figure 3(b)

(1) Suppose That $d_1 > 0$. When $h \in (h_2, 0)$, there exists two family of periodic travelling wave solutions which correspond

to two family of periodic orbits of (10). It follows from (11) that $y^2 = -d_3(P^2 - \gamma_1)(\gamma_2 - P^2)(P^2 - \gamma_3)$, where $\gamma_2 > \gamma_3 > 0 > \gamma_1$. Therefore, we have

$$P_8(\xi) = \pm \left(\frac{\gamma_2 \operatorname{dn}^2 \left(\sqrt{d_3 \gamma_2 (\gamma_1 - \gamma_3)} \xi, \sqrt{\gamma_1 (\gamma_3 - \gamma_2) / \gamma_2 (\gamma_3 - \gamma_1)} \right)}{1 - ((\gamma_3 - \gamma_2) / (\gamma_3 - \gamma_1)) \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_2 (\gamma_1 - \gamma_3)} \xi, \sqrt{\gamma_1 (\gamma_3 - \gamma_2) / \gamma_2 (\gamma_3 - \gamma_1)} \right)} \right)^{1/2}. \quad (28)$$

Thus, we obtain the periodic travelling wave solutions of (3) as follows:

$$\psi_8(x, \tau) = \pm \left(\frac{\gamma_2 \operatorname{dn}^2 \left(\sqrt{d_3 \gamma_2 (\gamma_1 - \gamma_3)} (vx + \tau), \sqrt{\gamma_1 (\gamma_3 - \gamma_2) / \gamma_2 (\gamma_3 - \gamma_1)} \right)}{1 - ((\gamma_3 - \gamma_2) / (\gamma_3 - \gamma_1)) \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_2 (\gamma_1 - \gamma_3)} (vx + \tau), \sqrt{\gamma_1 (\gamma_3 - \gamma_2) / \gamma_2 (\gamma_3 - \gamma_1)} \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (29)$$

(2) Suppose That $d_1 > 0$. When $h = 0$, there exists a pair of solitary wave solutions which correspond to two homoclinic orbits of (10). It follows from (11) that $y^2 = d_3 P^2 (\gamma_5 - P^2) (\gamma_6 - P^2)$, where $\gamma_5 > \gamma_6 > 0$, $\gamma_5 = (-d_2 + \sqrt{d_2^2 - 4d_1 d_3}) / 2d_3$, and $\gamma_6 = (-d_2 - \sqrt{d_2^2 - 4d_1 d_3}) / 2d_3$. Therefore, we have

$$P_9(\xi) = \pm \left(\frac{2\gamma_5 \gamma_6}{\gamma_5 + \gamma_6 + (\gamma_5 - \gamma_6) \cosh \left(2\sqrt{d_3 \gamma_5 \gamma_6} \xi \right)} \right)^{1/2}. \quad (30)$$

Thus, we obtain the solitary wave solutions of (3) as follows:

$$\psi_9(x, \tau) = \pm \left(\frac{2\gamma_5 \gamma_6}{\gamma_5 + \gamma_6 + (\gamma_5 - \gamma_6) \cosh \left(2\sqrt{d_3 \gamma_5 \gamma_6} (vx + \tau) \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (31)$$

(3) Suppose That $d_1 > 0$. When $h \in (0, +\infty)$, $h \in (0, h_3)$, $h = h_3$, and $h \in (h_3, +\infty)$, (3) has a family of periodic travelling wave solutions with the same solutions as (21), (23), (25), and (27), respectively.

Through the approach of dynamical system, we have studied the exact travelling wave solutions of the higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms. Firstly, through the travelling wave transformation, the higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms is reduced a planer Hamiltonian system. Then with the aid of Maple, the bifurcations of phase portraits of the planer Hamiltonian system

are given. By studying the bifurcations of phase portraits of the planer Hamiltonian system, we obtain exact travelling wave solutions of the higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms, which contain solitary wave solutions, kink and anti-kink wave solutions, and periodic travelling wave solutions. Note that our solutions in this paper are different from the given ones in previous references [1–4]. We have the hyperbolic function solutions, trigonometric function solutions, and the Jacobian elliptic function solutions. From the above discussions, obviously the dynamical system method is very powerful method to seek exact travelling wave solutions for nonlinear travelling wave equations. The method is concise, direct, and effective which reduces the large amount of calculations. It is a good method which allows us to solve complicated nonlinear evolution equations in mathematical physics.

4. Conclusion

By the results of Sections 2 and 3 and considering (6) and (7), we obtain the following main conclusion of this paper.

Theorem 2. Suppose that d_1 , d_2 , and d_3 are given by Section 1 and f_1 , f_2 , A_1 , B_1 , and γ_i , ($i = 1, \dots, 6$) are defined by Sections 2 and 3. The higher-order nonlinear Schrödinger equation with derivative non-Kerr nonlinear terms has the following 9 classes exact bounded travelling wave solutions.

(1) When $d_1 < 0$, $d_2 = 0$, and $d_3 = 0$,

$$\psi_1(x, \tau) = \sqrt{-\frac{2h}{d_1}} \cos \left(\sqrt{-d_1} (vx + \tau) \right) e^{i(kx - c\tau)}. \quad (32)$$

(2) When $d_1 < 0$, $d_2 > 0$, and $d_3 > 0$,

$$\psi_2(x, \tau) = \left(\frac{\gamma_2 \gamma_3 \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_1 (\gamma_3 - \gamma_2)} (vx + \tau), \sqrt{\gamma_3 (\gamma_1 - \gamma_2) / \gamma_1 (\gamma_3 - \gamma_2)} \right)}{\gamma_2 - \gamma_3 + \gamma_3 \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_1 (\gamma_3 - \gamma_2)} (vx + \tau), \sqrt{\gamma_3 (\gamma_1 - \gamma_2) / \gamma_1 (\gamma_3 - \gamma_2)} \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (33)$$

(3) When $d_1 < 0$, $d_2 > 0$, and $d_3 > 0$,

$$\psi_3(x, \tau) = \pm \left(f_1 + \frac{2f_1(f_1 - \gamma_4)}{\gamma_4 - 2f_1 + \gamma_4 \cosh \left(2\sqrt{d_3 f_1 (f_1 - \gamma_4)} (vx + \tau) \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (34)$$

(4) When $d_1 < d_2^2/4d_3$, $d_2 < 0$, $d_3 < 0$ or $d_1 < d_2^2/3d_3$, $d_2 < 0$, $d_3 < 0$,

$$\psi_4(x, \tau) = \left(\frac{\gamma_1 B_1 \left(1 - \operatorname{cn} \left(2\sqrt{-d_3 A_1 B_1} (vx + \tau), \sqrt{(\gamma_1^2 - (A_1 - B_1)^2) / 4A_1 B_1} \right) \right)}{A_1 + B_1 + (A_1 - B_1) \operatorname{cn} \left(2\sqrt{-d_3 A_1 B_1} (vx + \tau), \sqrt{(\gamma_1^2 - (A_1 - B_1)^2) / 4A_1 B_1} \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (35)$$

(5) When $d_2^2/4d_3 < d_1 \leq 0$, $d_2 < 0$, and $d_3 < 0$,

$$\psi_5(x, \tau) = \left(f_1 + \frac{2f_1(\gamma_4 - f_1)}{2f_1 - \gamma_4 - \gamma_4 \cos \left(2\sqrt{d_3 f_1 (\gamma_4 - f_1)} (vx + \tau) \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (36)$$

(6) When $d_2^2/4d_3 < d_1 \leq 0$, $d_2 < 0$, and $d_3 < 0$,

$$\psi_6(x, \tau) = \left(\frac{\gamma_2 \gamma_3 \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_1 (\gamma_2 - \gamma_3)} (vx + \tau), \sqrt{\gamma_2 (\gamma_3 - \gamma_1) / \gamma_1 (\gamma_3 - \gamma_2)} \right)}{\gamma_3 - \gamma_2 + \gamma_2 \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_1 (\gamma_2 - \gamma_3)} (vx + \tau), \sqrt{\gamma_2 (\gamma_3 - \gamma_1) / \gamma_1 (\gamma_3 - \gamma_2)} \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (37)$$

(7) When $d_2^2/4d_3 < d_1 \leq 0$, $d_2 < 0$, and $d_3 < 0$,

$$\psi_7(x, \tau) = \left(\frac{\gamma_2 B_2 \left(1 - \operatorname{cn} \left(2\sqrt{-d_3 A_2 B_2} (vx + \tau), \sqrt{(\gamma_2^2 - (A_2 - B_2)^2) / 4A_2 B_2} \right) \right)}{A_2 + B_2 + (A_2 - B_2) \operatorname{cn} \left(2\sqrt{-d_3 A_2 B_2} (vx + \tau), \sqrt{(\gamma_2^2 - (A_2 - B_2)^2) / 4A_2 B_2} \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (38)$$

(8) When $d_1 > 0$, $d_2 < 0$, and $d_3 < 0$,

$$\psi_8(x, \tau) = \pm \left(\frac{\gamma_2 \operatorname{dn}^2 \left(\sqrt{d_3 \gamma_3 (\gamma_1 - \gamma_3)} (vx + \tau), \sqrt{\gamma_1 (\gamma_3 - \gamma_2) / \gamma_2 (\gamma_3 - \gamma_1)} \right)}{1 - ((\gamma_3 - \gamma_2) / (\gamma_3 - \gamma_1)) \operatorname{sn}^2 \left(\sqrt{d_3 \gamma_3 (\gamma_1 - \gamma_3)} (vx + \tau), \sqrt{\gamma_1 (\gamma_3 - \gamma_2) / \gamma_2 (\gamma_3 - \gamma_1)} \right)} \right)^{1/2} e^{i(kx - c\tau)}. \quad (39)$$

(9) When $d_1 > 0$, $d_2 < 0$, and $d_3 < 0$,

$$\begin{aligned} \psi_9(x, \tau) &= \pm \left(\frac{2\gamma_5 \gamma_6}{\gamma_5 + \gamma_6 + (\gamma_5 - \gamma_6) \cosh \left(2\sqrt{d_3 \gamma_5 \gamma_6} (vx + \tau) \right)} \right)^{1/2} e^{i(kx - c\tau)}. \end{aligned} \quad (40)$$

Appendix

Here we describe the dynamical system method for finding traveling wave solutions of nonlinear wave equations. A $(n + 1)$ -dimensional nonlinear partial differential equation is given as follows:

$$P(t, x_i, u_t, u_{x_i}, u_{x_i x_i}, u_{x_i x_j}, u_{tt}, \dots) = 0, \quad (A.1)$$

$$i, j = 1, 2, \dots, n.$$

The main steps of the dynamical system method are as follows.

Step 1 (reduction of (A.1)). Making a transformation $u(t, x_1, x_2, \dots, x_n) = \phi(\xi)$, $\xi = \sum_{i=1}^n k_i x_i - ct$, (A.1) can be reduced to a nonlinear ordinary differential equation:

$$D(\xi, \phi, \phi_\xi, \phi_{\xi\xi}, \phi_{\xi\xi\xi}, \dots) = 0, \quad (A.2)$$

where k_i are nonzero constant and c is the wave speed. Integrating several times for (A.2), if it can be reduced to the following second-order nonlinear ordinary differential equation,

$$E(\phi, \phi_\xi, \phi_{\xi\xi}) = 0, \quad (A.3)$$

then let $\phi_\xi = d\phi/d\xi = y$, and (A.3) can be reduced to a two-dimensional dynamical system:

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= f(\phi, y), \end{aligned} \quad (A.4)$$

where $f(\phi, y)$ is an integral expression or a fraction. If $f(\phi, y)$ is a fraction such as $f(\phi, y) = F(\phi, y)/g(\phi)$ and $g(\phi_s) = 0$, $dy/d\xi$ does not exist when $\phi = \phi_s$. Then we will make a transformation $d\xi = g(\phi)d\zeta$; thus system (A.4) can be rewritten as

$$\begin{aligned} \frac{d\phi}{d\zeta} &= g(\phi) y, \\ \frac{dy}{d\zeta} &= F(\phi, y), \end{aligned} \quad (A.5)$$

where ζ is a parameter. If (A.1) can be reduced to the above system (A.4) or (A.5), then we can go on to the next step.

Step 2 (discussion of bifurcations of phase portraits of system (A.4)). If system (A.4) is an integral system, systems (A.4) and (A.5) can be reduced the differential equation:

$$\begin{aligned} \frac{dy}{d\phi} &= \frac{f(\phi, y)}{y}, \\ \frac{dy}{d\phi} &= \frac{F(\phi, y)}{g(\phi)y} = \frac{f(\phi, y)}{y}, \end{aligned} \quad (A.6)$$

and then systems (A.4) and (A.5) have the same first integral (that is Hamiltonian) as follows:

$$H(\phi, y) = h, \quad (A.7)$$

where h is an integral constant. According to the first integral, we can get all kinds of phase portraits in the parametric

space. Because the phase orbits defined the vector fields of system (A.4) (or system (A.5)) and determined all their travelling wave solutions of (A.1), we can investigate the bifurcations of phase portraits of system (A.4) (or system (A.5)) to seek the travelling wave solutions of (A.1). Usually, a periodic orbit always corresponds to a periodic wave solution; a homoclinic orbit always corresponds to a solitary wave solution; a heteroclinic orbit (or so called connecting orbit) always corresponds to kink (or anti-kink) wave solution. When we find all phase orbits, we can get the value of h or its range.

Step 3 (calculation of the first equation of system (A.4)). After h is determined, we can get the following relationship from (A.7):

$$y = y(\phi, h); \quad (\text{A.8})$$

that is, $d\phi/d\xi = y(\phi, h)$. If the expression (A.8) is an integral expression, then substituting it into the first term of (A.4) and integrating it, we obtain

$$\int_{\phi_0}^{\phi} \frac{d\phi}{y(\phi, h)} = \int_0^{\xi} d\tau, \quad (\text{A.9})$$

where $\phi(0)$ and 0 are initial constants. Usually, the initial constants can be taken by a root of (A.8) or inflection points of the travelling waves. Taking proper initial constants and integrating (A.9), through the Jacobian elliptic functions, we can obtain the exact travelling wave solutions of (A.1).

From the above description of “three-step method,” we can see that solutions of (A.1) can be obtained by studying and solving the dynamical system simplified by (A.1). Therefore, this approach is called dynamical system method. The different nonlinear wave equations correspond to different dynamical systems. The different dynamical systems correspond to different travelling wave solutions. This is the whole process of the dynamical system method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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