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Research Article

Sufficiency and Duality for Multiobjective Programming under New Invexity

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A class of multiobjective programming problems including inequality constraints is considered. To this aim, some new concepts of generalized (F, P)-type I and (F, P)-type II functions are introduced in the differentiable assumption by using the sublinear function F. These new functions are used to establish and prove the sufficient optimality conditions for weak efficiency or efficiency of the multiobjective programming problems. Moreover, two kinds of dual models are formulated. The weak dual, strong dual, and strict converse dual results are obtained under the aforesaid functions.

1. Introduction

The field of multiobjective programming, also called vector programming, has grown remarkably in different directions since the 1980s. Many researchers have been interested in the optimality conditions and duality results for the weak efficient solution and efficient solution of the multiobjective programming problems. A large literature was developed around the sufficiency and duality in multiobjective optimization [1]. In [2], Jayswal obtained the Kuhn-Tucker type sufficient optimality conditions for a feasible solution to be an efficient solution and the Mond-Weir type duality results are also presented. More specifically, Gao [3] considered the nonsmooth multiobjective semi-infinite programming and obtained several sufficient conditions and duality results. Also, Bae et al. [4] established duality theorems for nondifferentiable multiobjective programming problems under generalized convexity assumptions. Recently, Kim and Lee [5] introduced the nonsmooth multiobjective programming problems involving locally Lipschitz functions and support functions. For more descriptions of the multiobjective programming, we refer to [6–9].

Furthermore, various types of generalizations of convexity theory have played an important role in the evolution of the multiobjective programming. During the past decades, the generalizations of invexity were enriched with and without differentiability assumptions. For example, we can see [10–12]. In particular, Nahak and Mohapatra [13] introduced the concept of ρ-(η, θ)-invexity function and discussed a class of multiobjective programming problems by using the new generalized functions. Padhan and Nahak [14] introduced higher-order ρ-(η, θ)-invexity functions for studying two different pairs of higher-order symmetric dual programs. In [15], Antczak extended the concept of (ϕ, ρ)-invexity for differentiable optimization problems to the case of mathematical programming problems with locally Lipschitz functions. In [16], Antczak and Stasiak introduced the concept of (ϕ, ρ)-invexity for strong compact Lipschitz mappings in Banach spaces. Sufficient optimality conditions and Mond-Weir duality theorems are derived by the assumption of generalized nonsmooth (ϕ, ρ)-invexity between Banach spaces. In [17], based upon the F-convexity and ρ-convexity, the authors defined the (ϕ, ρ)-V-type I functions to consider a class of nonsmooth multiobjective programming problems. The invexity of functions is more useful in the research of optimization.

In this paper, we consider the multiobjective programming problems. The new class of generalized invexity functions, namely, pseudoinvex (F, P)-type I (pseudoinvex (F, P)-type II, etc.) are introduced. The sufficient optimality conditions are obtained. Then weak, strong, and strict converse dual results are also established for two types of dual models

related to multiobjective programming problems involving the new generalized invex functions.

2. Notations and Preliminaries

Throughout the paper, we use the following conventions for vectors in R^n:

$$\begin{aligned} &x \leqq y \quad \text{if and only } x_i \le y_i,\ \forall i = 1,2,\dots,n;\\ &x \le y \quad \text{if and only } x_i \leqq y_i,\ \forall i = 1,2,\dots,n,\ x \ne y;\\ &x < y \quad \text{if and only } x_i < y_i,\ \forall i = 1,2,\dots,n. \end{aligned} \tag{1}$$

In this paper, we consider the following multiobjective programming problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) = \left(f_1(x), f_2(x), \dots, f_k(x)\right)\\ &\text{subject to} \quad g_j(x) \leqq 0, \quad j = 1,2,\dots,m,\\ &\qquad\qquad x \in X, \end{aligned} \tag{MP}$$

where $X \subseteq R^n$ is an open set and $f_i : X \to R$, $i \in K = \{1,2,\dots,k\}$ and $g_j : X \to R$, $j \in M = \{1,2,\dots,m\}$ are differentiable on X. Let $X_0 = \{x \mid g_j(x) \leqq 0,\ j \in M\}$ be the set of all feasible solutions of (MP).

Definition 1. A feasible solution $\overline{x} \in X_0$ of (MP) is said to be a weakly efficient solution for (MP), if there exists no other $x \in X_0$, such that

$$f(x) < f(\overline{x}). \tag{2}$$

Definition 2. A feasible solution $\overline{x} \in X_0$ of (MP) is said to be an efficient solution for (MP), if there exists no other $x \in X_0$, such that

$$f(x) \le f(\overline{x}). \tag{3}$$

Definition 3. A function $F : X \times X \times R^n \to R$ is sublinear if, for any $x, \overline{x} \in X$,

$$\begin{aligned} &F(x,\overline{x};a_1 + a_2) \leqq F(x,\overline{x};a_1) + F(x,\overline{x};a_2), \quad \forall a_1, a_2 \in R^n,\\ &F(x,\overline{x};la) = lF(x,\overline{x};a), \quad \forall l \in R,\ l \geqq 0,\ a \in R^n. \end{aligned} \tag{4}$$

Remark 4. It should be noted that $F(x,\overline{x};0) = 0$.

Let f and g be differentiable at a given point $\overline{x} \in X$, $\eta : X \times X \to R^n$, $\alpha_i, \beta_j : X \times X \to R_+ \setminus \{0\}$, $d : X \times X \to R$, $\rho_i, \tau_j \in R$, $b : X \times X \to R_+ \setminus \{0\}$, $P \in R^n$, $i \in K$, $j \in M$.

Definition 5. (f,g) is said to be pseudoinvex (F,P)-type I at $\overline{x} \in X$, if there exists functions $b, \alpha_i, \beta_j, \eta, d$ and $\rho_i, \tau_j \in R$, $P \in R^n$, such that each $x \in X$; the following inequalities hold:

$$b(x,\overline{x})\left(f(x) - f(\overline{x})\right) < 0 \Longrightarrow \begin{cases} F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right) + \rho_i d^2(x,\overline{x}) < 0, & i \in K,\\ F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right) + \tau_j d^2(x,\overline{x}) < 0, & j \in M. \end{cases} \tag{5}$$

Example 6. Let $X = [-1,1] \subset R$. Let the functions $f : X \to R$, $g : X \to R$, and $b : X \times X \to R_+ \setminus \{0\}$ be defined by

$$\begin{aligned} f(x) &= e^x - 3x,\\ g(x) &= -x^3 - 2x - 5,\\ b(x,\overline{x}) &= x^2 + \overline{x}^2 + 1, \end{aligned} \tag{6}$$

and the functions $F : X \times X \times R \to R$, $\eta : X \times X \to R$ be given by

$$\begin{aligned} F(x,\overline{x};a) &= a\left(2 - \overline{x}^2\right),\\ \eta(x,\overline{x}) &= 2x - \overline{x}. \end{aligned} \tag{7}$$

Moreover, the functions $\alpha : X \times X \to R_+ \setminus \{0\}$, $\beta : X \times X \to R_+ \setminus \{0\}$, and $d : X \times X \to R$ are defined by

$$\begin{aligned} \alpha(x,\overline{x}) &= \overline{x}^2 + 1,\\ \beta(x,\overline{x}) &= \overline{x} + 2,\\ d(x,\overline{x}) &= x - \overline{x}. \end{aligned} \tag{8}$$

Then, let

$$\begin{aligned} \rho &= -1,\\ \tau &= -2,\\ P &= 1. \end{aligned} \tag{9}$$

Now, we have

$$b(x,\overline{x})\left(f(x) - f(\overline{x})\right) < 0, \quad \text{where } \overline{x} < x. \tag{10}$$

Then

$$\begin{aligned} &F\left(x,\overline{x};\alpha(x,\overline{x})\eta^T(x,\overline{x})\nabla f(\overline{x})P\right) + \rho d^2(x,\overline{x}) < 0,\\ &F\left(x,\overline{x};\beta(x,\overline{x})\eta^T(x,\overline{x})\nabla g(\overline{x})P\right) + \tau d^2(x,\overline{x}) < 0. \end{aligned} \tag{11}$$

Hence (f,g) is pseudoinvex (F,P)-type I at $u \in X$, where $u < x$, $x \in X$.

Definition 7. (f,g) is said to be pseudoinvex (F,P)-type II at $\overline{x} \in X$, if there exists functions $b, \alpha_i, \beta_j, \eta, d$ and $\rho_i, \tau_j \in R$, $P \in R^n$, such that each $x \in X$; the following inequalities hold:

$$b(x,\overline{x})\left(f(x)-f(\overline{x})\right)\leq 0 \Longrightarrow \begin{cases} F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})<0, & i\in K,\\ F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})<0, & j\in M.\end{cases} \tag{12}$$

Definition 8. (f,g) is said to be pseudoquasi-invex (F,P)-type I at $\overline{x}\in X$, if there exists functions $b,\alpha_i,\beta_j,\eta,d$ and $\rho_i,\tau_j\in R$, $P\in R^n$, such that each $x\in X$; the following inequalities hold:

$$b(x,\overline{x})\left(f(x)-f(\overline{x})\right)< 0 \Longrightarrow \begin{cases} F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})<0, & i\in K,\\ F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})\leqq 0, & j\in M.\end{cases} \tag{13}$$

Example 9. Let $f:R^2\to R^2$, $g:R^2\to R$, and $b:R^2\times R^2\to R_+\setminus\{0\}$ be defined by

$$\begin{aligned} f_1(x)&=x_1e^{2x_2},\\ f_2(x)&=x_2^3\left(x_1^2+1\right),\\ g(x)&=2x_1+x_2^2-3,\\ b(x,\overline{x})&=x_1^2+x_1+1,\end{aligned} \tag{14}$$

and the functions $F:R^2\times R^2\times R^2\to R$ and $\eta:R^2\times R^2\to R^2$ be given by

$$\begin{aligned} &F(x,\overline{x};a)=a^Tc(x,\overline{x}),\\ &\text{where } c(x,\overline{x})=\left(x_1-x_2,x_2\right)^T \text{ is a vector function.}\\ &\eta(x,\overline{x})=\left(x_1^2+x_2^2,x_1-2x_2+1\right)^T.\end{aligned} \tag{15}$$

Moreover, the functions $\alpha_i:R^2\times R^2\to R_+\setminus\{0\}$, $\beta:R^2\times R^2\to R_+\setminus\{0\}$, and $d:R^2\times R^2\to R$ are defined by

$$\begin{aligned} \alpha_1(x,\overline{x})&=\alpha_2(x,\overline{x})=x_1^2+e^{\cos x_2},\\ \beta(x,\overline{x})&=1,\\ d(x,\overline{x})&=x_1+x_2.\end{aligned} \tag{16}$$

Then, let

$$\begin{aligned} \rho_1&=\rho_2=\tau=-1,\\ P&=(1,1)^T.\end{aligned} \tag{17}$$

It is easy to see that (f,g) is pseudoquasi-invex (F,P)-type I at the point $\overline{x}=(0,0)$ with respect to b,α_i $(i=1,2)$, β,η,ρ_i $(i=1,2)$, τ,P, and d.

Definition 10. (f,g) is said to be pseudoquasi-invex (F,P)-type II at $\overline{x}\in X$, if there exists functions $b,\alpha_i,\beta_j,\eta,d$, and $\rho_i,\tau_j\in R$, $P\in R^n$, such that each $x\in X$; the following inequalities hold:

$$b(x,\overline{x})\left(f(x)-f(\overline{x})\right)\leq 0 \Longrightarrow \begin{cases} F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})<0, & i\in K,\\ F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})\leqq 0, & j\in M.\end{cases} \tag{18}$$

Definition 11. (f,g) is said to be quasipseudo-invex (F,P)-type I at $\overline{x}\in X$, if there exists functions $b,\alpha_i,\beta_j,\eta,d$, and $\rho_i,\tau_j\in R$, $P\in R^n$, such that each $x\in X$; the following inequalities hold:

$$b(x,\overline{x})\left(f(x)-f(\overline{x})\right)< 0 \Longrightarrow \begin{cases} F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})\leqq 0, & i\in K,\\ F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})<0, & j\in M.\end{cases} \tag{19}$$

Definition 12. (f,g) is said to be quasipseudo-invex (F,P)-type II at $\overline{x}\in X$, if there exists functions $b,\alpha_i,\beta_j,\eta,d$, and $\rho_i,\tau_j\in R$, $P\in R^n$, such that each $x\in X$; the following inequalities hold:

$$b(x,\overline{x})\left(f(x)-f(\overline{x})\right)\leq 0 \Longrightarrow \begin{cases} F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})\leqq 0, & i\in K,\\ F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})<0, & j\in M.\end{cases} \tag{20}$$

3. Sufficient Optimality Conditions

Now, we establish sufficient optimality conditions for the considered optimization problem (MP) under the new invexity.

Theorem 13. *Let $\overline{x} \in R^n$ be a feasible solution in problem* (MP). *Suppose that*

(i) *there exists $\lambda_i \geqq 0$, $\sum_{i=1}^{k} \lambda_i = 1$, $\mu_j \geqq 0$, $i \in K$, $j \in M$, such that*

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\overline{x}) = 0; \tag{21}$$

(ii) *(f, g) is pseudoinvex (F, P)-type I at $\overline{x}$;*

(iii) $\sum_{i=1}^{k}(\lambda_i \rho_i / \alpha_i(x, \overline{x})) + \sum_{j=1}^{m}(\mu_j \tau_j / \beta_j(x, \overline{x})) \geqq 0$.

Then $\overline{x}$ is a weakly efficient solution for (MP).

Proof. Suppose contrary to the result that $\overline{x}$ is not a weakly efficient solution to (MP). Then there exists $x \in X_0$ such that

$$f(x) < f(\overline{x}). \tag{22}$$

With $b(x, \overline{x}) > 0$, the above inequality yields

$$b(x, \overline{x})\left(f(x) - f(\overline{x})\right) < 0, \tag{23}$$

from hypothesis (ii), which implies

$$\begin{aligned} &F\left(x, \overline{x}; \alpha_i(x, \overline{x}) \eta^T(x, \overline{x}) \nabla f_i(\overline{x}) P\right) + \rho_i d^2(x, \overline{x}) < 0, \\ &\qquad i \in K, \\ &F\left(x, \overline{x}; \beta_j(x, \overline{x}) \eta^T(x, \overline{x}) \nabla g_j(\overline{x}) P\right) + \tau_j d^2(x, \overline{x}) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{24}$$

By sublinearity of F with $\alpha_i(x, \overline{x}) > 0$ and $\beta_j(x, \overline{x}) > 0$, inequalities (24) yield

$$\begin{aligned} &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \nabla f_i(\overline{x}) P\right) < -\frac{\rho_i}{\alpha_i(x, \overline{x})} d^2(x, \overline{x}), \\ &\qquad i \in K, \\ &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \nabla g_j(\overline{x}) P\right) < -\frac{\tau_j}{\beta_j(x, \overline{x})} d^2(x, \overline{x}), \\ &\qquad j \in M. \end{aligned} \tag{25}$$

Using $\lambda_i \geqq 0$, $\sum_{i=1}^{k} \lambda_i = 1$, and $\mu_j \geqq 0$, $j \in M$, along with the sublinearity of F, from inequality (25), we get

$$\begin{aligned} &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \left(\sum_{i=1}^{k} \lambda_i \nabla f_i(\overline{x})\right) P\right) \\ &\quad < -\sum_{i=1}^{k} \frac{\lambda_i \rho_i}{\alpha_i(x, \overline{x})} d^2(x, \overline{x}), \\ &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \left(\sum_{j=1}^{m} \mu_j \nabla g_j(\overline{x})\right) P\right) \\ &\quad \leqq -\sum_{j=1}^{m} \frac{\mu_j \tau_j}{\beta_j(x, \overline{x})} d^2(x, \overline{x}). \end{aligned} \tag{26}$$

By the sublinearity of F, we sum (26) to obtain

$$\begin{aligned} &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \left(\sum_{i=1}^{k} \lambda_i \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\overline{x})\right) P\right) \\ &\quad < -\left(\sum_{i=1}^{k} \frac{\lambda_i \rho_i}{\alpha_i(x, \overline{x})} + \sum_{j=1}^{m} \frac{\mu_j \tau_j}{\beta_j(x, \overline{x})}\right) d^2(x, \overline{x}), \end{aligned} \tag{27}$$

from hypothesis (iii), which follows

$$\begin{aligned} &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \left(\sum_{i=1}^{k} \lambda_i \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\overline{x})\right) P\right) \\ &\quad < 0. \end{aligned} \tag{28}$$

On the other hand, the hypothesis (i) implies

$$\begin{aligned} &F\left(x, \overline{x}; \eta^T(x, \overline{x}) \left(\sum_{i=1}^{k} \lambda_i \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\overline{x})\right) P\right) \\ &\quad = 0, \end{aligned} \tag{29}$$

which contradicts (28). Hence the conclusion of the theorem is established. □

Theorem 14. *Let $\overline{x} \in R^n$ be a feasible solution in problem* (MP). *Suppose that*

(i) *there exists $\lambda_i \geqq 0$, $\sum_{i=1}^{k} \lambda_i = 1$, $\mu_j \geqq 0$, $i \in K$, $j \in M$, such that*

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(\overline{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\overline{x}) = 0; \tag{30}$$

(ii) *(f, g) is pseudoinvex (F, P)-type II at $\overline{x}$;*

(iii) $\sum_{i=1}^{k}(\lambda_i \rho_i / \alpha_i(x, \overline{x})) + \sum_{j=1}^{m}(\mu_j \tau_j / \beta_j(x, \overline{x})) \geqq 0$.

Then $\overline{x}$ is an efficient solution for (MP).

Proof. By the way of contradiction, suppose that $\overline{x}$ is not an efficient solution for (MP). Then there exists $x \in X_0$ such that

$$f(x) \leq f(\overline{x}). \tag{31}$$

With $b(x, \overline{x}) > 0$, the above inequality yields

$$b(x, \overline{x})\left(f(x) - f(\overline{x})\right) \leq 0, \tag{32}$$

by hypothesis (ii), which follows

$$F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})<0,\quad i\in K,$$
$$F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})<0,\quad j\in M. \tag{33}$$

Using $\alpha_i(x,\overline{x})>0$, $\beta_j(x,\overline{x})>0$ and $\lambda_i\geqq 0$, $\sum_{i=1}^k\lambda_i=1$, $\mu_j\geqq 0$, $j\in M$, along with the sublinearity of F, inequality (33) yields

$$\begin{aligned}&F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{i=1}^k\lambda_i\nabla f_i(\overline{x})\right)P\right)\\&\quad<-\sum_{i=1}^k\frac{\lambda_i\rho_i}{\alpha_i(x,\overline{x})}d^2(x,\overline{x}),\\&F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{j=1}^m\mu_j\nabla g_j(\overline{x})\right)P\right)\\&\quad\leqq-\sum_{j=1}^m\frac{\mu_j\tau_j}{\beta_j(x,\overline{x})}d^2(x,\overline{x}).\end{aligned}\tag{34}$$

Summing inequalities (34) with the sublinearity of F, we obtain

$$\begin{aligned}&F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{i=1}^k\lambda_i\nabla f_i(\overline{x})+\sum_{j=1}^m\mu_j\nabla g_j(\overline{x})\right)P\right)\\&\quad<-\left(\sum_{i=1}^k\frac{\lambda_i\rho_i}{\alpha_i(x,\overline{x})}+\sum_{j=1}^m\frac{\mu_j\tau_j}{\beta_j(x,\overline{x})}\right)d^2(x,\overline{x}).\end{aligned}\tag{35}$$

From assumption (i), we have

$$\left(\sum_{i=1}^k\frac{\lambda_i\rho_i}{\alpha_i(x,\overline{x})}+\sum_{j=1}^m\frac{\mu_j\tau_j}{\beta_j(x,\overline{x})}\right)d^2(x,\overline{x})<0;\tag{36}$$

that is,

$$\sum_{i=1}^k\frac{\lambda_i\rho_i}{\alpha_i(x,\overline{x})}+\sum_{j=1}^m\frac{\mu_j\tau_j}{\beta_j(x,\overline{x})}<0,\tag{37}$$

which contradicts hypothesis (iii). That completes the proof. □

Theorem 15. *Let $\overline{x}\in R^n$ be a feasible solution in problem* (MP). *Suppose that*

(i) *there exists $\lambda_i\geqq 0$, $\sum_{i=1}^k\lambda_i=1$, $\mu_j\geqq 0$, $i\in K$, $j\in M$, such that*

$$\sum_{i=1}^k\lambda_i\nabla f_i(\overline{x})+\sum_{j=1}^m\mu_j\nabla g_j(\overline{x})=0;\tag{38}$$

(ii) *(f,g) is pseudoquasi-invex (F,P)-type I at $\overline{x}$;*

(iii) $\sum_{i=1}^k(\lambda_i\rho_i/\alpha_i(x,\overline{x}))+\sum_{j=1}^m(\mu_j\tau_j/\beta_j(x,\overline{x}))\geqq 0$.

Then $\overline{x}$ is a weakly efficient solution for (MP).

Proof. The proof follows the lines of Theorem 13. □

Theorem 16. *Let $\overline{x}\in R^n$ be a feasible solution in problem* (MP). *Suppose that*

(i) *there exists $\lambda_i\geqq 0$, $\sum_{i=1}^k\lambda_i=1$, $\mu_j\geqq 0$, $i\in K$, $j\in M$, such that*

$$\sum_{i=1}^k\lambda_i\nabla f_i(\overline{x})+\sum_{j=1}^m\mu_j\nabla g_j(\overline{x})=0;\tag{39}$$

(ii) *(f,g) is pseudoquasi-invex (F,P)-type II at $\overline{x}$;*

(iii) $\sum_{i=1}^k(\lambda_i\rho_i/\alpha_i(x,\overline{x}))+\sum_{j=1}^m(\mu_j\tau_j/\beta_j(x,\overline{x}))\geqq 0$.

Then $\overline{x}$ is an efficient solution for (MP).

Proof. The proof follows the lines of Theorem 14. □

Theorem 17. *Let $\overline{x}\in R^n$ be a feasible solution in problem* (MP). *Suppose that*

(i) *there exists $\lambda_i\geqq 0$, $\sum_{i=1}^k\lambda_i=1$, $\mu_j\geqq 0$ (at least one $\mu_j>0$), $i\in K$, $j\in M$, such that*

$$\sum_{i=1}^k\lambda_i\nabla f_i(\overline{x})+\sum_{j=1}^m\mu_j\nabla g_j(\overline{x})=0;\tag{40}$$

(ii) *(f,g) is quasipseudo-invex (F,P)-type I at $\overline{x}$;*

(iii) $\sum_{i=1}^k(\lambda_i\rho_i/\alpha_i(x,\overline{x}))+\sum_{j=1}^m(\mu_j\tau_j/\beta_j(x,\overline{x}))\geqq 0$.

Then $\overline{x}$ is a weakly efficient solution for (MP).

Proof. Suppose contrary to the result that $\overline{x}$ is not a weakly efficient solution to (MP). Then there exists $x\in X_0$ such that

$$f(x)<f(\overline{x}),\tag{41}$$

with $b(x,\overline{x})>0$; the above inequality yields

$$b(x,\overline{x})\left(f(x)-f(\overline{x})\right)<0,\tag{42}$$

from hypothesis (ii), which implies

$$F\left(x,\overline{x};\alpha_i(x,\overline{x})\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right)+\rho_i d^2(x,\overline{x})\leqq 0,\quad i\in K,$$
$$F\left(x,\overline{x};\beta_j(x,\overline{x})\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right)+\tau_j d^2(x,\overline{x})<0,\quad j\in M. \tag{43}$$

By sublinearity of F with $\alpha_i(x,\overline{x}) > 0$ and $\beta_j(x,\overline{x}) > 0$, inequalities (43) yield

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\nabla f_i(\overline{x})P\right) \leqq -\frac{\rho_i}{\alpha_i(x,\overline{x})}d^2(x,\overline{x}), \quad i \in K, \tag{44}$$

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\nabla g_j(\overline{x})P\right) < -\frac{\tau_j}{\beta_j(x,\overline{x})}d^2(x,\overline{x}), \quad j \in M. \tag{45}$$

Using $\lambda_i \geqq 0$, $\sum_{i=1}^{k}\lambda_i = 1$ along with the sublinearity of F, from inequality (44), we get

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{i=1}^{k}\lambda_i\nabla f_i(\overline{x})\right)P\right) \leqq -\sum_{i=1}^{k}\frac{\lambda_i\rho_i}{\alpha_i(x,\overline{x})}d^2(x,\overline{x}). \tag{46}$$

With $\mu_j \geqq 0$ (at least one $\mu_j > 0$), $j \in M$, and using the sublinearity of F, inequality (45) follows

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{j=1}^{m}\mu_j\nabla g_j(\overline{x})\right)P\right) < -\sum_{j=1}^{m}\frac{\mu_j\tau_j}{\beta_j(x,\overline{x})}d^2(x,\overline{x}). \tag{47}$$

By the sublinearity of F, we sum (46) and (47) to obtain

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{i=1}^{k}\lambda_i\nabla f_i(\overline{x}) + \sum_{j=1}^{m}\mu_j\nabla g_j(\overline{x})\right)P\right) < -\left(\sum_{i=1}^{k}\frac{\lambda_i\rho_i}{\alpha_i(x,\overline{x})} + \sum_{j=1}^{m}\frac{\mu_j\tau_j}{\beta_j(x,\overline{x})}\right)d^2(x,\overline{x}), \tag{48}$$

from hypothesis (iii), which follows

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{i=1}^{k}\lambda_i\nabla f_i(\overline{x}) + \sum_{j=1}^{m}\mu_j\nabla g_j(\overline{x})\right)P\right) < 0. \tag{49}$$

On the other hand, the hypothesis (i) implies

$$F\left(x,\overline{x};\eta^T(x,\overline{x})\left(\sum_{i=1}^{k}\lambda_i\nabla f_i(\overline{x}) + \sum_{j=1}^{m}\mu_j\nabla g_j(\overline{x})\right)P\right) = 0, \tag{50}$$

which contradicts (49). Hence the conclusion of theorem is established. □

Example 18. We consider the following programming problem:

$$\begin{aligned} &\text{Minimize} && f(x) = x^3 \\ &\text{subject to} && g(x) = 1 - e^x \leqq 0, \\ & && x \in R. \end{aligned} \tag{P}$$

The set of all feasible solutions of (P) can be given by $X_0 = \{x \mid x \geqq 0\}$.

Again, let $F(x,\overline{x};a)$ be the function defined by $F(x,\overline{x};a) = a(2\overline{x}+1)$.

It can be verified that (f,g) is quasipseudo-invex (F,P)-type I at $\overline{x} = 0$ with $\alpha(x,\overline{x}) = 1/(x^2+1)$, $\beta(x,\overline{x}) = 2/(x^2+1)$, $\eta(x,\overline{x}) = x^2+1$, $P = 1$, $d(x,\overline{x}) = \overline{x}-1$, and $\rho = 0$, $\tau = 2/3$, $\nabla f(0) = 0$, $\nabla g(0) = -1$.

Clearly, $\overline{x} = 0$ is a feasible solution for problem (P) and it satisfied the assumptions of Theorem 17, as there exist $\lambda = 1$, $\mu = 0$, such that

$$\begin{aligned} &\lambda\nabla f(0) + \mu\nabla g(0) = 0, \\ &\frac{\lambda\rho}{\alpha(x,\overline{x})} + \frac{\mu\tau}{\beta(x,\overline{x})} = 0. \end{aligned} \tag{51}$$

We observe that there exists no other $x \in X_0$, such that $f(x) < f(\overline{x})$. Hence, $\overline{x} = 0$ is an efficient solution for (P).

Similarly, we can establish the following theorem.

Theorem 19. *Let $\overline{x} \in R^n$ be a feasible solution in problem* (MP). *Suppose that*

(i) *there exists $\lambda_i \geqq 0$, $\sum_{i=1}^{k}\lambda_i = 1$, $\mu_j \geqq 0$ (at least one $\mu_j > 0$, as $j \in \overline{J}(\overline{x})$), $i \in K$, $j \in M$, such that*

$$\sum_{i=1}^{k}\lambda_i\nabla f_i(\overline{x}) + \sum_{j=1}^{m}\mu_j\nabla g_j(\overline{x}) = 0; \tag{52}$$

(ii) *(f,g) is quasipseudo-invex (F,P)-type II at $\overline{x}$;*

(iii) $\sum_{i=1}^{k}(\lambda_i\rho_i/\alpha_i(x,\overline{x})) + \sum_{j=1}^{m}(\mu_j\tau_j/\beta_j(x,\overline{x})) \geqq 0$.

Then $\overline{x}$ is an efficient solution for (MP).

4. Mond-Weir Duality

In this section, a dual problem is considered for the class of multiobjective programming problem with the new invex functions.

Consider the following Mond-Weir dual problem related to problem (MP):

$$\begin{aligned} &\text{Maximize} \quad f(u) = \left(f_1(u), f_2(u), \ldots, f_k(u)\right) \\ &\text{subject to} \quad \sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u) = 0, \\ &\qquad \overline{\lambda}_i \geqq 0, \quad \sum_{i=1}^{k} \overline{\lambda}_i = 1, \ i \in K, \\ &\qquad \overline{\mu}_j \geqq 0, \quad j \in M. \end{aligned} \tag{MDI}$$

Let $W_1 = \{(u, \overline{\lambda}, \overline{\mu}) \in X \times R^k \times R^m : \sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u) = 0, \overline{\lambda} \geq 0, \sum_{i=1}^{k} \overline{\lambda}_i = 1, \overline{\mu} \geqq 0\}$ be the set of all feasible solutions in problem (MDI).

Theorem 20 (weak duality). *Let x and $(u, \overline{\lambda}, \overline{\mu})$ be feasible solutions for* (MP) *and* (MDI), *respectively. Moreover, assume that*

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x,u)} + \sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(x,u)} \geqq 0. \tag{53}$$

If one of the following conditions is satisfied:

(a) *(f, g) is pseudoinvex (F, P)-type I at u,*

(b) *(f, g) is pseudoquasi-invex (F, P)-type I at u,*

then the following can not hold:

$$f(x) < f(u). \tag{54}$$

Proof. Suppose contrary to the result that

$$f(x) < f(u) \tag{55}$$

hold.

By $b(x, u) > 0$, the above inequality follows

$$b(x,u)\left(f(x) - f(u)\right) < 0; \tag{56}$$

by assumption (a), (f, g) is pseudoinvex (F, P)-type I at u, which yields

$$\begin{aligned} &F\left(x, u; \alpha_i(x,u)\, \eta^T(x,u)\, \nabla f_i(u)\, P\right) + \rho_i d^2(x,u) < 0, \\ &\qquad\qquad i \in K, \\ &F\left(x, u; \beta_j(x,u)\, \eta^T(x,u)\, \nabla g_j(u)\, P\right) + \tau_j d^2(x,u) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{57}$$

By sublinearity of F together with $\alpha_i(x, u) > 0$ and $\beta_j(x, u) > 0$, the above inequalities yield

$$\begin{aligned} &F\left(x, u; \eta^T(x,u)\, \nabla f_i(u)\, P\right) < -\frac{\rho_i}{\alpha_i(x,u)} d^2(x,u), \\ &\qquad\qquad i \in K, \\ &F\left(x, u; \eta^T(x,u)\, \nabla g_j(u)\, P\right) < -\frac{\tau_j}{\beta_j(x,u)} d^2(x,u), \\ &\qquad\qquad j \in M. \end{aligned} \tag{58}$$

From the feasibility of $(u, \overline{\lambda}, \overline{\mu})$ in Mond-Weir dual problem (MDI), it follows that $\overline{\lambda}_i \geqq 0$, $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, $\overline{\mu}_j \geqq 0$, $j \in M$. Multiplying inequalities (58) by $\overline{\lambda}_i$ and $\overline{\mu}_j$, respectively, together with the sublinearity of F, we get

$$\begin{aligned} &F\left(x, u; \eta^T(x,u)\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u)\right) P\right) \\ &\quad < -\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x,u)} d^2(x,u), \\ &F\left(x, u; \eta^T(x,u)\left(\sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u)\right) P\right) \\ &\quad \leqq -\sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(x,u)} d^2(x,u). \end{aligned} \tag{59}$$

Adding both sides of (59) with the sublinearity of F, we obtain

$$\begin{aligned} &F\left(x, u; \eta^T(x,u)\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u)\right) P\right) \\ &\quad < -\left(\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x,u)} + \sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(x,u)}\right) d^2(x,u). \end{aligned} \tag{60}$$

From assumption, $\sum_{i=1}^{k} (\overline{\lambda}_i \rho_i / \alpha_i(x,u)) + \sum_{j=1}^{m} (\overline{\mu}_j \tau_j / \beta_j(x,u)) \geqq 0$; inequality (60) implies

$$\begin{aligned} &F\left(x, u; \eta^T(x,u)\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u)\right) P\right) \\ &\quad < 0. \end{aligned} \tag{61}$$

By the constraint condition of dual problem (MDI) and $F(x, u; 0) = 0$, we have

$$\begin{aligned} &F\left(x, u; \eta^T(x,u)\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u)\right) P\right) \\ &\quad = 0, \end{aligned} \tag{62}$$

which contradicts inequality (61). Thus, the conclusion of theorem holds.

The proof of part (b) is similar to the proof of part (a). □

Theorem 21 (weak duality). *Let x and $(u, \overline{\lambda}, \overline{\mu})$ be feasible solutions for* (MP) *and* (MDI), *respectively. Moreover, assume that*

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x,u)} + \sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(x,u)} \geqq 0. \tag{63}$$

If one of the following conditions is satisfied:

(a) *(f, g) is pseudoinvex (F, P)-type II at u,*

(b) (f, g) *is pseudoquasi-invex* (F, P)*-type II at* u*, then the following can not hold:*

$$f(x) \leq f(u). \tag{64}$$

Proof. Suppose contrary to the result that

$$f(x) \leq f(u) \tag{65}$$

hold.

By $b(x, u) > 0$, the above inequality yields

$$b(x, u)\left(f(x) - f(u)\right) \leq 0; \tag{66}$$

by assumption (a), (f, g) is pseudoinvex (F, P)-type II at u, which yields

$$\begin{aligned} &F\left(x, u; \alpha_i(x, u)\, \eta^T(x, u)\, \nabla f_i(u)\, P\right) + \rho_i d^2(x, u) < 0, \\ &\qquad\qquad i \in K, \\ &F\left(x, u; \beta_j(x, u)\, \eta^T(x, u)\, \nabla g_j(u)\, P\right) + \tau_j d^2(x, u) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{67}$$

By sublinearity of F together with $\alpha_i(x, u) > 0$ and $\beta_j(x, u) > 0$, the above inequalities yield

$$\begin{aligned} &F\left(x, u; \eta^T(x, u)\, \nabla f_i(u)\, P\right) < -\frac{\rho_i}{\alpha_i(x, u)} d^2(x, u), \\ &\qquad\qquad i \in K, \\ &F\left(x, u; \eta^T(x, u)\, \nabla g_j(u)\, P\right) < -\frac{\tau_j}{\beta_j(x, u)} d^2(x, u), \\ &\qquad\qquad j \in M. \end{aligned} \tag{68}$$

From the feasibility of $(u, \overline{\lambda}, \overline{\mu})$ in Mond-Weir dual problem (MDI), we have

$$\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u) = 0. \tag{69}$$

For $\overline{\lambda}_i \geqq 0$, $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, $\overline{\mu}_j \geqq 0$, $j \in M$. Multiplying inequalities (68) by $\overline{\lambda}_i$ and $\overline{\mu}_j$, respectively, together with the sublinearity of F, we have

$$\begin{aligned} &F\left(x, u; \eta^T(x, u)\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u)\right) P\right) \\ &\quad < -\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x, u)} d^2(x, u), \\ &F\left(x, u; \eta^T(x, u)\left(\sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u)\right) P\right) \\ &\quad \leqq -\sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(x, u)} d^2(x, u). \end{aligned} \tag{70}$$

Adding both sides of (70) with the sublinearity of F, we get

$$\begin{aligned} &F\left(x, u; \eta^T(x, u)\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(u) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(u)\right) P\right) \\ &\quad < -\left(\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x, u)} + \sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(x, u)}\right) d^2(x, u). \end{aligned} \tag{71}$$

Combining (69) and (71), we obtain

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x, u)} + \sum_{j \in \overline{J}(u)} \frac{\overline{\mu}_j \tau_j}{\beta_j(x, u)} < 0, \tag{72}$$

which contradicts the assumption

$$\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(x, u)} + \sum_{j \in \overline{J}(u)} \frac{\overline{\mu}_j \tau_j}{\beta_j(x, u)} \geqq 0. \tag{73}$$

Thus, the conclusion of the theorem holds.

The proof of part (b) is similar to the proof of part (a). □

Theorem 22 (strong duality). *Assume that* $\overline{x}$ *is a weakly efficient solution of* (MP). *Suppose that there exists* $\overline{\lambda} \geq 0$ *and* $\overline{\mu} \geqq 0$*, such that* $(\overline{x}, \overline{\lambda}, \overline{\mu})$ *is feasible for* (MDI). *Furthermore, if the weak duality Theorem 20 holds for all feasible solutions of the problems* (MP) *and* (MDI)*, then* $(\overline{x}, \overline{\lambda}, \overline{\mu})$ *is a weakly efficient solution of* (MDI).

Proof. Suppose that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is not a weakly efficient solution of (MDI); then there exists another feasible solution (u, λ, μ) of (MDI) such that

$$f(\overline{x}) < f(u), \tag{74}$$

which is a contradiction to Theorem 20. Hence $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a weakly efficient solution of (MDI). □

Theorem 23 (strong duality). *Assume that* $\overline{x}$ *is an efficient solution of* (MP). *Suppose that there exist* $\overline{\lambda} \geq 0$ *and* $\overline{\mu} \geqq 0$*, such that* $(\overline{x}, \overline{\lambda}, \overline{\mu})$ *is feasible for* (MDI). *Furthermore, if the weak duality Theorem 21 holds for all feasible solutions of the problems* (MP) *and* (MDI)*, then* $(\overline{x}, \overline{\lambda}, \overline{\mu})$ *is an efficient solution of* (MDI).

Proof. Suppose that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is not an efficient solution of (MDI); then there exists another feasible solution (u, λ, μ) of (MDI) such that

$$f(\overline{x}) \leq f(u), \tag{75}$$

which is a contradiction to Theorem 21. Hence $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is an efficient solution of (MDI). □

Theorem 24 (strict converse duality). *Let* $\overline{x}$ *and* $(\overline{u}, \overline{\lambda}, \overline{\mu})$ *be feasible solutions for* (MP) *and* (MDI)*, respectively. Suppose that* $f(\overline{x}) \leq f(\overline{u})$*, and* $\sum_{i=1}^{k}(\overline{\lambda}_i \rho_i / \alpha_i(\overline{x}, \overline{u})) + \sum_{j=1}^{m}(\overline{\mu}_j \tau_j / \beta_j(\overline{x}, \overline{u})) \geqq 0$*. If one of the following conditions is satisfied:*

(a) (f, g) *is pseudoinvex* (F, P)*-type II at* $\overline{u}$,

(b) (f, g) *is pseudoquasi-invex* (F, P)*-type II at* $\overline{u}$,

then $\overline{x} = \overline{u}$.

Proof. Suppose that $\overline{x} \neq \overline{u}$.

By $b(\overline{x}, \overline{u}) > 0$, the condition $f(\overline{x}) \leq f(\overline{u})$ yields

$$b(\overline{x}, \overline{u})\left(f(\overline{x}) - f(\overline{u})\right) \leq 0, \tag{76}$$

using assumption (a), (f, g) is pseudoinvex (F, P)-type II at $\overline{u}$, which yields

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \alpha_i(\overline{x}, \overline{u})\eta^T(\overline{x}, \overline{u})\nabla f_i(\overline{u})P\right) + \rho_i d^2(\overline{x}, \overline{u}) < 0, \\ &\qquad i \in K, \\ &F\left(\overline{x}, \overline{u}; \beta_j(\overline{x}, \overline{u})\eta^T(\overline{x}, \overline{u})\nabla g_j(\overline{u})P\right) + \tau_j d^2(\overline{x}, \overline{u}) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{77}$$

By sublinearity of F together with $\alpha_i(\overline{x}, \overline{u}) > 0$ and $\beta_j(\overline{x}, \overline{u}) > 0$, the above inequalities yield

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\nabla f_i(\overline{u})P\right) < -\frac{\rho_i}{\alpha_i(\overline{x}, \overline{u})} d^2(\overline{x}, \overline{u}), \\ &\qquad i \in K, \\ &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\nabla g_j(\overline{u})P\right) < -\frac{\tau_j}{\beta_j(\overline{x}, \overline{u})} d^2(\overline{x}, \overline{u}), \\ &\qquad j \in M. \end{aligned} \tag{78}$$

Because $(\overline{u}, \overline{\lambda}, \overline{\mu})$ is feasible for (MDI), then

$$\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{u}) = 0. \tag{79}$$

For $\overline{\lambda}_i \geqq 0$, $\sum_{i=1}^{k} \overline{\lambda}_i = 1$, $\overline{\mu}_j \geqq 0$, $j \in M$. Multiplying inequalities (78) by $\overline{\lambda}_i$ and $\overline{\mu}_j$, respectively, together with the sublinearity of F, we have

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(\overline{u})\right)P\right) \\ &\quad < -\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(\overline{x}, \overline{u})} d^2(\overline{x}, \overline{u}), \\ &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\left(\sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{u})\right)P\right) \\ &\quad \leqq -\sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(\overline{x}, \overline{u})} d^2(\overline{x}, \overline{u}). \end{aligned} \tag{80}$$

Adding both sides of (80) with the sublinearity of F, we get

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{u})\right)P\right) \\ &\quad < -\left(\sum_{i=1}^{k} \frac{\overline{\lambda}_i \rho_i}{\alpha_i(\overline{x}, \overline{u})} + \sum_{j=1}^{m} \frac{\overline{\mu}_j \tau_j}{\beta_j(\overline{x}, \overline{u})}\right) d^2(\overline{x}, \overline{u}). \end{aligned} \tag{81}$$

Using the assumption $\sum_{i=1}^{k}(\overline{\lambda}_i \rho_i / \alpha_i(\overline{x}, \overline{u})) + \sum_{j=1}^{m}(\overline{\mu}_j \tau_j / \beta_j(\overline{x}, \overline{u})) \geqq 0$, inequality (81) follows that

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{u})\right)P\right) \\ &\quad < 0. \end{aligned} \tag{82}$$

Using inequality (79) together with the sublinearity of F, we obtain

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \eta^T(\overline{x}, \overline{u})\left(\sum_{i=1}^{k} \overline{\lambda}_i \nabla f_i(\overline{u}) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(\overline{u})\right)P\right) \\ &\quad = 0, \end{aligned} \tag{83}$$

which is a contradiction to (82). Then $\overline{x} = \overline{u}$.

The proof of part (b) is similar to the proof of part (a). □

Theorem 25 (strict converse duality). *Let* $\overline{x}$ *and* $(\overline{u}, \overline{\lambda}, \overline{\mu})$ *be feasible solutions for* (MP) *and* (MDI), *respectively. Suppose that* $f(\overline{x}) < f(\overline{u})$, *and* $\sum_{i=1}^{k}(\overline{\lambda}_i \rho_i / \alpha_i(\overline{x}, \overline{u})) + \sum_{j=1}^{m}(\overline{\mu}_j \tau_j / \beta_j(\overline{x}, \overline{u})) \geqq 0$. *If one of the following conditions is satisfied:*

(a) (f, g) *is pseudoinvex* (F, P)*-type I at* $\overline{u}$,

(b) (f, g) *is pseudoquasi-invex* (F, P)*-type I at* $\overline{u}$,

then $\overline{x} = \overline{u}$.

Proof. Suppose that $\overline{x} \neq \overline{u}$.

By $b(\overline{x}, \overline{u}) > 0$, the condition $f(\overline{x}) < f(\overline{u})$ yields

$$b(\overline{x}, \overline{u})\left(f(\overline{x}) - f(\overline{u})\right) < 0; \tag{84}$$

using assumption (a), (f, g) is pseudoinvex (F, P)-type I at $\overline{u}$, which yields

$$\begin{aligned} &F\left(\overline{x}, \overline{u}; \alpha_i(\overline{x}, \overline{u})\eta^T(\overline{x}, \overline{u})\nabla f_i(\overline{u})P\right) + \rho_i d^2(\overline{x}, \overline{u}) < 0, \\ &\qquad i \in K, \\ &F\left(\overline{x}, \overline{u}; \beta_j(\overline{x}, \overline{u})\eta^T(\overline{x}, \overline{u})\nabla g_j(\overline{u})P\right) + \tau_j d^2(\overline{x}, \overline{u}) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{85}$$

By using the sublinearity of F with $\alpha_i(\overline{x},\overline{u}) > 0$ and $\beta_j(\overline{x},\overline{u}) > 0$ and $(\overline{u},\overline{\lambda},\overline{\mu}) \in W_1$, inequalities (85) imply

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})\right)P\right) \\ &\quad < -\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(\overline{x},\overline{u})}d^2(\overline{x},\overline{u}), \\ &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad \leqq -\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(\overline{x},\overline{u})}d^2(\overline{x},\overline{u}). \end{aligned} \tag{86}$$

Adding both sides of (86) with the sublinearity of F, we have

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad < -\left(\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(\overline{x},\overline{u})}+\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(\overline{x},\overline{u})}\right)d^2(\overline{x},\overline{u}). \end{aligned} \tag{87}$$

Using the condition $\sum_{i=1}^{k}(\overline{\lambda}_i\rho_i/\alpha_i(\overline{x},\overline{u})) + \sum_{j=1}^{m}(\overline{\mu}_j\tau_j/\beta_j(\overline{x},\overline{u})) \geqq 0$, inequality (87) yields

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad < 0. \end{aligned} \tag{88}$$

According to the constraint condition of (MDI) with $F(\overline{x},\overline{u};0) = 0$, we get

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad = 0, \end{aligned} \tag{89}$$

which is a contradiction to (88). Then $\overline{x} = \overline{u}$.

The proof of part (b) is similar to the proof of part (a). □

5. Wolfe Type Duality

In this section, we consider the Wolfe type dual for (MP) and establish various duality theorems. Let e be the vector of R^k whose components are all ones.

$$\begin{aligned} &\text{Maximize} \quad G(u) = f(u) + \overline{\mu}^T g(u)e = \left(f_1(u)+\sum_{j=1}^{m}\overline{\mu}_j g_j(u), f_2(u)+\sum_{j=1}^{m}\overline{\mu}_j g_j(u), \ldots, f_k(u)+\sum_{j=1}^{m}\overline{\mu}_j g_j(u)\right) \\ &\text{subject to} \quad \sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u)+\sum_{j=1}^{m}\overline{t}_j\nabla g_j(u) = 0, \\ &\qquad \overline{t}_j = 2\overline{\mu}_j, \quad j\in M, \\ &\qquad \overline{\lambda}_i \geqq 0, \quad \sum_{i=1}^{k}\overline{\lambda}_i = 1, \ i\in K, \\ &\qquad \overline{\mu}_j \geqq 0, \quad j\in M. \end{aligned} \tag{MDII}$$

Let $W_2 = \{(u,\overline{\lambda},\overline{\mu},\overline{t}) \in X\times R^k\times R^m\times R^m : \sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u) + \sum_{j=1}^{m}\overline{t}_j g_j(u) = 0, \overline{\lambda}\geq 0, \sum_{i=1}^{k}\overline{\lambda}_i = 1, \overline{\mu}\geqq 0, \overline{t}\geqq 0\}$ be the set of all feasible solutions in problem (MDII).

Theorem 26 (weak duality). *Let $x \in X_0$ and $(u,\overline{\lambda},\overline{\mu},\overline{t}) \in W_2$. Suppose that*

$$\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)}+\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)} \geqq 0. \tag{90}$$

If one of the following conditions is satisfied:

(a) *(G,g) is pseudoinvex (F,P)-type I at u,*

(b) *(G,g) is pseudoquasi-invex (F,P)-type I at u,*

then the following can not hold:

$$f(x) < G(u). \tag{91}$$

Proof. If

$$f(x) < G(u) \tag{92}$$

holds, then we have

$$f(x) < f(u) + \overline{\mu}^T g(u)e. \tag{93}$$

Since $x \in X_0$ and $\overline{\mu} \geqq 0$, the above inequality yields

$$f(x) + \overline{\mu}^T g(x)e < f(u) + \overline{\mu}^T g(u)e; \tag{94}$$

that is,

$$G(x) < G(u); \tag{95}$$

by using $b(x,u) > 0$, we have

$$b(x,u)(G(x) - G(u)) < 0; \tag{96}$$

by assumption (a), (G,g) is pseudoinvex (F,P)-type I at u, which implies

$$\begin{aligned} &F\left(x,u;\alpha_i(x,u)\eta^T(x,u)\left(\nabla f_i(u) + \sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(u)\right)\right. \\ &\quad\left.\cdot P\right) + \rho_i d^2(x,u) < 0, \quad i \in K, \\ &F\left(x,u;\beta_j(x,u)\eta^T(x,u)\nabla g_j(u)P\right) + \tau_j d^2(x,u) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{97}$$

By sublinearity of F together with $\alpha_i(x,u) > 0$, $\beta_j(x,u) > 0$, and the feasibility of $(u,\overline{\lambda},\overline{\mu},\overline{t})$ in (MDII), the above inequalities yield

$$\begin{aligned} &F\left(x,u;\eta^T(x,u)\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u) + \sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(u)\right)P\right) \\ &\quad < -\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)}d^2(x,u), \\ &F\left(x,u;\eta^T(x,u)\left(\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(u)\right)P\right) \\ &\quad \leqq -\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)}d^2(x,u). \end{aligned} \tag{98}$$

Adding both sides of (98) with the sublinearity of F, we get

$$\begin{aligned} &F\left(x,u;\eta^T(x,u)\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u) + \sum_{j=1}^{m}\overline{t}_j\nabla g_j(u)\right)P\right) \\ &\quad < -\left(\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)} + \sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)}\right)d^2(x,u), \\ &\qquad\qquad \left(\overline{t}_j = 2\overline{\mu}_j,\ j \in M\right). \end{aligned} \tag{99}$$

From assumption, $\sum_{i=1}^{k}(\overline{\lambda}_i\rho_i/\alpha_i(x,u)) + \sum_{j=1}^{m}(\overline{\mu}_j\tau_j/\beta_j(x,u)) \geqq 0$; inequality (99) implies

$$\begin{aligned} &F\left(x,u;\eta^T(x,u)\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u) + \sum_{j=1}^{m}\overline{t}_j\nabla g_j(u)\right)P\right) \\ &\quad < 0. \end{aligned} \tag{100}$$

By the constraint condition of (MDII) and $F(x,u;0) = 0$, we have

$$\begin{aligned} &F\left(x,u;\eta^T(x,u)\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u) + \sum_{j=1}^{m}\overline{t}_j\nabla g_j(u)\right)P\right) \\ &\quad = 0, \end{aligned} \tag{101}$$

which contradicts inequality (100). Thus, the conclusion of theorem holds.

The proof of part (b) is similar to the proof of part (a). □

Theorem 27 (weak duality). *Let $x \in X_0$ and $(u,\overline{\lambda},\overline{\mu},\overline{t}) \in W_2$. Suppose that*

$$\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)} + \sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)} \geqq 0. \tag{102}$$

If one of the following conditions is satisfied:

(a) *(G,g) is pseudoinvex (F,P)-type II at u,*

(b) *(G,g) is pseudoquasi-invex (F,P)-type II at u,*

then the following can not hold:

$$f(x) \leq G(u). \tag{103}$$

Proof. Suppose contrary to the result that

$$f(x) \leq G(u) \tag{104}$$

hold. That is,

$$f(x) \leq f(u) + \overline{\mu}^T g(u)e. \tag{105}$$

Since $x \in X_0$ and $\overline{\mu} \geqq 0$, the above inequality yields

$$f(x) + \overline{\mu}^T g(x)e \leq f(u) + \overline{\mu}^T g(u)e; \tag{106}$$

that is,

$$G(x) \leq G(u). \tag{107}$$

By using $b(x,u) > 0$, we get

$$b(x,u)(G(x) - G(u)) \leq 0. \tag{108}$$

By assumption (a), (G,g) is pseudoinvex (F,P)-type II at u, which yields

$$\begin{aligned} &F\left(x,u;\alpha_i(x,u)\eta^T(x,u)\left(\nabla f_i(u) + \sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(u)\right)\right. \\ &\quad\left.\cdot P\right) + \rho_i d^2(x,u) < 0, \quad i \in K, \\ &F\left(x,u;\beta_j(x,u)\eta^T(x,u)\nabla g_j(u)P\right) + \tau_j d^2(x,u) \\ &\quad < 0, \quad j \in M. \end{aligned} \tag{109}$$

By sublinearity of F together with $\alpha_i(x,u) > 0$, $\beta_j(x,u) > 0$, and the feasibility of $(u,\overline{\lambda},\overline{\mu},\overline{t})$ in (MDII), the above inequalities yield

$$\begin{aligned} &F\left(x,u;\eta^T(x,u)\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u)+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(u)\right)P\right)\\ &\quad<-\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)}d^2(x,u),\\ &F\left(x,u;\eta^T(x,u)\left(\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(u)\right)P\right)\\ &\quad\leqq-\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)}d^2(x,u). \end{aligned} \tag{110}$$

Adding both sides of (110) with the sublinearity of F, we get

$$\begin{aligned} &F\left(x,u;\eta^T(x,u)\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(u)+\sum_{j=1}^{m}\overline{t}_j\nabla g_j(u)\right)P\right)\\ &\quad<-\left(\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)}+\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)}\right)d^2(x,u),\\ &\qquad\left(\overline{t}_j=2\overline{\mu}_j,\ j\in M\right). \end{aligned} \tag{111}$$

By using the constraint condition of (MDII) and $F(x,u;0)=0$, inequality (111) implies

$$\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(x,u)}+\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(x,u)}<0, \tag{112}$$

which contradicts the assumption $\sum_{i=1}^{k}(\overline{\lambda}_i\rho_i/\alpha_i(x,u))+\sum_{j=1}^{m}(\overline{\mu}_j\tau_j/\beta_j(x,u))\geqq 0$.

Thus, the conclusion of theorem holds.

The proof of part (b) is similar to the proof of part (a). □

Theorem 28 (strong duality). *Assume that $\overline{x}$ is a weakly efficient solution of* (MP). *Suppose that there exist $\overline{\lambda}\geq 0$, $\overline{\mu}\geqq 0$, and $\overline{t}\geqq 0$, such that $(\overline{x},\overline{\lambda},\overline{\mu},\overline{t})$ is feasible for* (MDII). *Furthermore, if the weak duality Theorem 26 holds for all feasible solutions of the problems* (MP) *and* (MDII), *then $(\overline{x},\overline{\lambda},\overline{\mu},\overline{t})$ is a weakly efficient solution of* (MDII).

Proof. Suppose that $(\overline{x},\overline{\lambda},\overline{\mu},\overline{t})$ is not a weakly efficient solution of (MDII); then there exists another feasible solution (x,λ,μ,t) of (MDII) such that

$$G(\overline{x})<G(u); \tag{113}$$

that is,

$$f(\overline{x})+\overline{\mu}^T g(\overline{x})e<f(u)+\mu^T g(u)e. \tag{114}$$

For $\overline{x}\in X_0$ and $(\overline{x},\overline{\lambda},\overline{\mu},\overline{t})\in W_2$, the above inequality yields

$$f(\overline{x})<f(u)+\mu^T g(u)e; \tag{115}$$

that is,

$$f(\overline{x})<G(u), \tag{116}$$

which is a contradiction to Theorem 26. Hence $(\overline{x},\overline{\lambda},\overline{\mu},\overline{t})$ is a weakly efficient solution of (MDII). □

Theorem 29 (strict converse duality). *Let $\overline{x}$ and $(\overline{u},\overline{\lambda},\overline{\mu},\overline{t})$ be feasible solutions for* (MP) *and* (MDII), *respectively. Suppose that $f(\overline{x})<G(\overline{u})$, and $\sum_{i=1}^{k}(\overline{\lambda}_i\rho_i/\alpha_i(\overline{x},\overline{u}))+\sum_{j=1}^{m}(\overline{\mu}_j\tau_j/\beta_j(\overline{x},\overline{u}))\geqq 0$. If one of the following conditions is satisfied:*

(a) *(f,g) is pseudoinvex (F,P)-type I at $\overline{u}$,*

(b) *(f,g) is pseudoquasi-invex (F,P)-type I at $\overline{u}$,*

then $\overline{x}=\overline{u}$.

Proof. Suppose that $\overline{x}\neq\overline{u}$.

From $\overline{x}\in X_0$ and $(\overline{u},\overline{\lambda},\overline{\mu},\overline{t})\in W_2$, the condition $f(\overline{x})<G(\overline{u})$ yields

$$f(\overline{x})+\overline{\mu}^T g(\overline{x})e<f(\overline{u})+\overline{\mu}^T g(\overline{u})e; \tag{117}$$

that is,

$$G(\overline{x})<G(\overline{u}), \tag{118}$$

by $b(\overline{x},\overline{u})>0$, which implies

$$b(\overline{x},\overline{u})(G(\overline{x})-G(\overline{u}))<0; \tag{119}$$

using assumption (a), (G,g) is pseudoinvex (F,P)-type I at $\overline{u}$, which yields

$$\begin{aligned} &F\left(\overline{x},\overline{u};\alpha_i(\overline{x},\overline{u})\eta^T(\overline{x},\overline{u})\left(\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)\right.\\ &\quad\left.\cdot P\right)+\rho_i d^2(\overline{x},\overline{u})<0,\quad i\in K;\\ &F\left(\overline{x},\overline{u};\beta_j(\overline{x},\overline{u})\eta^T(\overline{x},\overline{u})\nabla g_j(\overline{u})P\right)+\tau_j d^2(\overline{x},\overline{u})\\ &\quad<0,\quad j\in M. \end{aligned} \tag{120}$$

By sublinearity of F together with $\alpha_i(\overline{x},\overline{u}) > 0$, $\beta_j(\overline{x},\overline{u}) > 0$, and $(\overline{u},\overline{\lambda},\overline{\mu},\overline{t}) \in W_2$, the above inequalities yield

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad < -\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(\overline{x},\overline{u})}d^2(\overline{x},\overline{u}), \\ &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{j=1}^{m}\overline{\mu}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad \leqq -\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(\overline{x},\overline{u})}d^2(\overline{x},\overline{u}). \end{aligned} \tag{121}$$

Adding both sides of (121) with the sublinearity of F, we have

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{t}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad < -\left(\sum_{i=1}^{k}\frac{\overline{\lambda}_i\rho_i}{\alpha_i(\overline{x},\overline{u})}+\sum_{j=1}^{m}\frac{\overline{\mu}_j\tau_j}{\beta_j(\overline{x},\overline{u})}\right)d^2(\overline{x},\overline{u}), \\ &\qquad \left(\overline{t}_j = 2\overline{\mu}_j,\ j\in M\right). \end{aligned} \tag{122}$$

Using the assumption $\sum_{i=1}^{k}(\overline{\lambda}_i\rho_i/\alpha_i(\overline{x},\overline{u})) + \sum_{j=1}^{m}(\overline{\mu}_j\tau_j/\beta_j(\overline{x},\overline{u})) \geqq 0$, inequality (122) implies

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{t}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad < 0. \end{aligned} \tag{123}$$

By using the constraint condition of (MDII) and $F(\overline{x},\overline{u};0) = 0$, we obtain

$$\begin{aligned} &F\left(\overline{x},\overline{u};\eta^T(\overline{x},\overline{u})\left(\sum_{i=1}^{k}\overline{\lambda}_i\nabla f_i(\overline{u})+\sum_{j=1}^{m}\overline{t}_j\nabla g_j(\overline{u})\right)P\right) \\ &\quad = 0, \end{aligned} \tag{124}$$

which is a contradiction to (123). Then $\overline{x} = \overline{u}$.

The proof of part (b) is similar to the proof of part (a). □

6. Discussion and Conclusion

In this paper, we study the multiobjective programming problems and two kinds of dual models. Then the sufficient optimality conditions, weak dual, strong dual, and strict converse dual, results are obtained and proved under a class of new generalized invex functions assumptions for the multiobjective programming.

Competing Interests

The authors declare that they have no competing interests.

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