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Research Article

Maximum Temperature in Dry Surface Grinding for High Peclet Number and Arbitrary Heat Flux Profile

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Regarding heat transfer in dry surface grinding, simple asymptotic expressions of the maximum temperature for large Peclet numbers are derived. For this purpose, we consider the most common heat flux profiles reported in the literature, such as constant, linear, triangular, and parabolic. In the constant case, we provide a refinement of the expression given in the literature. In the linear case, we derive the same expression found in the literature, being the latter fitted by using a linear regression. The expressions for the triangular and parabolic cases are novel.

1. Introduction

The computation of the maximum temperature from the grinding conditions in dry surface grinding is essential for predicting thermal damage [1, Section 6.2]. For grinding regimes in which we find a high Peclet number, normally the use of coolant is required to reduce the risk of thermal damage. However, the pressure to reduce production costs and the considerable increase in environmental awareness have led manufacturers to reduce or eliminate cutting fluids in machining operations. In fact, coolant costs are in the range of 10–17% of total manufacturing costs [2]. Therefore, the prediction of the maximum temperature in order to evaluate the risk of thermal damage in dry grinding for a large Peclet number is not negligible. For this purpose, considering a constant heat flux distribution (with a value of q Wm^{-2} in SI units) within the contact width 2ℓ (m) between wheel and workpiece, the maximum temperature $T_c^{\max}$ (K) can be approximated for large Peclet numbers ($L \to \infty$) as

$$T_c^{\max} \approx \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{q\sqrt{k\ell}}{k_0\sqrt{v_f}}, \qquad L = \frac{v_f \ell}{2k} \longrightarrow \infty, \tag{1}$$

where v_f (ms^{-1}) is the constant velocity of the heat source sliding over the workpiece, k is the thermal diffusivity of the workpiece ($\mathrm{m}^2\mathrm{s}^{-1}$), and k_0 is its thermal conductivity ($\mathrm{Jm}^{-1}\mathrm{K}^{-1}\mathrm{s}^{-1}$). This result is first given by Jaeger in [3, Eq. 33], being reported also by Malkin and Guo in [4, Eq. 2]. It is worth noting, on the one hand, that Jaeger omits the derivation of the asymptotic formula (1), being this given in dimensionless form. On the other hand, Malkin and Guo follow Jaeger's analysis replacing the constant factor $2\sqrt{2/\pi}$ by the approximation 1.595 and stating that (1) provides a good approximation for $L > 5$, although in practice it can be applied down to $L = 1$, which includes most of the actual grinding situations (see also [5]).

Despite the fact that (1) has been used widely to predict thermal damage in surface grinding [1, Section 6.2], it seems to be more realistic to assume a linear heat flux distribution with its maximum at the leading edge. Therefore, considering this linear heat flux profile and assuming a geometrical contact length within the wheel and the workpiece [1, Eq. 3-4]

$$2\ell = \sqrt{aD}, \tag{2}$$

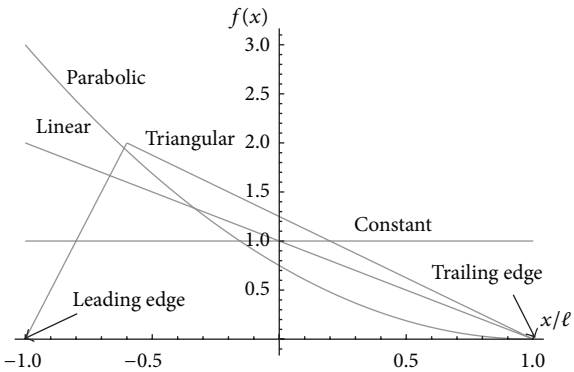

FIGURE 1: Linear, triangular, and parabolic heat flux profiles.

where D is the diameter of the grinding wheel and a is the depth of cut, the maximum temperature $T_l^{\max}$ can be approximated for large Peclet numbers as [6, Eq. 3]

$$T_l^{\max} \approx 1.06 \frac{q k^{1/2} a^{1/4} D^{1/4}}{k_0 v_f^{1/2}}, \tag{3}$$

where q is now the average heat flux along the contact zone. It is worth noting that in [6] the constant factor 1.06 is calculated by performing a linear regression, without any *ab initio* derivation.

Since the most common heat flux profiles reported in the literature are constant [7, 8], linear [8, 9], triangular [10, 11], and parabolic [12] (see Figure 1), this paper is intended, on the one hand, to provide a sound proof for the asymptotic formulas (1) and (3) and, on the other hand, to derive new asymptotic formulas for the triangular and parabolic cases.

This paper is organized as follows. In Section 2, the heat transfer model for surface grinding is presented in order to provide the framework in which the asymptotic formulas of the maximum temperature for large Peclet numbers have to be derived. Section 3 carries out the calculation of these asymptotic expressions considering constant, linear, triangular, and parabolic heat flux profiles. Also, in the constant case, a refinement of (1) is also calculated. Section 4 is devoted to evaluating numerically the accuracy of the asymptotic formulas in order to justify the range in which they are valid. Finally, the conclusions are collected in Section 5.

2. Heat Transfer Model in Surface Grinding

In dry surface grinding, friction due to contact between wheel and workpiece is modelled by an infinite strip heat source of width 2ℓ that slides over the workpiece surface at the plane $z = 0$ and moves at a constant velocity $\vec{v}_f = -|v_f|\vec{i}$ [3]. Assuming a two-dimensional model (see Figure 2), the temperature field of the workpiece $T(x, z)$ with respect to the room temperature in the stationary regime must satisfy the following equation [13, Eqns. 1.6(6) & 1.7(2)]:

$$k\nabla^2 T(x, z) + v_f \partial_x T(x, z) = 0, \tag{4}$$

where $-\infty < x < \infty$ and $z \geq 0$. Considering a dimensionless heat flux profile $f(x)$ within the contact area between wheel and workpiece, (4) is subjected to the following boundary condition:

$$k_0 \partial_z T(x, 0) = q f(x) \theta(\ell - x) \theta(x + \ell), \tag{5}$$

where $\theta(x)$ denotes the Heaviside function and q is the average heat flux entering into the workpiece along the contact width 2ℓ; thus

$$\frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)\, dx = 1. \tag{6}$$

Setting the dimensionless quantities

$$\begin{aligned} X &= \frac{x}{s}, \\ L &= \frac{\ell}{s}, \\ Z &= \frac{z}{s}, \\ \mathcal{T} &= \frac{\pi k_0}{q s} T, \end{aligned} \tag{7}$$

where s is a characteristic length:

$$s = \frac{2k}{v_f}, \tag{8}$$

the solution of (4) and (5) is expressed as [14]

$$\mathcal{T}(X, Z) = \int_{X-L}^{X+L} f(s[X - u]) e^u K_0\left(\sqrt{u^2 + Z^2}\right) du, \tag{9}$$

with K_0 being the zeroth-order *modified Bessel function of the second kind*. On the surface ($Z = 0$), (9) reduces to

$$\mathcal{T}(X, 0) = \int_{X-L}^{X+L} f(s[X - u]) e^u K_0(|u|)\, du. \tag{10}$$

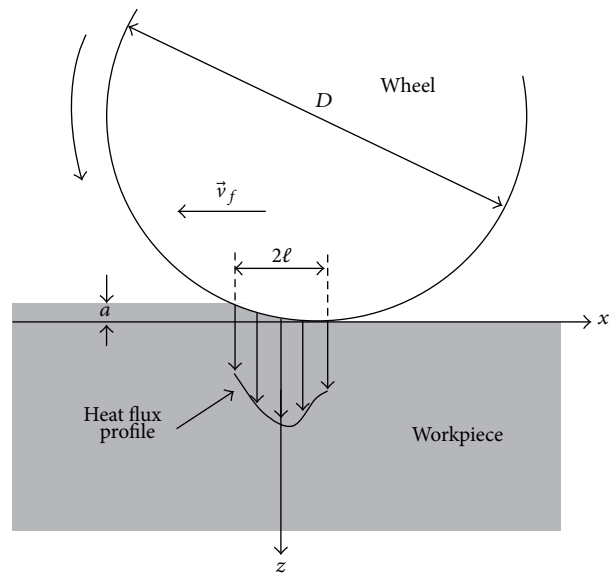

FIGURE 2: Heat transfer modelling in surface grinding.

Assuming that $f(x)$ is an analytic function $\forall x \in [-\ell, \ell]$, then it can be expanded in its Taylor series as

$$f(sX - su) = \sum_{n=0}^{\infty} \frac{f^{(n)}(sX)}{n!} (-su)^n. \tag{11}$$

Substituting (11) in (10), we obtain

$$\mathcal{T}(X, 0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(sX)}{n!} (-s)^n \, \mathrm{Jg}_n(z)\big|_{z=X-L}^{X+L}, \tag{12}$$

where we have defined the following function:

$$\mathrm{Jg}_n(x) = \int_0^x u^n e^u K_0(|u|)\, du, \quad n = 0, 1, 2, \ldots. \tag{13}$$

According to [14], we can calculate the integral given in (13) as

$$\mathrm{Jg}_n(x) = \begin{cases} e^x x^{n+1} \{K_0(|x|)\Psi_n(x) + K_1(|x|)\Phi_n(x)\} + A_n, & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{14}$$

where the following polynomials in $1/x$ in terms of hypergeometric functions are defined as

$$\Psi_n(x) = \frac{1}{n+1}\left[1 - \frac{n}{2n+1} {}_3F_1\left(\begin{matrix} 1, 1-n, -1-n \\ \frac{1}{2} - n \end{matrix}; \frac{1}{2x}\right)\right], \tag{15}$$

$$\Phi_n(x) = \frac{\mathrm{sgn}(x)}{2n+1} {}_3F_1\left(\begin{matrix} 1, -n, -n \\ -n + \frac{1}{2} \end{matrix}; \frac{1}{2x}\right),$$

as well as the constant

$$A_n = \frac{(-2)^{n+1}(n+1)!\,(n!)^2}{(2n+2)!}. \tag{16}$$

In [14], the solution given in (12) is used for computing the maximum temperature in dry surface grinding for the most common heat flux profiles reported in the literature, provided that this maximum must be found in the stationary regime and on the workpiece surface, within the contact zone. Despite the fact that this computation is very rapid, asymptotic expressions of the maximum temperature for large Peclet number ($L \to \infty$) for any dimensionless heat flux profile $f(x)$ can be calculated. This is quite useful not only because these expressions are extremely rapid to compute, but mainly because they provide a starting point for the development of a model of thermal damage in surface grinding as the one elaborated by Malkin and Guo considering a constant heat flux profile [1, Section 6.2].

3. Large Peclet Number Approximations

3.1. Asymptotic Formula for Jg_n *within the Grinding Zone.* Notice that, taking into account the definition of the generalized hypergeometric function [15, Eq. 2.1.2]

$${}_pF_q\left(\begin{matrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{\prod_{r=1}^{p} (\alpha_r)_k}{\prod_{s=1}^{q} (\gamma_s)_k} \frac{z^k}{k!} \tag{17}$$

and the definition of the Pochhammer symbol

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}, \tag{18}$$

we have the following asymptotic behavior of $\Psi_n(x)$:

$$\begin{aligned} \lim_{x \to \pm\infty} \Psi_n(x) &= \lim_{x \to \pm\infty} \frac{1}{n+1}\left[1 - \frac{n}{2n+1} \sum_{k=0}^{\infty} \frac{[(1-n)_k]^2}{(1/2 - n)_k} \left(\frac{1}{2x}\right)^k\right] \\ &= \frac{1}{2n+1}, \end{aligned} \tag{19}$$

and $\Phi_n(x)$

$$\begin{aligned} \lim_{x \to \pm\infty} \Phi_n(x) &= \lim_{x \to \pm\infty} \frac{\pm 1}{2n+1} \sum_{k=0}^{\infty} \frac{[(-n)_k]^2}{(1/2 - n)_k} \left(\frac{1}{2x}\right)^k \\ &= \frac{\pm 1}{2n+1}, \end{aligned} \tag{20}$$

where in both sums given in (19) and (20) only survives the first term (i.e., $k = 0$), when we perform the limit $x \to \pm\infty$.

Now taking into account the asymptotic behavior of the Macdonald function [16, Eq. 5.16.5],

$$K_\nu(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \longrightarrow \infty, \tag{21}$$

and the results (19) and (20), we can calculate the asymptotic behavior of the function $\mathrm{Jg}_n(x)$ given in (14) as

$$\begin{aligned} \lim_{x \to +\infty} \mathrm{Jg}_n(x) &= \lim_{x \to +\infty} e^x x^{n+1} \{K_0(x)\Psi_n(x) + K_1(x)\Phi_n(x)\} \\ &\quad + A_n \\ &= \lim_{x \to +\infty} \frac{x^{n+1}}{2n+1} \sqrt{2\pi x} + A_n. \end{aligned} \tag{22}$$

Similarly

$$\begin{aligned} \lim_{x \to -\infty} \mathrm{Jg}_n(x) &= \lim_{x \to -\infty} e^x x^{n+1} \{K_0(|x|)\Psi_n(x) + K_1(|x|)\Phi_n(x)\} \\ &\quad + A_n \\ &= A_n. \end{aligned} \tag{23}$$

Notice that within the grinding zone

$$-L < X < L \longrightarrow \begin{cases} X + L > 0 \longrightarrow \lim\limits_{L\to\infty} X + L = +\infty, \\ X - L < 0 \longrightarrow \lim\limits_{L\to\infty} X - L = -\infty; \end{cases} \tag{24}$$

thus, according to (22)–(24), we conclude that

$$\mathrm{Jg}_n(z)\big|_{z=X-L}^{X+L} \approx \frac{\sqrt{2\pi}}{2n+1}(X+L)^{n+1/2}, \qquad L \longrightarrow \infty,\ X \in (-L, L). \tag{25}$$

3.2. Constant Profile. Notice that the dimensionless function,

$$f_c(x) = 1, \tag{26}$$

provides a constant heat flux profile satisfying (6) (see Figure 1). Therefore, substituting (26) in (12), we have

$$\boxed{\mathcal{T}_c(X,0) = \mathrm{Jg}_0(z)\big|_{z=X-L}^{X+L}}, \tag{27}$$

so, according to (25), the asymptotic behavior for large Peclet number within the grinding zone is

$$\boxed{\mathcal{T}_c(X,0) \approx \sqrt{2\pi(X+L)}}, \qquad L \longrightarrow \infty,\ X \in (-L, L). \tag{28}$$

Considering both constant and linear heat flux profiles, the maximum temperature in wet grinding is located on the surface within the grinding zone, in the stationary regime, as it is proved in [17]. Also in [17], wet grinding is modelled assuming a constant heat transfer coefficient h over all the workpiece surface. Therefore, setting $h = 0$, we can apply this result of the location of the maximum temperature $\mathcal{T}_c^{\max}$ also to dry grinding in order to find an estimation of $\mathcal{T}_c^{\max}$ for a large Peclet number. Notice that, in (28), the maximum is reached at

$$X_c^{\max} \approx L; \tag{29}$$

since we have a monotonic increasing function within the grinding zone, thus

$$\mathcal{T}_c^{\max}(L) \approx 2\sqrt{\pi L}, \quad L \longrightarrow \infty, \tag{30}$$

and in dimension variables, recalling the definitions (7)-(8), we obtain the result given by Jaeger (1). It is remarkable that Jaeger found this approximation from the integral representation of $\mathrm{Jg}_0(z)$, that is to say (13), and not from its solution, given in (14).

We can improve the approximation given in (28) taking into account the following asymptotic expansion of the Macdonald function [18, Eq. 10.40.2]:

$$K_\nu(z) \approx \sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{\infty}\frac{a_k(\nu)}{z^k}, \quad z \longrightarrow \infty, \tag{31}$$

where, according to [18, Eq. 10.17.1],

$$a_k(\nu) = \frac{1}{k!8^k}\prod_{l=1}^{k}\left(4\nu^2 - (2l-1)^2\right), \quad a_0(\nu) = 1, \tag{32}$$

so

$$\begin{aligned} K_0(z) &\approx \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 - \frac{1}{8z} + \cdots\right), \quad z \longrightarrow \infty, \\ K_1(z) &\approx \sqrt{\frac{\pi}{2z}}e^{-z}\left(1 + \frac{3}{8z} + \cdots\right), \quad z \longrightarrow \infty. \end{aligned} \tag{33}$$

Therefore, the result given in (23) can be refined to the following one:

$$\lim_{x\to-\infty}\mathrm{Jg}_n(x) = \lim_{x\to-\infty}\frac{(-1)^n}{2}\sqrt{\frac{\pi}{2}}\frac{e^{2x}|x|^{n-1/2}}{2n+1} + A_n, \tag{34}$$

and then, taking into account (22), (24), and (34), we arrive at

$$\boxed{\mathcal{T}_c(X,0) = \mathrm{Jg}_0(z)\big|_{z=X-L}^{X+L} \approx \sqrt{2\pi}\left(\sqrt{X+L} - \frac{e^{2(X-L)}}{4\sqrt{L-X}}\right)}, \qquad L \longrightarrow \infty,\ X \in (-L, L). \tag{35}$$

Despite the fact that we cannot solve exactly the maximum of the function given in (35), let us solve it approximately. First, let us perform the change of variables:

$$u = L - X, \tag{36}$$

so that we have to find the maximum of the function:

$$g(u) = \sqrt{2\pi}\left(\sqrt{2L-u} - \frac{e^{-2u}}{4\sqrt{u}}\right). \tag{37}$$

Second, notice that, according to (29),

$$u_{\max} = L - X_c^{\max} \approx 0, \tag{38}$$

so, expanding in Taylor series, we have

$$\begin{aligned} \sqrt{2L-u} &\approx \sqrt{2L} - \frac{u}{2\sqrt{2L}}, \quad u \longrightarrow 0, \\ e^{-2u} &\approx 1, \quad u \longrightarrow 0, \end{aligned} \tag{39}$$

and (37) can be approximated as

$$g(u) \approx \sqrt{2\pi}\left(\sqrt{2L} - \frac{u}{2\sqrt{2L}} - \frac{1}{4\sqrt{u}}\right), \quad u \longrightarrow 0. \tag{40}$$

In order to find out the maximum, let us solve

$$0 = g'(u) \approx \sqrt{2\pi}\left(-\frac{1}{2\sqrt{2L}} + \frac{1}{8u^{3/2}}\right); \tag{41}$$

thus

$$u_{\max} \approx \frac{L^{1/3}}{2} \longrightarrow X_c^{\max} \approx L - \frac{L^{1/3}}{2}. \tag{42}$$

Finally, substituting (42) in (28), we obtain the following refinement of (30):

$$\boxed{\mathcal{T}_c^{\max}(L) \approx \sqrt{\pi\left(4L - L^{1/3}\right)}, \quad L \longrightarrow \infty}. \tag{43}$$

3.3. Linear Profile. The following linear dimensionless function,

$$f_l(x) = 1 - \frac{x}{\ell}, \tag{44}$$

provides null heat flux at the trailing edge $f(\ell) = 0$ and also satisfies (6) (see Figure 1). Therefore, according to (12), we have

$$\boxed{\mathcal{T}_l(X,0) = \left(1 - \frac{X}{L}\right) \mathrm{Jg}_0(z) + \frac{1}{L}\mathrm{Jg}_1(z)\Big|_{z=X-L}^{X+L}}. \tag{45}$$

Applying (25) to (45), we arrive at

$$\boxed{\mathcal{T}_l(X,0) \approx \frac{2\sqrt{2\pi}}{3L}\sqrt{X+L}\,(2L-X)}, \quad L \longrightarrow \infty,\ X \in (-L,L). \tag{46}$$

In order to estimate the maximum temperature for large Peclet number, we can apply again the mathematical proof given in [17], in order to assure that such maximum must occur in the stationary regime and on the surface, within the grinding zone. Therefore, since $\mathcal{T}_l(X,0)$ is a differentiable function in $X \in (-L,L)$, let us calculate the extrema points of the function given in (46); that is to say,

$$0 = \frac{\partial \mathcal{T}_l}{\partial X} \approx -\frac{2\sqrt{2\pi}X}{L\sqrt{X+L}} \longrightarrow X_l^{\max} \approx 0, \quad L \longrightarrow \infty. \tag{47}$$

Note that since (47) has got a unique solution, it must correspond to the maximum. Therefore

$$\mathcal{T}_l^{\max}(L) \approx \frac{4}{3}\sqrt{2\pi L}, \quad L \longrightarrow \infty, \tag{48}$$

and, in dimensional variables, we have

$$T_l^{\max} \approx \frac{8}{3\sqrt{\pi}}\frac{q\sqrt{k\ell}}{k_0\sqrt{v_f}}, \quad \frac{v_f\ell}{2k} \longrightarrow \infty. \tag{49}$$

It is worth noting that assuming a geometrical contact length within the wheel and the workpiece, that is, (2), (49) is rewritten as

$$T_l^{\max} \approx \frac{8}{3\sqrt{2\pi}}\frac{qk^{1/2}a^{1/4}D^{1/4}}{k_0 v_f^{1/2}}, \quad \frac{v_f\ell}{2k} \longrightarrow \infty, \tag{50}$$

which is, in fact, formula (3) given by Malkin and Guo, provided that the factor $8/3\sqrt{2\pi} \approx 1.06384$ has been fitted to 1.06.

3.4. Triangular Profile. The dimensionless triangular heat flux profile satisfying (6) is given by

$$f_\triangle(x) = \begin{cases} \dfrac{2}{1+\lambda}\left(1 + \dfrac{x}{\ell}\right), & x \in (-\ell, \ell_{\max}), \\ \dfrac{2}{1-\lambda}\left(1 - \dfrac{x}{\ell}\right), & x \in (\ell_{\max}, \ell), \end{cases} \tag{51}$$

where $f_\triangle(\pm\ell) = 0$ and the profile apex is located in $x = \ell_{\max} \in [-\ell, \ell]$ (see Figure 1). Also, we have set the dimensionless parameter:

$$\lambda = \frac{\ell_{\max}}{\ell} \in [-1, 1]. \tag{52}$$

Notice that when the heat flux occurs in an arbitrary interval, say $x \in (a,b)$ instead of $x \in (-\ell, \ell)$, the dimensionless surface temperature is given by

$$\mathcal{T}(X,0) = \int_{X-B}^{X-A} f(s[X-u])\, e^u K_0(|u|)\, du, \tag{53}$$

where we have set the following dimensionless parameters:

$$A = \frac{a}{s}, \qquad B = \frac{b}{s}. \tag{54}$$

Since (4) is a linear differential equation, the field temperature is given by the superposition of both parts of (51), so according to (12), we have

$$\begin{aligned} \mathcal{T}_\triangle(X,0) &= \frac{2}{1+\lambda}\int_{X-\lambda L}^{X+L}\left(1 + \frac{X-u}{L}\right) e^u K_0(|u|)\, du \\ &\quad + \frac{1}{1-\lambda}\int_{X-L}^{X-\lambda L}\left(1 - \frac{X-u}{L}\right) e^u K_0(|u|)\, du. \end{aligned} \tag{55}$$

Applying (13) and reordering terms, we arrive at

$$\boxed{\mathcal{T}_\triangle(X,0) = \frac{H_L(X)}{1+\lambda} + \frac{H_{-L}(X)}{1-\lambda} + \frac{2\lambda}{1-\lambda^2}H_{-\lambda L}(X)}, \tag{56}$$

where we have defined

$$H_\Lambda(X) = 2\mathrm{Jg}_0(X+\Lambda)\left(1 + \frac{X}{\Lambda}\right) - \frac{2}{\Lambda}\mathrm{Jg}_1(X+\Lambda). \tag{57}$$

On the one hand, notice that calculating the limit in (57) when $\Lambda = L \to \infty$, according to (22) and (24), we can conclude that

$$H_L(X) \approx \frac{4\sqrt{2\pi}}{3L}(X+L)^{3/2}, \quad L \to \infty,\ X \in (-L,L). \tag{58}$$

Similarly, taking into account (23) and (24), we arrive at

$$H_{-L}(X) \approx 2\frac{X-L}{L}, \quad L \to \infty,\ X \in (-L,L). \tag{59}$$

On the other hand, note that within grinding zone, on the left-hand side of the apex, we have

$$-L < X < \lambda L \longrightarrow X - \lambda L < 0 \longrightarrow \lim_{L\to\infty} X - \lambda L = -\infty, \tag{60}$$

and, on the right-hand side of the apex, we have

$$\lambda L < X < L \longrightarrow X - \lambda L > 0 \longrightarrow \lim_{L\to\infty} X - \lambda L = +\infty; \tag{61}$$

thus

$$H_{-\lambda L}(X) \approx \begin{cases} 2\dfrac{X-\lambda L}{\lambda L}, & X \in (-L, \lambda L), \\ -\dfrac{4\sqrt{2\pi}}{3\lambda L}(X+L)^{3/2}, & X \in (\lambda L, L), \end{cases} \quad L \longrightarrow \infty. \tag{62}$$

Taking into account (58), (59), and (62) in (56), after some algebra, we arrive at

$$\boxed{\mathcal{T}_{\triangle}(X,0) \approx \begin{cases} \dfrac{4\sqrt{2\pi}}{3L(1+\lambda)}(X+L)^{3/2}, & X \in (-L, \lambda L), \\ \dfrac{4\sqrt{2\pi}}{3L(1+\lambda)}\left[(X+L)^{3/2} - \dfrac{2(X-\lambda L)^{3/2}}{1-\lambda}\right], & X \in (\lambda L, L) \end{cases}}, \quad L \longrightarrow \infty. \tag{63}$$

Despite the fact we do not possess any mathematical proof for the location of the maximum temperature, we have to assume now that this one must be found in the stationary regime and on the surface, within the contact zone, as in the constant and linear cases. In fact, this is a quite natural assumption, physically speaking. Bearing this in mind, notice that, according to (63), the maximum cannot lie on the left-hand side of the apex. Therefore, since the maximum must have null derivative, let us solve

$$0 = \frac{\partial \mathcal{T}_{\triangle}}{\partial X} \approx \frac{2\sqrt{2\pi}}{L(1+\lambda)}\left[(X+L)^{1/2} - \frac{2}{1-\lambda}(X-\lambda L)^{1/2}\right]; \tag{64}$$

thus

$$X_{\triangle}^{\max} \approx \frac{L(1+\lambda)}{3-\lambda}, \quad L \longrightarrow \infty, \tag{65}$$

and the maximum temperature is then given by

$$\mathcal{T}_{\triangle}^{\max}(L) \approx \frac{8}{3}\sqrt{\frac{2\pi L}{3-\lambda}}, \quad L \longrightarrow \infty. \tag{66}$$

Notice that if the apex of the triangular profile tends to the leading edge, that is to say $\lambda \to -1$, then the profile is linear and (66) becomes (48).

3.5. Parabolic Profile. The dimensionless parabolic heat flux profile satisfying (6) is given by

$$f_p(x) = \frac{3}{4}\left(1 - \frac{x}{\ell}\right)^2, \tag{67}$$

where the heat flux at the trailing edge fulfills that $f_p(\ell) = 0$ and $f_p'(\ell) = 0$. Therefore, according to (12), we obtain

$$\boxed{\mathcal{T}_p(X,0) = \frac{3}{4L^2}\left[(L-X)^2 \operatorname{Jg}_0(z) + 2(L-X)\operatorname{Jg}_1(z) + \operatorname{Jg}_2(z)\right]_{z=X-L}^{X+L}}. \tag{68}$$

Taking into account (25), we have the following asymptotic behavior for large Peclet number within the grinding zone:

$$\boxed{\mathcal{T}_p(X,0) \approx \frac{\sqrt{2\pi}}{5L^2}\sqrt{X+L}\left(2X^2 - 6LX + 7L^2\right)}, \quad L \longrightarrow \infty, \ X \in (-L, L). \tag{69}$$

As in the triangular case, let us consider the fact that the maximum temperature is located on the surface, within the grinding zone, in the stationary regime. Therefore, in order to find out the maximum, let us solve

$$0 = \frac{\partial \mathcal{T}_p}{\partial X} \approx \frac{\sqrt{2\pi}\left(2X^2 - 2LX - L^2\right)}{2L^2\sqrt{X+L}}, \tag{70}$$

so

$$X_p^{\max} \approx \frac{L}{2}\left(1 \pm \sqrt{3}\right). \tag{71}$$

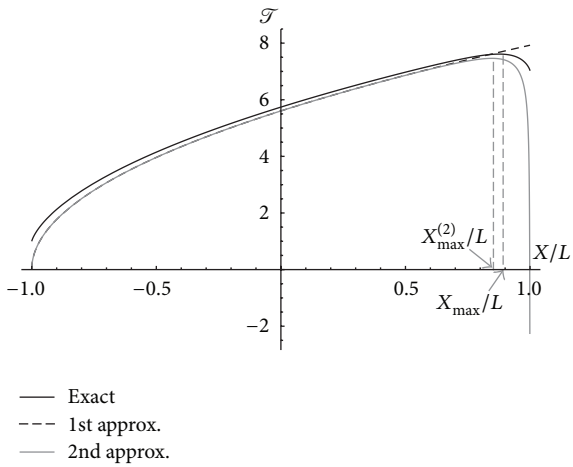

Figure 3: Dimensionless surface temperature for a constant heat flux profile taking $L = 5$.

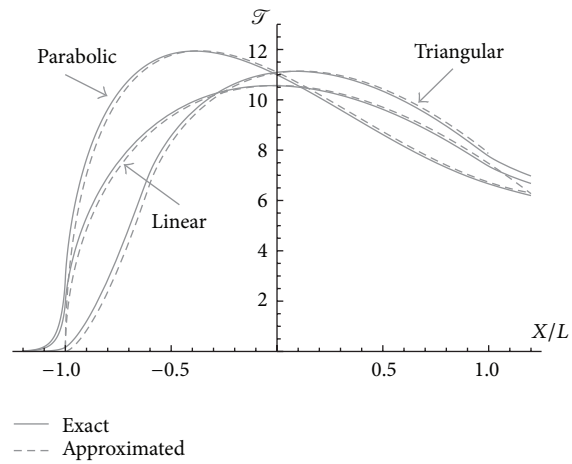

Figure 4: Dimensionless temperature (solid line) and its approximation (dashed line) taking $L = 5$.

Since the "+" sign in (71) leads to a value out of the interval $(-L, L)$, we have to choose the "−" sign. Substitution of the latter value in (69) yields

$$\mathcal{T}_p^{\max}(L) \approx \frac{2}{5}\sqrt{6\pi L\left(3+\sqrt{3}\right)}, \quad L \longrightarrow \infty. \tag{72}$$

It is worth noting that substitution of (71) in (69) taking the "+" sign leads to the same result as (72), but with a "$-\sqrt{3}$", so anyway it is clear that (72) is the approximation to the maximum.

4. Numerical Results

In this section we examine the goodness of the asymptotic approximations given in Section 3 for large Peclet number. Figure 3 shows the dimensionless temperature in the stationary regime on the surface $\mathcal{T}(X, 0)$, considering a constant heat flux profile (27), as well as the asymptotic approximations (28) and (35). The graphs are plotted within the grinding zone $X \in (-L, L)$ and taking $L = 5$. Note that the maximum lies nearby the trailing edge within the grinding zone. Also, the first and second approximations, (28) and (35), respectively, overlap everywhere except for the zone nearby the maximum, where the second approximation has got a vertical asymptote. Note as well in Figure 3 that the location of the maximum $X_{\max}$ can be better estimated by the second approximation $X^{(2)}_{\max}$, given in (42), than by the first approximation $X^{(1)}_{\max} = L$, given in (29). However, the maximum temperature is better estimated by using the first approximation (28) at $X^{(2)}_{\max}$, that is to say, the approximation given in (43). These considerations justify the procedure followed for the refinement of the maximum temperature approximation performed in Section 3.2.

In Figure 4 is plotted the dimensionless temperature on the surface $\mathcal{T}(X, 0)$ for linear, triangular, and parabolic heat flux profiles, (45), (56), and (68), respectively, as well as its asymptotic approximations, (46), (63), and (69), taking $L = 10$ for all the graphs and $\lambda = -0.6$ for the triangular case. We can appreciate that the asymptotic approximations are quite near to the exact solutions within the contact zone $X \in (-L, L)$.

In order to evaluate quantitatively the distance between the exact solution and the asymptotic approximation as a function of L, we can use the functional defined in [19]. If $f(x)$ and $g(x)$ are two nonnegative functions defined in an interval I, the relative distance between them is defined as

$$\Delta_I(f, g) = \begin{cases} \dfrac{\int_I |f(x) - g(x)|\, dx}{\int_I [f(x) + g(x)]\, dx}, & f \text{ or } g \neq 0, \\ 0, & f = g = 0. \end{cases} \tag{73}$$

It is easy to prove that (see [19])

$$0 \le \Delta_I(f, g) \le 1, \tag{74}$$

where a value close to 0 means relatively near, but a value close to 1 means relatively infinitely far. Figure 5 shows the computation of (73), taking as functions f and g the exact solution $\mathcal{T}(X, 0)$ given in (12) and the asymptotic approximation $\mathcal{T}_{\text{approx}}(X, 0)$ of the surface temperature for the different heat flux profiles considered, within the interval $I = (-L, L)$. We can see that, for $L \ge 10$, the asymptotic approximation is relatively quite near to the exact solution, regardless of the heat flux profile considered. Note that second approximation of the constant case (35) is worse than the first approximation (28) because we are considering the distance within the grinding zone $I = (-L, L)$; and nearby the maximum, the first approximation fits better (see Figure 3).

Nevertheless, the behavior of the maximum is a little bit different from the global behavior of the asymptotic approximation with respect to the exact solution. To see it quantitatively, let us define the relative error as

$$\begin{aligned} \delta_{\max}(L) &= \frac{\left|T^{\max}_{\text{approx}}(L) - T^{\max}_{\text{exact}}(L)\right|}{T^{\max}_{\text{exact}}(L)} \\ &= \left|\frac{\mathcal{T}^{\max}_{\text{approx}}(L)}{\mathcal{T}^{\max}_{\text{exact}}(L)} - 1\right|. \end{aligned} \tag{75}$$

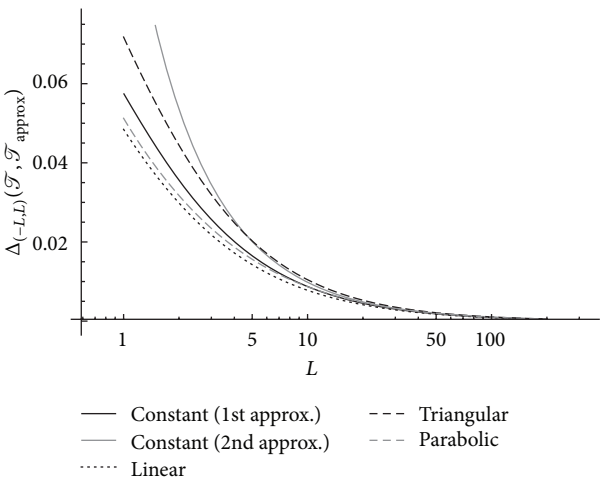

Figure 5: Relative distance between $\mathcal{T}(X, 0)$ and its approximation for large Peclet number within the interval $X \in (-L, L)$.

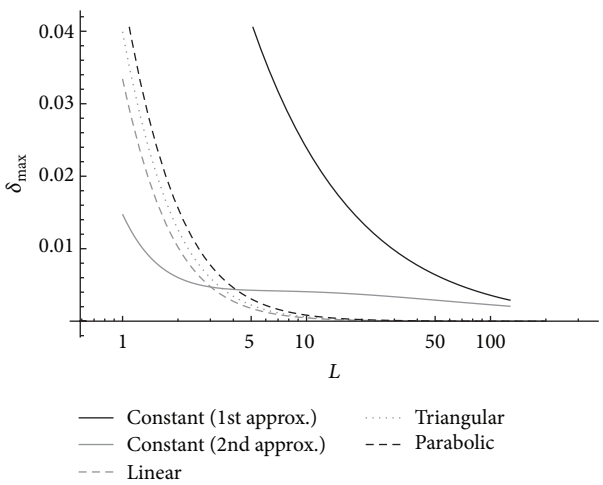

Figure 6: Relative error of the maximum temperature approximation.

Figure 6 presents the computation of (75) for the different heat flux profiles considered, where the exact value of the maximum $\mathcal{T}_{\text{exact}}^{\max}$ has been computed according to [14]. It is worth noting that, eliminating the absolute value bars in (75), we can check that all the asymptotic approximations of the maximum temperatures overestimate the exact value, except for the second approximation of the constant case. Notice as well that the behavior of the second approximation for the constant case substantially improves the first approximation, which confirms the derivation given in the previous section.

5. Conclusions

We have considered the heat transfer in dry surface grinding in the stationary regime. Assuming the most common heat flux profiles proposed in the literature (constant, linear, triangular, and parabolic), we have derived very simple asymptotic expressions of the maximum temperature for high Peclet numbers. In the constant case, we have found a refinement of the expression found in the literature, which is quite accurate (an error less than 1% for $L \geq 2$). In the linear case, we have arrived at an expression which is reported in the literature by using a linear regression to fit the multiplicative constant. The expressions for the triangular and parabolic cases seem not to be reported in the literature.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. Malkin and C. Guo, *Grinding Technology: Theory and Application of Machining with Abrasives*, Industrial Press Inc, New York, NY, USA, 2008.

[2] U. Segurajauregui and P. J. Arrazola, "Heat-flow determination through inverse identification in drilling of aluminium workpieces with MQL," *Production Engineering*, vol. 9, no. 4, pp. 517–526, 2015.

[3] J. C. Jaeger, "Moving sources of heat and the temperature at sliding contracts," *Proceedings of the Royal Society of New South Wales*, vol. 76, pp. 204–224, 1942.

[4] S. Malkin and C. Guo, "Thermal analysis of grinding," *CIRP Annals—Manufacturing Technology*, vol. 56, no. 2, pp. 760–782, 2007.

[5] S. Malkin and E. Lenz, "Burning limits for surface and cylindrical grinding of steels," *CIRP Annals—Manufacturing Technology*, vol. 27, pp. 233–236, 1978.

[6] C. Guo and S. Malkin, "Analytical and experimental investigation of burnout in creep-feed grinding," *CIRP Annals—Manufacturing Technology*, vol. 43, no. 1, pp. 283–286, 1994.

[7] A. S. Lavine, S. Malkin, and T. C. Jen, "Thermal aspects of grinding with CBN wheels," *CIRP Annals—Manufacturing Technology*, vol. 38, no. 1, pp. 557–560, 1989.

[8] C. Guo and S. Malkin, "Analysis of energy partition in grinding," *Journal of Engineering for Industry*, vol. 117, no. 1, pp. 55–61, 1995.

[9] C. Guo, Y. Wu, V. Varghese, and S. Malkin, "Temperatures and energy partition for grinding with vitrified CBN wheels," *CIRP Annals—Manufacturing Technology*, vol. 48, no. 1, pp. 247–250, 1999.

[10] M. Mahdi and L. Zhang, "Applied mechanics in grinding—VI. Residual stresses and surface hardening by coupled thermoplasticity and phase transformation," *International Journal of Machine Tools and Manufacture*, vol. 38, no. 10-11, pp. 1289–1304, 1998.

[11] I. Zarudi and L. C. Zhang, "A revisit to some wheel-workpiece interaction problems in surface grinding," *International Journal of Machine Tools and Manufacture*, vol. 42, no. 8, pp. 905–913, 2002.

[12] W. B. Rowe, S. C. E. Black, B. Mills, H. S. Qi, and M. N. Morgan, "Experimental investigation of heat transfer in grinding," *CIRP Annals—Manufacturing Technology*, vol. 44, no. 1, pp. 329–332, 1995.

[13] H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, Oxford, UK, 1988.

[14] J. L. González-Santander, "Calculation of an integral arising in dry flat grinding for a general heat flux profile. Application to maximum temperature evaluation," *Journal of Engineering Mathematics*, vol. 88, pp. 137–160, 2014.

[15] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, UK, 1999.

[16] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications, New York, NY, USA, 1972.

[17] J. L. González-Santander and G. Martín, "A theorem for finding maximum temperature in wet grinding," *Mathematical Problems in Engineering*, vol. 2015, Article ID 150493, 13 pages, 2015.

[18] F. W. J. Olver, DW. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, NY, USA, 2010.

[19] J. L. González-Santander and G. M. González, "Relative distance between two scalar fields. Application to mathematical modelling approximation," *Mathematical Methods in the Applied Sciences*, vol. 37, no. 18, pp. 2906–2922, 2014.