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Research Article

New Exponential Stability Results for Neutral Stochastic Systems with Distributed Delays

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This paper studies the stability problem of a grey neutral stochastic system with distributed delays. The main feature of a grey system is that the parameters of system are evaluated by grey numbers, and system has the strong background in engineering applications. However, the literature dealing with the stability problem for grey system seems to be scarce. The main aim of this paper is to fill the gap. Based on the Lyapunov stability theorem and Itô's formula, especially, using the decomposition technique of the continuous matrix-covered sets of grey matrix, we propose several novel sufficient conditions, which ensure our considered grey system in mean-square exponential stability and almost surely exponential stability. Furthermore, two examples are provided to show the effectiveness of the obtained results.

1. Introduction

In various physical and engineering systems, time-delay frequently causes oscillation, bifurcation, or instability. Meanwhile, the influence of stochastic phenomenon also cannot be ignored. So, time-delay stochastic system has gained many scholars' attention and interest (see [1–9]). As pointed out in [10, 11], if the differences between nearby argument values are reduced, but the number of summands is incremental, we will find distributed delays phenomenon. In addition, we can describe many practical systems by neutral differential equations, Hence, neutral stochastic system with distributed delays has been extensively investigated by many scholars (see [12–20]). For instance, in [12], stability analysis of neutral type neural networks has been studied. Authors have proposed several delay-dependent criteria to ensure the networks to have a balance point. In [13], by the Lyapunov-Krasovkii functional, authors have studied the global exponential stability of neutral type neural networks with distributed time delays, and authors have presented new sufficient conditions of the system.

It is well known that we cannot obtain parameters of systems accurately. Naturally, in order to reflect many real systems, we have to evaluate the parameters of systems. When we evaluate the parameters of system by grey numbers, the system will be described uncertainly and turn into grey system (see [21]). Recently, grey system has come to play the important role in many fields and can be widely applied to physical and engineering systems, such as population ecology, communication systems, electric power systems, and so on. Due to the strong background in engineering applications, the stability analysis for grey system has become a focused topic of theoretical as well as practical importance. But, little attention has been taken to the stability problem for grey system except some paper [21–25]. In [21], the problem of exponential stability for a grey stochastic system with distributed delays has been studied, and the authors have proposed the delay-dependent criteria, which ensure the system in p-moment exponential robust stability. In [22], the authors have investigated robust stability problem for a grey stochastic nonlinear system with distributed delays, and several exponential stability criteria have been derived in terms of LMIs.

Motivated by the above discussion, using a similar method of [24], we will address the stability problem of grey system here. The main work and contribution of this paper are given as follows. (1) Compared with [24], this paper studies exponential stability problem of grey nonlinear systems with distributed delays for the first time, and the

obtained results are more effective and general. (2) Different from traditional methods [10, 14, 20], for the uncertain systems, combining with concepts and conclusions of the Grey Theory (see Definitions 1 and 2 and Lemma 3), we adopt the new decomposition technique of the continuous matrix-covered sets of grey matrix. Then, several sufficient conditions are derived to ensure the systems in mean-square exponential stability and almost surely exponential stability. (3) Since the obtained criteria are presented LMIs, it can be easily solved by the Matlab LMI Control Toolbox. Meanwhile, two simple examples to illustrate the effectiveness of the obtained criteria are provided and solved.

The notation in this paper is fairly standard. The superscript "T" represents the transpose and $\|\cdot\|$ denotes the Euclidean norm. The notation $X > 0$, where X is real symmetric matrix, means that X is positive definite. Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a probability space with a filtration $\{F_t\}_{t\geq 0}$ and $C([-\tau,0];R^n)$ be the family of all continuous R^n-valued functions φ on $[-\tau,0]$ and $L^2_{F_0}([-\tau,0];R^n)$ be the family of all F_0-measurable bounded $C([-\tau,0];R^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$.

2. Preliminaries and Problem Formulation

In this section, we consider the following grey system:

$$\begin{aligned} &d\left[x(t) - H_I x(t-\tau)\right] \\ &\quad = \left[A_I x(t) + B_I x(t-\tau) + \int_0^\tau G(s)\, x(t-s)\, ds\right] dt \\ &\qquad + f(x(t), x(t-\tau), t)\, dw(t), \quad t \geq 0 \\ &x_0 = \xi, \quad \xi \in L^2_{F_0}([-\tau,0];R^n), \ -\tau \leq t \leq 0, \end{aligned} \tag{1}$$

where A_I, B_I, and H_I are interval matrices, and

$$\begin{aligned} A_I &= [L_a, U_a] \\ &= \left\{A = (a_{ij}) : \underline{a_{ij}} \leq a_{ij} \leq \overline{a_{ij}},\ i,j = 1,2,\ldots,n\right\}, \\ B_I &= [L_b, U_b] \\ &= \left\{B = (b_{ij}) : \underline{b_{ij}} \leq b_{ij} \leq \overline{b_{ij}},\ i,j = 1,2,\ldots,n\right\}, \\ H_I &= [L_h, U_h] \\ &= \left\{H = (h_{ij}) : \underline{h_{ij}} \leq h_{ij} \leq \overline{h_{ij}},\ i,j = 1,2,\ldots,n\right\} \end{aligned} \tag{2}$$

are called the continuous matrix-covered sets of grey matrices A, B, and H.

For grey system (1), we make the following assumptions and definitions:

(H1) $f : R^n \times R^n \times R_+ \to R^{n\times n}$ is locally Lipschitz continuous and satisfies the linear growth condition.

(H2) There are constants $\alpha \geq 0$, $\beta \geq 0$, for arbitrary $(x,y,t) \in f : R^n \times R^n \times R_+$; we have

$$\operatorname{Trace}\left[f^T(x,y,t)\, f(x,y,t)\right] \leq \alpha |x|^2 + \beta |y|^2. \tag{3}$$

Definition 1 (see [21]). Grey system (1) is mean square exponentially stable, if for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$ and arbitrary $A \in A_I$, $B \in B_I$, $H \in H_I$, there exist constants $r > 0$ and $C > 0$, such that

$$E\left|x(t;\xi)\right|^2 \leq Ce^{-rt} \sup_{-\tau\leq\theta\leq 0} E\left|\xi(\theta)\right|^2, \quad t \geq 0. \tag{4}$$

Definition 2 (see [21]). Grey system (1) is almost surely exponentially robustly stable, if for all $\xi \in L^2_{F_0}([-\tau,0];R^n)$ and arbitrary $A \in A_I$, $B \in B_I$, $H \in H_I$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left|x(t;\xi)\right| \leq -\frac{\hat{r}}{2}, \quad \text{a.s.} \tag{5}$$

To prove our main results, the following lemmas are very important (in particular, Lemma 3).

Lemma 3 (see [21]). *If $A = (a_{ij})_{n\times n}$ is a grey matrix, $[\underline{a_{ij}}, \overline{a_{ij}}]$ is a number-covered sets of a_{ij}; then for arbitrary matrix $A \in A_I$, we have*

(1) $A = (U_a + L_a)/2 + \Delta A$,

(2) $0 \leq \Delta A \leq (U_a - L_a)/2$,

(3) $\|A\| \leq \|(U_a + L_a)/2\| + \|(U_a - L_a)/2\|$,

where $L_a = (\underline{a_{ij}})_{n\times n}$, $U_a = (\overline{a_{ij}})_{n\times n}$, $\Delta A = (((\overline{a_{ij}} - \underline{a_{ij}})/2) r_{ij})_{n\times n}$, and $r_{ij} \in [-1,1]$.

Lemma 4 (see [26]). *For any real vectors x, y and any matrix $P > 0$ with appropriate dimensions, $M, N \in R^{n\times n}$, $\varepsilon > 0$, we have*

$$2x^T M^T P N y \leq \varepsilon x^T M^T P M x + \varepsilon^{-1} y^T N^T P N y. \tag{6}$$

Lemma 5 (see [27]). *For a given matrix $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$ with $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$, the following conditions are equivalent:*

(1) $S < 0$,

(2) $S_{22} < 0$, $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

3. Main Results and Proofs

Several novel stability conditions are proposed to guarantee grey system (1) in the mean-square exponential stability and almost surely exponential stability. The main results are given in the following theorems and corollaries.

Theorem 6. *System (1) is mean square exponentially stable, if $k = \|(U_h+L_h)/2\| + \|(U_h-L_h)/2\| < 1$ and there exist symmetric matrices $P > 0$, $Q > 0$, $R > 0$ and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, such that*

$$\lambda_{\min}(R) \geq \left(\varepsilon_2^{-1} + \varepsilon_4^{-1}\right) \tau \sup_{0\leq s\leq\tau} \|G(s)\|^2, \tag{7}$$

$$\begin{pmatrix} \Delta_1 & P\dfrac{U_b + L_b}{2} & M \\ \dfrac{U_b^T + L_b^T}{2} P & \Delta_2 & 0 \\ M^T & 0 & -J \end{pmatrix} < 0, \tag{8}$$

where

$$\begin{aligned}\Delta_1 &= Q + \tau R + P\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}P + \lambda_{\max}(P) \\ &\quad \times \left[2\left\|\frac{U_a - L_a}{2}\right\| + \varepsilon_2 + \alpha\right] I_n + \varepsilon_3^{-1}\left(\left\|\frac{U_a + L_a}{2}\right\| \right. \\ &\quad \left. + \left\|\frac{U_a - L_a}{2}\right\|\right)^2 I_n + \varepsilon_1 P^2, \\ \Delta_2 &= -Q + \left[\varepsilon_1^{-1}\left\|\frac{U_b - L_b}{2}\right\|^2 + \varepsilon_3\left(\left\|\frac{U_h + L_h}{2}\right\| \right.\right. \\ &\quad \left. + \left\|\frac{U_h - L_h}{2}\right\|\right)^2 + \varepsilon_4\left(\left\|\frac{U_h + L_h}{2}\right\| + \left\|\frac{U_h - L_h}{2}\right\|\right)^2 \\ &\quad + 2\left(\left\|\frac{U_h + L_h}{2}\right\| \cdot \left\|\frac{U_b + L_b}{2}\right\| + \left\|\frac{U_h + L_h}{2}\right\| \right. \\ &\quad \cdot \left\|\frac{U_b - L_b}{2}\right\| + \left\|\frac{U_h - L_h}{2}\right\| \cdot \left\|\frac{U_b + L_b}{2}\right\| + \left\|\frac{U_h - L_h}{2}\right\| \\ &\quad \left.\left. \cdot \left\|\frac{U_b - L_b}{2}\right\|\right) + \lambda_{\max}(P) \times \beta\right] I_n, \\ M &= \begin{pmatrix} P & P & P & P \end{pmatrix}, \\ J &= \operatorname{diag}\left(\varepsilon_1 I_n \quad \varepsilon_2 I_n \quad \varepsilon_3 I_n \quad \varepsilon_4 I_n\right).\end{aligned} \tag{9}$$

Then, for all $\xi \in L^2_{F_0}([-\tau, 0]; R^n)$, *we have*

$$E\left|x(t,\xi)\right|^2 \le \frac{C_0}{(1-k)(1-ke^{r\tau})} e^{-rt} \sup_{-\tau\le\theta\le 0} E\left|\xi(\theta)\right|^2; \tag{10}$$

here,

$$\begin{aligned} C_0 &= \lambda_{\max}(P)\left(1+k^2\right)\left(2 + r\tau e^{r\tau}\right) \\ &\quad + \left(r\tau^2 e^{r\tau} + \tau\right)\lambda_{\max}(Q) \\ &\quad + \left(r\tau^3 e^{r\tau} + \tau^2\right)\lambda_{\max}(R), \end{aligned} \tag{11}$$

and r is the unique positive solution of the following equation

$$\begin{aligned} &r\lambda_{\max}(P)\left(1+k^2\right)\left(1+e^{r\tau}\right) \\ &\quad + \left[\lambda_{\max}(Q) + \tau\lambda_{\max}(R)\right] r\tau e^{r\tau} + \lambda_{\max}(\Psi) = 0, \end{aligned} \tag{12}$$

$$\text{with } \Psi := \begin{pmatrix} \Delta_1 + \sum_{i=1}^{4}\varepsilon_i^{-1}P^2 & P\dfrac{U_b + L_b}{2} \\ \dfrac{U_b^T + L_b^T}{2}P & \Delta_2 \end{pmatrix} < 0, \tag{13}$$

and $1 + r\tau > 0$, $1 - ke^{r\tau} > 0$.

Proof. Using Lemma 5,

$$\begin{pmatrix} \Delta_1 & P\dfrac{U_b + L_b}{2} & M \\ \dfrac{U_b^T + L_b^T}{2}P & \Delta_2 & 0 \\ M^T & 0 & -J \end{pmatrix} < 0 \tag{14}$$

implies that

$$\psi := \begin{pmatrix} \Delta_1 + \sum_{i=1}^{4}\varepsilon_i^{-1}P^2 & P\dfrac{U_b + L_b}{2} \\ \dfrac{U_b^T + L_b^T}{2}P & \Delta_2 \end{pmatrix} < 0. \tag{15}$$

For convenience, let

$$\begin{aligned} x(t,\xi) &= x(t), \\ y(t) &= x(t-\tau), \\ z(t) &= \int_0^\tau G(s)\, x(t-s)\, ds, \\ \eta(t) &= x(t) - H_I x(t-\tau). \end{aligned} \tag{16}$$

Firstly, arbitrarily fix $\xi \in L^2_{F_0}([-\tau, 0]; R^n)$, and $A \in A_I$, $B \in B_I$, $H \in H_I$; we introduce the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V(x(t), t) &= \eta^T(t) P\eta(t) + \int_{t-\tau}^{t} x^T(s) Q x(s)\, ds \\ &\quad + \int_{-\tau}^{0}\int_{t+\theta}^{t} x^T(s) R x(s)\, ds\, d\theta. \end{aligned} \tag{17}$$

Using Itô's differential formula, we have

$$\begin{aligned} &LV(x(t), t) \\ &\quad = x^T(t)(Q + \tau R)x(t) - y^T(t) Q y(t) \\ &\qquad - \int_{t-\tau}^{t} x^T(s) R x(s)\, ds + 2x^T(t) PAx(t) \\ &\qquad + 2x^T(t) PBy(t) + 2x^T(t) Pz(t) \\ &\qquad - 2y^T(t) H^T Ax(t) - 2y^T(t) H^T By(t) \\ &\qquad - 2y^T(t) H^T z(t) \\ &\qquad + \operatorname{Trace}\left[f^T(x(t), y(t), t) Pf(x(t), y(t), t)\right]. \end{aligned} \tag{18}$$

By applying Lemmas 3 and 4, it follows that

$$\begin{aligned} &2x^T(t) PAx(t) \le x^T(t)\left(P\frac{U_a + L_a}{2} + \frac{U_a^T + L_a^T}{2}P\right) \\ &\quad \cdot x(t) + \lambda_{\max}(P) \times 2\left\|\frac{U_a - L_a}{2}\right\| x^T(t) x(t), \end{aligned} \tag{19}$$

$$\begin{aligned} &2x^T(t) PBy(t) \le x^T(t)\left(P\frac{U_b + L_b}{2}\right) y(t) + y^T(t) \\ &\quad \cdot \left(\frac{U_b^T + L_b^T}{2}P\right) x(t) + \varepsilon_1 x^T(t) P^2 x(t) \\ &\quad + \varepsilon_1^{-1}\left\|\frac{U_b - L_b}{2}\right\|^2 y^T(t) y(t), \end{aligned} \tag{20}$$

$$\begin{aligned} &2x^T(t) Pz(t) \le \lambda_{\max}(P) \times \left[\varepsilon_2 x^T(t) x(t) \right. \\ &\quad \left. + \varepsilon_2^{-1} z^T(t) z(t)\right], \end{aligned} \tag{21}$$

$$\begin{aligned}
&-2y^T(t)H^TAx(t) \le \varepsilon_3 y^T(t)H^THy(t) + \varepsilon_3^{-1}x^T(t) \\
&\quad \cdot A^TAx(t) \le \varepsilon_3\left(\left\|\frac{U_h+L_h}{2}\right\| + \left\|\frac{U_h-L_h}{2}\right\|\right)^2 y^T(t) \\
&\quad \cdot y(t) + \varepsilon_3^{-1}\left(\left\|\frac{U_a+L_a}{2}\right\| + \left\|\frac{U_a-L_a}{2}\right\|\right)^2 x^T(t) \\
&\quad \cdot x(t),
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-2y^T(t)H^TBy(t) \le 2y^T(t)L_h^TL_by(t) + 2y^T(t) \\
&\quad \cdot L_h^T\Delta By(t) + 2y^T(t)\Delta^THL_by(t) + 2y^T(t) \\
&\quad \cdot \Delta^TH\Delta By(t) \le 2\left(\left\|\frac{U_h+L_h}{2}\right\| \cdot \left\|\frac{U_b+L_b}{2}\right\|\right. \\
&\quad + \left\|\frac{U_h+L_h}{2}\right\| \cdot \left\|\frac{U_b-L_b}{2}\right\| + \left\|\frac{U_h-L_h}{2}\right\| \\
&\quad \left.\cdot \left\|\frac{U_b+L_b}{2}\right\| + \left\|\frac{U_h-L_h}{2}\right\| \cdot \left\|\frac{U_b-L_b}{2}\right\|\right) y^T(t)y(t),
\end{aligned} \tag{23}$$

$$\begin{aligned}
&-2y^T(t)H^Tz(t) \le \varepsilon_4 y^T(t)H^THy(t) + \varepsilon_4^{-1}z^T(t)z(t) \\
&\quad \le \varepsilon_4\left(\left\|\frac{U_h+L_h}{2}\right\| + \left\|\frac{U_h-L_h}{2}\right\|\right)^2 y^T(t)y(t) \\
&\quad + \varepsilon_4^{-1}z^T(t)z(t),
\end{aligned} \tag{24}$$

$$\begin{aligned}
&z^T(t)z(t) \le \tau\int_0^\tau |G(s)x(t-s)|^2\,ds \\
&\quad \le \tau\sup_{0\le s\le\tau}\|G(s)\|^2\int_{t-\tau}^t x^T(s)x(s)\,ds.
\end{aligned} \tag{25}$$

From assumptions (H2), we can get

$$\begin{aligned}
&\operatorname{Trace}\left[f^T(x(t),y(t),t)Pf(x(t),y(t),t)\right] \\
&\quad \le \lambda_{\max}(P)\left[\alpha x^T(t)x(t) + \beta y^T(t)y(t)\right].
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\text{Obviously, } -\int_{t-\tau}^t x^T(s)Rx(s)\,ds \\
&\quad \le -\lambda_{\min}(R)\int_{t-\tau}^t x^T(s)x(s)\,ds.
\end{aligned} \tag{27}$$

Then, combining (18)–(27) and noting (7), we have

$$\begin{aligned}
LV(x(t),t) &\le \left(x^T(t), y^T(t)\right)\Psi\begin{pmatrix} x(t) \\ y(t)\end{pmatrix} \\
&\le \lambda_{\max}(\Psi)\left(|x(t)|^2 + |y(t)|^2\right) \\
&\le \lambda_{\max}(\Psi)|x(t)|^2,
\end{aligned} \tag{28}$$

where Ψ is defined in (13).

Now, let $\Phi = e^{rt}V(x(t),t)$, and by (28), we can get

$$\begin{aligned}
&L\Phi(x(t),t) \\
&\quad \le re^{rt}\left[\lambda_{\max}(P)\left(1+k^2\right)\left[|x(t)|^2 + |y(t)|^2\right]\right. \\
&\quad + \lambda_{\max}(Q)\int_{t-\tau}^t |x(s)|^2\,ds \\
&\quad \left. + \tau\lambda_{\max}(R)\int_{t-\tau}^t |x(s)|^2\,ds\right] + e^{rt}\lambda_{\max}(\Psi)|x(t)|^2.
\end{aligned} \tag{29}$$

Thus, it can be seen that

$$\begin{aligned}
&\int_0^t EL\Phi(x(s),s)\,ds \le \left[r\lambda_{\max}(P)\left(1+k^2\right)\left(1+e^{r\tau}\right)\right. \\
&\quad \left. + r\tau e^{r\tau}\left(\lambda_{\max}(Q) + \tau\lambda_{\max}(R)\right) + \lambda_{\max}(\Psi)\right] \\
&\quad \cdot \int_0^t e^{rs}E|x(s)|^2\,ds + \left[r\tau\lambda_{\max}(P)\left(1+k^2\right)e^{r\tau}\right. \\
&\quad \left. + r\tau^2e^{r\tau}\left(\lambda_{\max}(Q) + \tau\lambda_{\max}(R)\right)\right]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2.
\end{aligned} \tag{30}$$

Set $f(r) = r\lambda_{\max}(P)(1+k^2)(1+e^{r\tau}) + r\tau e^{r\tau}[\lambda_{\max}(Q) + \tau\lambda_{\max}(R)] + \lambda_{\max}(\Psi)$.

We have $f'(r) = \lambda_{\max}(P)(1+k^2) + [\tau\lambda_{\max}(Q) + \tau^2\lambda_{\max}(R)](1+r\tau)e^{r\tau}$.

Since $f'(r) > 0$, $f(0) = \lambda_{\max}(\Psi) < 0$ and $f(+\infty) = +\infty$, then (12) has a unique solution $r > 0$.

By (30) and if (12) holds, we can obtain

$$\begin{aligned}
&\int_0^t EL\Phi(x(s),s)\,ds \le \left[r\tau\lambda_{\max}(P)\left(1+k^2\right)e^{r\tau}\right. \\
&\quad \left. + r\tau^2e^{r\tau}\left(\lambda_{\max}(Q) + \tau\lambda_{\max}(R)\right)\right]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2.
\end{aligned} \tag{31}$$

On the other hand, it is easy to see

$$\begin{aligned}
&E\Phi(x(0),0) \\
&\quad \le \left[\lambda_{\max}(P)\left(2+2k^2\right) + \tau\lambda_{\max}(Q) + \tau^2\lambda_{\max}(R)\right] \\
&\quad \cdot \sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2.
\end{aligned} \tag{32}$$

Then, it follows from (31) and (32) that

$$\begin{aligned}
&E\Phi(x(t),t) = E\Phi(x(0),0) + \int_0^t EL\Phi(x(s),s)\,ds \\
&\quad \le \left[\lambda_{\max}(P)\left(1+k^2\right)\left(2+r\tau e^{r\tau}\right)\right. \\
&\quad + \left(r\tau^2e^{r\tau} + \tau\right)\lambda_{\max}(Q)\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2 \\
&\quad \left. + \left(r\tau^3e^{r\tau} + \tau^2\right)\lambda_{\max}(R)\right]\sup_{-\tau\le\theta\le 0}E|\xi(\theta)|^2.
\end{aligned} \tag{33}$$

Note that

$$\begin{aligned} &E\left[e^{rt}V(x(t),t)\right] \\ &\quad \geq e^{rt}\left[(1-k)E|x(t)|^2-\left(\frac{1}{k}-1\right)k^2E\left|y(t)\right|^2\right], \\ &|H|\leq k=\left\|\frac{U_h+L_h}{2}\right\|+\left\|\frac{U_h-L_h}{2}\right\|<1. \end{aligned} \tag{34}$$

Then, we can derive

$$\begin{aligned} e^{rt}E|(x(t),t)|^2 \leq{}& \frac{1}{1-k}E\left[e^{rt}V(x(t),t)\right] \\ &+ke^{rt}\sup_{-\tau\leq\theta\leq 0}E|x(t+\theta)|^2. \end{aligned} \tag{35}$$

By (33), and note that definition of C_0, it is easy to obtain

$$\begin{aligned} e^{rt}E|(x(t),t)|^2 \leq{}& \frac{C_0}{1-k}\sup_{-\tau\leq\theta\leq 0}E\left|\xi(\theta)\right|^2 \\ &+ke^{r\tau}\sup_{-\tau\leq t\leq T}\left(e^{rt}E|x(t)|^2\right). \end{aligned} \tag{36}$$

So, we also have

$$\begin{aligned} \sup_{-\tau\leq t\leq T}e^{rt}E|(x(t),t)|^2 \leq{}& \frac{C_0}{1-k}\sup_{-\tau\leq\theta\leq 0}E\left|\xi(\theta)\right|^2 \\ &+ke^{r\tau}\sup_{-\tau\leq t\leq T}\left(e^{rt}E|x(t)|^2\right). \end{aligned} \tag{37}$$

By (37) and noting $1-ke^{r\tau}>0$, we can get

$$\begin{aligned} &\sup_{-\tau\leq t\leq T}e^{rt}E|(x(t),t)|^2 \\ &\quad\leq \frac{C_0}{(1-k)(1-ke^{r\tau})}\sup_{-\tau\leq\theta\leq 0}E\left|\xi(\theta)\right|. \end{aligned} \tag{38}$$

That is, (10) holds. This completes the proof of Theorem 6. □

Corollary 7. *System (1) is mean square exponentially stable, if $k=\|(U_h+L_h)/2\|+\|(U_h-L_h)/2\|<1$ and there exist symmetric matrices $Q>0$ and scalars $m>0$, $\varepsilon_1>0$, $\varepsilon_2>0$, $\varepsilon_3>0$, $\varepsilon_4>0$, such that*

$$\begin{pmatrix} \Sigma_1 & m\dfrac{U_b+L_b}{2} & N \\ m\dfrac{U_b^T+L_b^T}{2} & \Sigma_2 & 0 \\ N^T & 0 & -J \end{pmatrix}<0, \tag{39}$$

where

$$\begin{aligned} \Sigma_1 ={}& Q+\tau^2\left[\left(\varepsilon_2^{-1}+\varepsilon_4^{-1}\right)\sup_{0\leq s\leq\tau}\|G(s)\|^2\right]I_n \\ &+m\left(\frac{U_a+L_a}{2}+\frac{U_a^T+L_a^T}{2}\right)+\left(2\left\|\frac{U_a-L_a}{2}\right\|+\varepsilon_2\right. \\ &\left.+\alpha\right)mI_n+\varepsilon_3^{-1}\left(\left\|\frac{U_a+L_a}{2}\right\|+\left\|\frac{U_a-L_a}{2}\right\|\right)^2I_n \\ &+\varepsilon_1m^2I_n, \\ \Sigma_2 ={}& -Q+\left[\varepsilon_1^{-1}\left\|\frac{U_b-L_b}{2}\right\|^2+\varepsilon_3\left(\left\|\frac{U_g+L_g}{2}\right\|\right.\right. \\ &\left.+\left\|\frac{U_g-L_g}{2}\right\|\right)^2+\varepsilon_4\left(\left\|\frac{U_h+L_h}{2}\right\|+\left\|\frac{U_h-L_h}{2}\right\|\right)^2 \\ &+2\left(\left\|\frac{U_h+L_h}{2}\right\|\cdot\left\|\frac{U_b+L_b}{2}\right\|+\left\|\frac{U_h+L_h}{2}\right\|\right. \\ &\cdot\left\|\frac{U_b-L_b}{2}\right\|+\left\|\frac{U_h-L_h}{2}\right\|\cdot\left\|\frac{U_b+L_b}{2}\right\| \\ &\left.\left.+\left\|\frac{U_h-L_h}{2}\right\|\cdot\left\|\frac{U_b-L_b}{2}\right\|\right)+m\beta\right]I_n, \\ N ={}& \left(mI_n\ \ mI_n\ \ mI_n\ \ mI_n\right), \\ J ={}& \operatorname{diag}\left(\varepsilon_1I_n\ \ \varepsilon_2I_n\ \ \varepsilon_3I_n\ \ \varepsilon_4I_n\right). \end{aligned} \tag{40}$$

Then, for all $\xi\in L^2_{F_0}([-\tau,0];R^n)$, we have

$$E\left|x(t,\xi)\right|^2\leq\frac{C_1}{(1-k)(1-ke^{r\tau})}e^{-rt}\sup_{-\tau\leq\theta\leq 0}E\left|\xi(\theta)\right|^2; \tag{41}$$

here,

$$\begin{aligned} C_1 ={}& m\left(1+k^2\right)\left(2+r\tau e^{r\tau}\right)+\left(r\tau^2e^{r\tau}+\tau\right)\lambda_{\max}(Q) \\ &+\left(r\tau^4e^{r\tau}+\tau^3\right)\left(\varepsilon_2^{-1}+\varepsilon_4^{-1}\right)\sup_{0\leq s\leq\tau}\|G(s)\|^2, \end{aligned} \tag{42}$$

and r is the unique positive solution of the following equation

$$\begin{aligned} &rm\left(1+k^2\right)\left(1+e^{r\tau}\right) \\ &\quad+r\tau e^{r\tau}\left[\lambda_{\max}(Q)+\tau^2\left(\varepsilon_2^{-1}+\varepsilon_4^{-1}\right)\sup_{0\leq s\leq\tau}\|G(s)\|^2\right] \\ &\quad+\lambda_{\max}(\Phi)=0, \end{aligned} \tag{43}$$

$$\text{with } \Phi:=\begin{pmatrix} \Sigma_1+\sum_{i=1}^{4}\varepsilon_i^{-1}m^2I_n & m\dfrac{U_b+L_b}{2} \\ m\dfrac{U_b^T+L_b^T}{2} & \Sigma_2 \end{pmatrix}<0,$$

and $1+r\tau>0$, $1-ke^{r\tau}>0$.

Proof. Let $P = mI_n$, $R = [(\varepsilon_2^{-1} + \varepsilon_4^{-1})\tau \sup_{0\le s\le\tau}\|G(s)\|^2]I_n$; similar to the method used in Theorem 6, the proof of Corollary 7 is completed. □

Remark 8. If the interval matrices A_I, B_I, and H_I are replaced by deterministic matrices $\widehat{A}$, $\widehat{B}$, and $\widehat{H}$, then grey system (1) will become the deterministic stochastic system:

$$\begin{aligned} &d\left[x(t) - \widehat{H}x(t-\tau)\right] \\ &\quad = \left[\widehat{A}x(t) + \widehat{B}x(t-\tau) + \int_0^\tau G(s)\,x(t-s)\,ds\right]dt \\ &\qquad + f(x(t), x(t-\tau), t)\,dw(t), \quad t \ge 0, \\ &x_0 = \xi, \quad \xi \in L^2_{F_0}([-\tau, 0]; R^n), \ -\tau \le t \le 0. \end{aligned} \tag{44}$$

Let $L_a = U_a = \widehat{A}$, $L_b = U_b = \widehat{B}$, and $L_h = U_h = \widehat{H}$, and similar to the method used in Corollary 7, the criterion of the deterministic system (44) can be obtained.

Corollary 9. *System (44) is mean square exponentially stable, if $p = \|H\| < 1$ and there exist symmetric matrices $Q > 0$ and scalars $m > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, such that*

$$\begin{pmatrix} \Omega_1 & m\widehat{B} & N \\ m\widehat{B}^T & \Omega_2 & 0 \\ N^T & 0 & -J \end{pmatrix} < 0, \tag{45}$$

where

$$\begin{aligned} \Omega_1 &= Q + \tau^2\left[\left(\varepsilon_2^{-1} + \varepsilon_4^{-1}\right)\sup_{0\le s\le\tau}\|G(s)\|^2\right]I_n \\ &\quad + m\left(\widehat{A} + \widehat{A}^T\right) + (\varepsilon_2 + \alpha)mI_n + \varepsilon_3^{-1}\left\|\widehat{A}\right\|^2 I_n \\ &\quad + \varepsilon_1 m^2 I_n, \\ \Omega_2 &= -Q \\ &\quad + \left(\varepsilon_3\left\|\widehat{H}\right\|^2 + \varepsilon_4\left\|\widehat{H}\right\|^2 + 2\left\|\widehat{H}\right\|\cdot\left\|\widehat{B}\right\| + m\beta\right)I_n, \\ N &= \left(mI_n \ \ mI_n \ \ mI_n \ \ mI_n\right), \\ J &= \operatorname{diag}\left(\varepsilon_1 I_n \ \ \varepsilon_2 I_n \ \ \varepsilon_3 I_n \ \ \varepsilon_4 I_n\right). \end{aligned} \tag{46}$$

Then, for all $\xi \in L^2_{F_0}([-\tau, 0]; R^n)$, we have

$$E\left|x(t,\xi)\right|^2 \le \frac{C_2}{(1-p)(1-pe^{r\tau})}e^{-rt}\sup_{-\tau\le\theta\le 0}E\left|\xi(\theta)\right|^2; \tag{47}$$

here,

$$\begin{aligned} C_2 &= m\left(1+p^2\right)\left(2 + r\tau e^{r\tau}\right) + \left(r\tau^2 e^{r\tau} + \tau\right)\lambda_{\max}(Q) \\ &\quad + \left(r\tau^4 e^{r\tau} + \tau^3\right)\left(\varepsilon_2^{-1} + \varepsilon_4^{-1}\right)\sup_{0\le s\le\tau}\|G(s)\|^2, \end{aligned} \tag{48}$$

and r is the unique positive solution of the following equation

$$\begin{aligned} &rm\left(1+p^2\right)\left(1+e^{r\tau}\right) \\ &\quad + r\tau e^{r\tau}\left[\lambda_{\max}(Q) + \tau^2\left(\varepsilon_2^{-1} + \varepsilon_4^{-1}\right)\sup_{0\le s\le\tau}\|G(s)\|^2\right] \\ &\quad + \lambda_{\max}(\Pi) = 0, \\ &\text{with } \Pi := \begin{pmatrix} \Omega_1 + \sum_{i=1}^4 \varepsilon_i^{-1} m^2 I_n & m\widehat{B} \\ m\widehat{B}^T & \Omega_2 \end{pmatrix} < 0, \end{aligned} \tag{49}$$

and $1 + r\tau > 0$, $1 - ke^{r\tau} > 0$.

Theorem 10. *Under the conditions of Theorem 6, grey system (1) is almost surely exponential stability. That is, for all $\xi \in L^2_{F_0}([-\tau, 0]; R^n)$, we have*

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left|x(t;\xi)\right| \le -\frac{\widehat{r}}{2}, \quad \text{a.s.}, \tag{50}$$

where $\widehat{r} = \min\{r, \tau^{-1}, \ln k^{-1}\}$, $k = \|(U_h + L_h)/2\| + \|(U_h - L_h)/2\| < 1$, and r satisfies (12).

Proof. Under the conditions of Theorem 6, system (1) is mean square exponentially stable, so we have

$$E\left|x(t;\xi)\right|^2 \le K\sup_{-\tau\le\theta\le 0}E\left|\xi(\theta)\right|^2 e^{-rt}. \tag{51}$$

Using Doob's martingale inequality, Cauchy inequality, from (51), we have

$$\begin{aligned} &P\left(\omega : \sup_{0\le s\le\tau}\left|x(k\tau + s) - G(\widehat{\otimes})x_{k\tau+s}\right|^2 > e^{-(r-\delta)k\tau}\right) \\ &\quad \le K\left(1+k^2\right)\left(1+e^{-r\tau}\right)e^{-\delta k\tau}\sup_{-\tau\le\theta\le 0}E\left|\xi(\theta)\right|^2. \end{aligned} \tag{52}$$

Then, using Borel-Cantelli lemma, and similar to the method used in Theorem 2 (see [23, 24]), we complete the proof of Theorem 10. □

4. Numerical Examples

Example 1. Consider a grey neutral systems with distributed delays:

$$\begin{aligned} &d\left[x(t) - H_I x(t-\tau)\right] \\ &\quad = \left[A_I x(t) + B_I x(t-\tau) + \int_0^{0.5} G(s)\,x(t-s)\,ds\right]dt \\ &\qquad + f(x(t), x(t-\tau), t)\,dw(t), \quad t \ge 0, \\ &x_0 = \xi, \quad \xi \in L^2_{F_0}([-0.5, 0]; R^n), \ -0.5 \le t \le 0. \end{aligned} \tag{53}$$

Table 1: Calculation values of ε_1, ε_2, ε_3, ε_4, and r in Example 1.

	Constants				The Lyapunov exponent
	ε_1	ε_2	ε_3	ε_4	r
Values	0.5281	0.7163	0.8210	0.8516	1.1025

Here, the lower bound and upper bound matrices of A_I, B_I, and H_I are

$$\begin{aligned} L_a &= \begin{bmatrix} -3.75 & -0.52 \\ 0.53 & -3.35 \end{bmatrix}, \\ U_a &= \begin{bmatrix} -3.63 & -0.48 \\ 0.58 & -3.25 \end{bmatrix}, \\ L_b &= \begin{bmatrix} -0.65 & 0.20 \\ 0.13 & -0.66 \end{bmatrix}, \\ U_b &= \begin{bmatrix} -0.62 & 0.22 \\ 0.18 & -0.60 \end{bmatrix}, \\ L_h &= \begin{bmatrix} -0.35 & 0.13 \\ 0.15 & -0.28 \end{bmatrix}, \\ U_h &= \begin{bmatrix} -0.30 & 0.19 \\ 0.21 & -0.25 \end{bmatrix}. \end{aligned} \tag{54}$$

Moreover,

$$\begin{aligned} & f(x(t), x(t-0.5), t) \\ & \quad = \begin{bmatrix} \frac{1}{2} x_1(t) \sin(x_2(t-0.5)) \\ \frac{1}{2} x_2(t) \sin(x_1(t-0.5)) \end{bmatrix}, \\ & G(s) = \begin{bmatrix} e^{-(s+3)/6} & 0 \\ 0 & e^{-(s+2)/3} \end{bmatrix}. \end{aligned} \tag{55}$$

Then, we have

$$\begin{aligned} & \operatorname{Trace}\left[f^T(x(t), x(t-0.5), t) \right. \\ & \quad \left. \cdot f(x(t), x(t-0.5), t) \right] \le 0.25 x^2(t), \\ & \sup_{0 \le s \le 0.5} \|G(s)\|^2 = e^{-1}. \end{aligned} \tag{56}$$

By the calculation procedure programmed in [24], we can obtain and optimize scalars ε_1, ε_2, ε_3, and ε_4, which satisfy the criteria conditions in Theorem 6, and make the Lyapunov exponent much smaller. Now, the computed allowable values of constants and the Lyapunov exponent are shown in Table 1.

According to Theorem 6, and from Table 1, it is obvious that grey system (53) is exponentially stable in mean square, although the given system is described uncertainly. Moreover, it should be pointed out that [24] merely studies the exponential stability of a linear grey stochastic system. But, in this example, we can see that our work is suited to more general nonlinear grey stochastic systems. In the meantime, Theorem 6 only needs four positive scalars.

Table 2: Comparison of the Lyapunov exponent r for different methods.

Methods	The Lyapunov exponent
Sun and Liu [7]	0.8985
Song et al. [15]	0.9962
Corollary 9	1.0296

Example 2. Consider the stochastic system (44) with

$$\begin{aligned} \widehat{A} &= \begin{bmatrix} -3.70 & -0.55 \\ 0.49 & -3.26 \end{bmatrix}, \\ \widehat{B} &= \begin{bmatrix} -0.60 & 0.28 \\ 0.16 & -0.72 \end{bmatrix}, \\ \widehat{H} &= \begin{bmatrix} -0.32 & 0.17 \\ 0.20 & -0.26 \end{bmatrix}. \end{aligned} \tag{57}$$

For this deterministic system, if Corollary 9 in this paper is used, and by the Matlab LMI Control Toolbox, we can know that the Lyapunov exponent r of this example is 1.0296.

For a comparison with the results of other methods and ours, the Lyapunov exponent r is shown in Table 2.

It can be seen from Table 2 that,the Lyapunov exponent r obtained by Corollary 9 is larger. That is, the result obtained by Corollary 9 is less conservative than that in [7, 15] and is an improvement of those results. Therefore, the criterion in Corollary 9 is valid. In addition, it is worth noting that since the parameters of given system are deterministic, the criterion in [7, 15] cannot be applied to Example 1. But Corollary 9 can be generalized, and we can obtain Corollary 7 to solve Example 1. Meanwhile, in this example, Corollary 9 only seeks a positive define matrix.

5. Conclusion

This paper has focused on the exponential stability problem for a grey neutral stochastic system with distributed delays. By the Lyapunov stability theorem, Itô's formula, and decomposition technique of the continuous matrix-covered sets of grey matrix, some novel stability criteria have been obtained, which ensure our grey system in the mean-square exponential stability and almost surely exponential stability. Finally, we provide two examples to show the effectiveness of the proposed method.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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